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pp-waves in modified gravity

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Abstract: The family of metrics corresponding to the plane-fronted gravitational waves with parallel propagation, commonly referred to as the family of pp-wave metrics, is studied in the context of various modified gravitational models in a self-contained and coherent manner by using a variant of the null coframe formulation of Newman and Penrose and the exterior algebra of differential forms on pseudo-Riemannian manifolds.

Key words: Classical general relativity, Einstein–Maxwell spacetimes, spacetimes with fluids, radiation or classical fields, wave propagation and interactions, modified theories of gravity, Chern–Simons gauge theory, exact solutions

1. Introduction

The purpose of the present work is to present a brief and self-contained discussion of the pp-wave solutions to the modified gravitational models in connection with the corresponding solutions in the general theory of relativity (GR). In particular, the previous results reported in [1, 2, 3, 4, 5, 6] and some other results long known in the literature are presented in a unified manner by making use of the algebra exterior forms and the complex null tetrad formalism [7, 8] of Newman and Penrose (NP).

The layout of the paper is as follows. The paper is roughly divided into two main parts. In the first part, the geometrical techniques of null coframe formalism in connection with a variant of the spin coefficient formalism of Newman and Penrose is developed in Section 2. The mathematical properties of the gravitational plane fronted waves with parallel propagation, so-called pp-waves, are briefly reviewed in the notation developed. The second part is composed of the application of the geometrical techniques to various metric theories of modified gravity that allow a unified treatment presented in Section 3. In particular, the derivation of the field equations for the Brans–Dicke (BD) theory and the Chern–Simons modified gravity are discussed in a relatively more detailed manner compared to the discussions of the other models such as the metric $f(R)$ gravity and the gravity model involving a nonminimally coupled Maxwell field and a tensor-tensor gravity theory with a torsion. The pp-waves solutions to the general quadratic curvature gravity in four dimensions are also discussed in some detail as well.

The references are not by any means complete and the reader is referred to more authoritative books [9, 10, 11] that introduce the NP technique in full detail and apply it extensively in the context of GR.

2. Geometrical preliminaries

In this preliminary section, the essentials of the null tetrad formalism that are required to derive almost any geometrical quantity from scratch are presented. However, because the family of pp-wave metrics has a relatively

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simple form to deal with, a thorough presentation of the NP formalism, which would be excessive to the purposes in what follows, is avoided.

The use of exterior algebra of differential forms on pseudo-Riemannian manifolds is made use of throughout the paper, and it proves to be practical and powerful in calculations relative to an orthonormal or a null coframe and a coordinate coframe as well. The expressions for the geometrical quantities relative to a coordinate coframe are only discussed briefly in connection with the corresponding expressions relative to a null coframe.

2.1. The definitions of connection and curvature forms relative to a null coframe

All the calculations in the following will be carried out relative to a set of orthonormal and/or complex null basis coframe 1-forms, denoted by $\{\theta^a\}$, for which the metric reads $g = \eta_{ab}\theta^a \otimes \theta^b$ with the metric components η_{ab} as constants. The mathematical conventions closely follow those of the "Exact Solutions" books [9, 10]. The signature of the metric is assumed to be mostly plus. The set of basis frame fields is $\{e_a\}$ and the abbreviation $i_{e_a} \equiv i_a$ is used for the contraction operator with respect to the basis frame field e_a. $*$ denotes the Hodge dual operator acting on basis forms and $*1 = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$ is the oriented volume element. When the Einstein summation convention is used, the exterior products of the basis 1-forms are also abbreviated as $\theta^a \wedge \theta^b \wedge \cdots \wedge \theta_c \wedge \theta_d \cdots \equiv \theta^{ab\cdots}{}_{cd\cdots}$ for the sake of notational simplicity. The complex null coframe basis will also be denoted by $\{\theta^a\} = \{k, l, m, \bar{m}\}$ with $a = 0, 1, 2, 3$ and an overbar denotes a complex conjugation. The associated complex basis frame fields will be denoted by $\{e_a\}$. In terms of the NP-type null coframe basis, the invariant volume element explicitly reads

$$*1 = \frac{1}{4!}\epsilon_{abcd}\theta^{abcd} = +ik \wedge l \wedge m \wedge \bar{m}, \tag{1}$$

where the completely antisymmetric permutation symbol admits the numerical values values $0, \mp i$ with $\epsilon_{0123} = +i$ relative to a null coframe and in this case the indices are raised and lowered by the metric having nondiagonal and constant elements $\eta_{01} = \eta_{10} = -\eta_{23} = -\eta_{32} = 1$. For example, a numerical index 1 goes to 0 accompanied by a sign change whereas a numerical index 2 goes to 3 retaining the sign.

In particular, it follows from these definitions that the self-dual 2-forms that diagonalize the Hodge dual operator are

$$\begin{aligned} *(k \wedge m) &= ik \wedge m \\ *(l \wedge \bar{m}) &= il \wedge \bar{m} \\ *(k \wedge l - m \wedge \bar{m}) &= i(k \wedge l - m \wedge \bar{m}). \end{aligned} \tag{2}$$

The anti-self-dual 2-forms follow from the complex conjugation of the Hodge duality relations given in Eq. (2) above.

The first structure equations of Cartan with vanishing torsion read

$$\Theta^a = D\theta^a = d\theta^a + \omega^a{}_b \wedge \theta^b = 0 \tag{3}$$

with $\Theta^a = \frac{1}{2}T^a{}_{bc}\theta^b \wedge \theta^c$ where $T^a{}_{bc}$ represents the components of the torsion tensor. D is the covariant exterior derivative acting on tensor-valued forms. A suitable definition and its relation to covariant derivative can be found, for example, in [13, 12]. In terms of the complex connection 1-forms and the null coframe, the first structure equations of Cartan explicitly read

$$\begin{aligned} &dk + \omega^0{}_0 \wedge k + \bar{\omega}^0{}_3 \wedge m + \omega^0{}_3 \wedge \bar{m} = 0, \\ &dl - \omega^0{}_0 \wedge l + \omega^1{}_2 \wedge m + \bar{\omega}^1{}_2 \wedge \bar{m} = 0, \\ &dm + \bar{\omega}^1{}_2 \wedge k + \omega^0{}_3 \wedge l - \omega^3{}_3 \wedge m = 0, \end{aligned} \tag{4}$$

where the complex conjugate of the last equation has been omitted for convenience. Because the Levi-Civita connection is metric-compatible, one has $D\eta_{ab} = d\eta_{ab} - \eta_{ac}\omega^c{}_b - \eta_{bc}\omega^c{}_a$, which implies the antisymmetry $\omega_{ab} + \omega_{ab} = 0$ for a null (and also for an orthonormal) coframe. Consequently, $\omega^1{}_0 = \omega^0{}_1 = \omega^2{}_3 = \omega^3{}_2 = 0$ and there are three complex connection 1-forms

$$\omega^0{}_3, \quad \omega^1{}_2, \quad \tfrac{1}{2}(\omega^0{}_0 - \omega^3{}_3) \tag{5}$$

related to the other connection 1-forms by complex conjugation. Explicitly, one has the conjugation relations $\bar{\omega}^0{}_2 = \omega^0{}_3$, $\bar{\omega}^1{}_2 = \omega^1{}_3$, $\bar{\omega}^0{}_0 = \omega^0{}_0$ and $\bar{\omega}^3{}_3 = \bar{\omega}^2{}_2 = -\bar{\omega}^3{}_3$. The charge conjugation amounts to the interchange $2 \leftrightarrow 3$ of the null coframe indices. The twelve NP spin coefficients can be identified as the components of the above complex connection 1-forms [9]. In contrast to the structure equations expressed in terms of six real connection 1-forms relative to an orthonormal coframe, it is possible to write the structure equations using only three complex connection 1-forms displayed in Eq. (5) as in Eq. (4).

The curvature 2-form $\Omega^a{}_b$ with $\Omega^a{}_b = \frac{1}{2}R^a{}_{bcd}\theta^c \wedge \theta^d$ satisfies the second structure of equations of Cartan

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{6}$$

and in terms of the complex connection 1-forms defined above and the complex curvature 2-forms defined accordingly, the complex structure equations can be written in the form

$$\begin{aligned} \Omega^0{}_3 &= d\omega^0{}_3 - \omega^0{}_3 \wedge \left(\omega^0{}_0 - \omega^3{}_3\right), \\ \Omega^1{}_2 &= d\omega^1{}_2 + \omega^1{}_2 \wedge \left(\omega^0{}_0 - \omega^3{}_3\right), \\ \Omega^0{}_0 - \Omega^3{}_3 &= d(\omega^0{}_0 - \omega^3{}_3) + 2\omega^0{}_3 \wedge \omega^1{}_2, \end{aligned} \tag{7}$$

where the use of complex differential forms reduces the number of independent tensor-valued 2-forms by half compared to the corresponding components defined relative to an orthonormal coframe.

In four dimensions, the curvature 2-form can be decomposed into the irreducible parts in the form [9]

$$\Omega^a{}_b = C^a{}_b + \tfrac{1}{2}\left(\theta^a \wedge S_b - \theta_b \wedge S^a\right) + \tfrac{1}{12}R\theta^a \wedge \theta_b, \tag{8}$$

where $C^a{}_b$ is the traceless fourth-rank Weyl tensor, and the second term on the right-hand side is expressed in terms of second-rank traceless Ricci 1-form $S^a \equiv R^a - \frac{1}{4}R\theta^a$ and the last term is the scalar trace. It is possible to show (see, for example, Ref. [14]) that each term has definite self-duality/self anti-self-duality property by using the defining relation of Eq. (8). Namely, the first and the third terms on the right-hand side are self-dual, whereas the traceless Ricci part constitutes the anti-self-dual. Now, using the above definitions, it is possible to express the decomposition of the curvature 2-form of Eq. (8) relative to a complex null coframe in terms of the original NP curvature scalars as

$$
\begin{aligned}
\Omega^0{}_3 &= -\Psi_0 l\wedge\bar{m} - \Psi_1(k\wedge l - m\wedge\bar{m}) + \Psi_2 k\wedge m \\
&\quad - \Phi_{00} l\wedge m - \Phi_{01}(k\wedge l + m\wedge\bar{m}) + \Phi_{02} k\wedge\bar{m} + \tfrac{1}{12}Rk\wedge m, \\
\tfrac{1}{2}(\Omega^0{}_0 - \Omega^3{}_3) &= +\Psi_1 l\wedge\bar{m} + \Psi_2(k\wedge l - m\wedge\bar{m}) - \Psi_3 k\wedge m \\
&\quad + \Phi_{10} l\wedge m + \Phi_{11}(k\wedge l + m\wedge\bar{m}) - \Phi_{12} k\wedge\bar{m} - \tfrac{1}{24}R(k\wedge l - m\wedge\bar{m}), \\
\Omega^1{}_2 &= +\Psi_2 l\wedge\bar{m} + \Psi_3(k\wedge l - m\wedge\bar{m}) - \Psi_4 k\wedge m \\
&\quad + \Phi_{20} n\wedge m + \Phi_{21}(k\wedge l + m\wedge\bar{m}) - \Phi_{22} k\wedge\bar{m} + \tfrac{1}{12}Rl\wedge\bar{m}.
\end{aligned}
\tag{9}
$$

The scalar NP field equations, i.e. the Ricci identities in component form, can be reproduced from Eqs. (7) and (9) by also taking the original definitions of the NP spin coefficients into account.

The above exterior algebra equations and the definitions belonging to the null coframe formalism closely follow the variant of the NP formalism presented in [9]. The mathematical formula introduced above is sufficient for the description of gravitational wave metrics in the context of modified gravity models starting from scratch provided that the field equations are formulated accordingly by using the algebra of differential forms. Thus, the above geometrical formulas are sufficient for the formulation of the field equations in a form suitable for the discussion below. Except for Section 3.6, in which a particular tensor-tensor model of gravity allowing a torsion that can consistently be set to zero is studied, the discussion on the modified gravity models is confined to the pseudo-Riemannian case.

2.2. The geometrical description of pp-wave metrics

The pp-wave metrics were introduced quite a long time ago by Brinkman [15] and shortly after that by Jeffrey and Baldwin [16]. Subsequently, they were interpreted as the metrics representing gravitational waves [17] by Peres. From an idealized point of view, pp-waves metrics can be regarded as a far-field description of an isolated astrophysical source radiating gravitational waves.

It is well known that the family of pp-wave metrics can conveniently be defined by introducing a covariantly constant geodesic null congruence [18, 19, 20] with all the optical scalars corresponding to shear, divergence, and twist vanishing. The Killing symmetries of the pp-wave metrics were studied by Sippel and Goenner, and by Bondi et al. [21, 22], for a variety of profile functions in both pseudo-Riemannian and Riemann-Cartan geometry settings. Another well-known peculiar property of the pp-wave metric is that all the polynomial scalar invariants vanish [23, 24, 25]. It is of Petrov type N with only one nonvanishing complex Weyl curvature spinor [9]. The classical gravitational plane waves are shown [26] to be unaffected by the vacuum polarization effects to all loop orders.

In terms of the global null coordinates $\{x^\alpha\} = \{u, v, \zeta, \bar{\zeta}\}$ for $\alpha = 0, 1, 2, 3$, respectively, the pp-wave ansatz in so-called Kerr–Schild form can be expressed as

$$
g = -du\otimes dv - dv\otimes du - 2H du\otimes du + d\zeta\otimes d\bar{\zeta} + d\bar{\zeta}\otimes d\zeta \tag{10}
$$

with a real profile function $H = H(u, \zeta, \bar{\zeta})$. For $H = 0$, the metric becomes the Minkowski background. The real null vector ∂_v, the four-fold repeated principle null direction of the Weyl tensor, defines the direction of propagation and that $u =$ constant surfaces are the flat transverse planes.

The particular form of the pp-wave metric given in Eq. (33) is also known to be the Brinkman form and there is yet another useful form that is known as the Rosen form [27]. By a suitable coordinate transformation, the metric in the Brinkman form in Eq. (33) can be related to a corresponding Rosen form, which explicitly illustrates the transverse character of the pp-wave metrics. For the particular case of the plane wave metrics, the explicit coordinate transformations relating the two forms can be found in [10]. In the discussion that follows, the Brinkman form for the family of pp-wave metrics will be used.

To begin with, the metric of Eq. (33) is to be cast into the following familiar complex null form

$$g = -k \otimes l - l \otimes k + m \otimes \bar{m} + \bar{m} \otimes m \tag{11}$$

in terms of null basis coframe 1-forms $k, l, m, \bar{m}$. In practical calculations, such a null coframe can be constructed with the help of a set of orthonormal basis 1-forms as well. Although there are different possible choices for the frames and associated coframes in the literature for the pp-wave metric, a natural choice for the set of basis coframe 1-forms is

$$\theta^0 = k = du, \qquad \theta^1 = l = Hdu + dv, \qquad \theta^2 = m = d\zeta \tag{12}$$

in terms of the complex null coordinates and that $\theta^3 = \bar{\theta}^2$. The other choices of the basis coframe 1-forms can be related to Eq. (12) by, for example, the interchanges $k \leftrightarrow l$ and $m \leftrightarrow \bar{m}$ (see the Appendix). The corresponding volume 4-form defined up to an orientation is explicitly given by

$$*1 = idu \wedge dv \wedge d\zeta \wedge d\bar{\zeta}, \tag{13}$$

which is identical to that of the Minkowski volume 4-form. The set of orthonormal basis frame fields associated with the above coframe can be written as

$$e_0 = -\partial_v, \qquad e_1 = -\partial_u + H\partial_v, \qquad e_2 = \partial_{\bar{\zeta}}, \qquad e_3 = \partial_\zeta. \tag{14}$$

The definition of the frame fields of Eq. (14) are identical to the frame fields adopted in [9]. The set of frame fields are useful in the calculations making use of the tensorial methods, and when this is the case, both of the minus signs in the defining relations of e_0 and e_1 are usually transferred to the coframe fields. The geometrical quantities for the pp-wave metric then can be calculated readily by using the commonplace techniques of the exterior algebra of differential forms.

By definition, the only nonvanishing exterior derivative of basis coframe 1-forms can be expressed in the form

$$dl = -H_\zeta k \wedge m - H_{\bar{\zeta}} k \wedge \bar{m}. \tag{15}$$

In Eq. (15) and in the expressions that follow, a coordinate subscript to a function denotes the partial derivative with respect to the coordinate. Consequently, by making use of the derivative expression in connection with the first structure equations, one readily finds that the only nonvanishing connection 1-form is

$$\omega^1{}_2 = H_\zeta k. \tag{16}$$

As a result, there is only one nonvanishing spin coefficient for the pp-wave metric ansatz. Now using Eq. (16), it is straightforward to show that the vector field $k^a e_a$, associated to the basis 1-form $\theta^0 = k$, is covariantly constant and real.

Turning now to the second structure equations of Eq. (7), it is easy to see that relative to the NP-type complex null coframe of Eq. (12), there are only two nonvanishing curvature components of the curvature

2-form $\Omega^1{}_2$. The pp-wave metric ansatz is known to linearize the curvature 2-forms and, accordingly, the only nonvanishing curvature 2-form components then take the form

$$\Omega^1{}_2 = d\omega^1{}_2 = -H_{\zeta\zeta} k \wedge m - H_{\zeta\bar{\zeta}} k \wedge \bar{m} \tag{17}$$

with $\Omega^0{}_3 = \Omega^0{}_0 = \Omega^3{}_3 = 0$. The nonvanishing components of the Riemann tensor are $R^{13}{}_{02}$ and $R^{13}{}_{03}$ relative to the null coframe. Consequently, one finds $R = 0$. The set of basis 2-form $k \wedge m$, $l \wedge \bar{m}$ and $\frac{1}{2}(k \wedge l - m \wedge \bar{m})$ defined by Eq. (2) are self-dual, whereas their complex conjugates define the set of anti-self-dual 2-forms relative to the null coframe and the volume element defined in Eq. (1). The set of all self-dual and anti-self-dual 2-forms form a convenient basis for the 2-forms.

Thus, for example, the $k \wedge m$ component of the curvature 2-form of Eq. (17) corresponds to the component of the complex Weyl 2-form. The curvature spinors can be obtained by comparing the general expression of Eq. (8) with the curvature expression in Eq. (17). One finds

$$C^1{}_2 = -H_{\zeta\zeta} k \wedge m, \qquad R^1 = -2H_{\zeta\bar{\zeta}} k \tag{18}$$

for the pp-wave metric ansatz of Eq. (10). Accordingly, the nonvanishing curvature scalars are given by

$$\Psi_4 = H_{\zeta\zeta}, \qquad \Phi_{22} = H_{\zeta\bar{\zeta}}. \tag{19}$$

As a side remark, note that in general the Einstein field equations $G_{ab} = \kappa^2 T_{ab}$ can conveniently be implemented directly into the curvature expression as the anti-self-dual part of the curvature expansion of Eq. (8), e.g., Φ_{ik} components of the curvature 2-forms in Eq. (9), in the NP formalism. In addition, the use of exterior algebra also offers some alternate means to calculate the Ricci spinors, as will be exemplified below in the case of the pp-wave metrics.

The explicit form of the Weyl spinor ψ_4 can be obtained after the profile function is determined from the field equations that determines the Ricci components. In a vacuum, the amplitude of a pp-waves is then determined by the nonvanishing component of the Weyl tensor $|\Psi_4|$, whereas its polarization is determined by the angle φ in $\Psi_4 = |\Psi_4| e^{i\varphi}$. By introducing a suitable frame, it is possible to show that the real and imaginary parts can be identified with the usual "+" and "×" transverse polarization modes obtained by linearizing the Einstein field equations around the Minkowski spacetime.

To facilitate the comparison with the coordinate expressions in the literature, it is possible to relate the curvature 2-form expression relative to the orthonormal coframe easily in the particular case of the pp-wave metric ansatz. For this purpose, first note that the curvature expression $\Omega^1{}_2 = d\omega^1{}_2$ can be written in the form $\Omega^1{}_2 = dH_\zeta \wedge k$. Considering the coordinate expressions of the tensorial quantities labeled by the Greek indices, one arrives at the relation

$$\Omega^1{}_2 = \partial_\zeta \partial_{[\alpha} H k_{\beta]} dx^\alpha \wedge dx^\beta, \tag{20}$$

where $\partial_\alpha \equiv \partial/\partial x^\alpha$ and the expansion $k = k_\beta dx^\beta$ of the basis 1-form k are used. The square brackets imply the antisymmetrized indices. Furthermore, it is now convenient to rewrite the partial derivative of the profile function with respect to the complex coordinate ζ in the form

$$\Omega^1{}_2 = \bar{m}^\nu (\partial_\nu \partial_{[\alpha} H) k_{\beta]} dx^\alpha \wedge dx^\beta \tag{21}$$

using the definition of the basis frame vectors. As the curvature 2-form $\Omega^1{}_2$ can be expanded into the coordinate basis with respect to the last pair of indices as in Eq. (20), the first pair of indices can also be expressed in terms of the contractions of the Riemann tensor with the coordinate basis frame fields. Consequently, it follows from the definition of the curvature 2-form that

$$\Omega^1{}_2 = \tfrac{1}{2} l^\mu \bar{m}^\nu R_{\mu\nu\alpha\beta} dx^\alpha \wedge dx^\beta. \tag{22}$$

Finally, by making use of the fact that $g(k,l) = g_{\mu\nu}k^\mu l^\nu = -1$, one readily finds that the coordinate components of the Riemann tensor are given by the relation

$$R_{\mu\nu\alpha\beta} = -4k_{[\mu}(\partial_{\nu]}\partial_{[\alpha}H)k_{\beta]}. \tag{23}$$

The corresponding expression for the Ricci tensor components can be found, by the contraction $R_{\mu\nu} = g^{\alpha\beta}R_{\mu\alpha\nu\beta}$, as

$$R_{\mu\nu} = k_\mu k_\nu g^{\alpha\beta}\partial_\alpha\partial_\beta H = 2k_\mu k_\nu \partial_\zeta \partial_{\bar{\zeta}} H \tag{24}$$

by using the fact that $k^\mu \partial_\mu H = 0$ by definition. Consequently, the scalar curvature defined by $R = g^{\mu\nu}R_{\mu\nu}$ vanishes identically as a consequence of the fact that $k^\mu\partial_\mu$ is a null vector. It is now straightforward to see that any contraction of the vector $k^\mu\partial_\mu$ with the curvature tensor of Eq. (23)vanishes identically. Explicitly, the curvature tensor expression implies the relations $k^\mu R_{\mu\nu\alpha\beta} = 0$ and $k^\mu R_{\mu\nu} = 0$.

Alternatively, Eq. (23) can also be obtained by using the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$. Although the orthonormal coframe expression of Eq. (17) for the curvature tensor is not particularly less convenient than the corresponding expression relative to the coordinate basis from a merely technical point of view, the coordinate expression (23) is more frequently used in the literature.

Now returning to the calculation of the Einstein form relative to the orthonormal coframe, the Einstein 3-form can be calculated from the general formula

$$*G^a = \tfrac{1}{2}\Omega_{bc} \wedge *\theta^{abc}, \tag{25}$$

where $G_a = G_{ab}\theta^b$ is a covector-valued 1-form that can be expressed in terms of Ricci tensor and curvature scalar as $G_a \equiv (R_{ab} - \frac{1}{2}\eta_{ab}R)\theta^b$. Explicitly, by specializing the indices to the null coframe introduced above and with the help of Hodge duality relations, one arrives at

$$*G^1 = \bar{\Omega}^1{}_2 \wedge i\bar{m} - \Omega^1{}_2 \wedge im = -2H_{\zeta\bar{\zeta}} * k. \tag{26}$$

As was noted previously, because the scalar curvature vanishes identically, one has $G^1 = R^1$ and thus the Ricci tensor is of the form $R_{ab}\theta^a \otimes \theta^b = 2H_{\zeta\bar{\zeta}} k \otimes k$ or equivalently in components $R_{\mu\nu} = 2H_{\zeta\bar{\zeta}}k_\mu k_\nu$ after expanding the basis 1-form k to its coordinate components. The expression for the Ricci tensor components is consistent with the previous coordinate expression of Eq. (24). In addition, by using the Ricci spinor definitions [10], one finds $\Phi_{22} \equiv \frac{1}{2}l^\mu l^\nu R_{\mu\nu} = H_{\zeta\bar{\zeta}}$.

It is also worth noting that the contracted second Bianchi identity reduces to $D * G^1 = d * G^1 = 0$ and, consequently, the Einstein 3-form can be rewritten as an exact form [28]. For this purpose, one first notes that

$$*G^1 = id(\bar{\omega}^1{}_2 \wedge m - \omega^1{}_2 \wedge \bar{m}) \tag{27}$$

by making use of $dm = dd\zeta \equiv 0$. This alternate expression can be simplified further. It is possible to verify by direct calculation that for the pp-wave ansatz of Eq. (10), the real null basis 1-forms k, l satisfy the following relation:

$$d * dl = -2H_{\zeta\bar{\zeta}} * k, \tag{28}$$

by making use of (15). Note also that $d*l=0$ by definition. Consequently, one ends up with the relation

$$*G^1=-d*dl \tag{29}$$

illustrating the linearization of the components of the metric ansatz in an explicit and compact form.

The vacuum equations for the pp-wave metric then reduce to the Laplace equation $H_{\zeta\bar{\zeta}}=0$ on the transverse plane spanned by the complex coordinate ζ. This equation has the general solution of the form $H(u,\zeta,\bar{\zeta})=h(u,\zeta)+\bar{h}(u,\bar{\zeta})$ with $h(u,\zeta)$ being an arbitrary complex function of the coordinates u,ζ and it is analytic in ζ. Because the field equations do not determine the profile function fully, it is possible to construct various metrics, for example those involving distribution functions of the form $\delta(u)$.

The pp-wave ansatz in the form above is restrictive in admitting a matter source. However, the discussion can be generalized to include a null electromagnetic field in a straightforward way as follows. The self-dual Faraday 2-form can be defined as

$$\mathcal{F}=\tfrac{1}{2}(F+i*F)=\Phi_0 l\wedge\bar{m}+\Phi_1(k\wedge l-m\wedge\bar{m})-\Phi_2 k\wedge m \tag{30}$$

in terms of the complex Maxwell spinor scalars Φ_k. Maxwell's equations can be written as equations for 3-forms in the form $dF=0$ and $d*F=0$. These equations can also be rewritten in the complex form as $d\mathcal{F}=0$ in terms of self-dual Maxwell 2-form defined by $\mathcal{F}\equiv\frac{1}{2}(F+i*F)$. In terms of F, the components of the energy-momentum 3-forms for the Faraday 2-form field read

$$*T^a[F]=\tfrac{1}{2}(i^aF\wedge *F-F\wedge i^a*F), \tag{31}$$

and note that these expressions are valid relative to an orthonormal frame and to a null coframe. $*T^a[F]$ can also be expressed in terms of self-dual 2-form $\mathcal{F}$ as well by making use of Eq. (30).

For the pp-wave metric, it is straightforward to see that a 2-form $\mathcal{F}$ compatible with the metric ansatz of Eq. (33) can have only one nonvanishing component. For $\mathcal{F}=\Phi_2 k\wedge m$ with the nonvanishing Maxwell spinor $\Phi_2=\Phi_2(u,\zeta,\bar{\zeta})$. Maxwell's equations in the complex form

$$d\mathcal{F}=\partial_{\bar{\zeta}}\Phi_2 k\wedge m\wedge\bar{m}=0 \tag{32}$$

imply that $\partial_{\bar{\zeta}}\Phi_2=0$ and therefore one can define a convenient four-potential 1-form. Explicitly, it is possible to define $\Phi_2=\Phi_2(u,\bar{\zeta})\equiv\partial_\zeta f(u,\zeta)$ with $f(u,\zeta)$ being an arbitrary function analytic in the variable ζ. Consequently, the Faraday 2-form F can be derived from the gauge potential $A=A_a\theta^a$ that has the expression

$$A=[f(u,\zeta)+\bar{f}(u,\bar{\zeta})]du. \tag{33}$$

In terms of the self-dual Faraday 2-form $\mathcal{F}$, and relative to the complex null coframe, the energy-momentum form has the only nonvanishing component

$$*T^1[F]=-2\Phi_2\bar{\Phi}_2*k=-2[\partial_\zeta f(u,\zeta)\partial_{\bar{\zeta}}\bar{f}(u,\bar{\zeta})]*k. \tag{34}$$

The electrovacuum Einstein field equations $*G^a=\kappa^2*T^a[F]$ reduce to $*G^1=\kappa^2*T^1[F]$ or equivalently can be written in the form $\Phi_{22}=\kappa^2\Phi_2\bar{\Phi}_2$, which explicitly reads

$$H_{\zeta\bar{\zeta}}=\kappa^2\partial_\zeta f(u,\zeta)\partial_{\bar{\zeta}}\bar{f}(u,\bar{\zeta}) \tag{35}$$

where $\kappa^2 \equiv 8\pi G/c^4$ with G and c standing for Newton's gravitational constant and the speed of light, respectively. Note that the field equations can also be written as an equation for 3-forms as

$$d * dl = \kappa^2 f_\zeta \bar{f}_{\bar{\zeta}} * k. \tag{36}$$

In general, one can show that the Einstein–Maxwell field equations can be expressed in the form $\Phi_{ik} = \kappa^2 \Phi_i \bar{\Phi}_k$ in terms of the Ricci and the Maxwell spinors with $i, k = 0, 1, 2$ provided that the complex components of the curvature 2-forms of Eq. (7) are identified in terms of the Ricci spinors Φ_{ik} with $i, k = 0, 1, 2$ given in Eq. (9).

The general solution to the field equation of Eq. (35) can be written in the form

$$H(u, \zeta, \bar{\zeta}) = h(u, \zeta) + \bar{h}(u, \bar{\zeta}) + \kappa^2 f(u, \zeta) \bar{f}(u, \bar{\zeta}) \tag{37}$$

with h and f being arbitrary functions of the coordinates u and ζ and analytic in ζ. As has been stated above, the field equations do not determine the u-dependence of the profile function and the complex function h and because the superposition principle holds as an exception, h can be expanded as

$$h(u, \zeta) = \sum_{n=2}^{\infty} a_n(u)\zeta^n + \mu \ln \zeta + \sum_{n=1}^{\infty} b_n(u)\zeta^{-n}. \tag{38}$$

The superposition principle also implies that the pp-waves propagating in the same direction do not interact. $n = 0$ and $n = 1$ terms in the first sum are omitted because these terms can be eliminated by the coordinate transformation defined by

$$\zeta = \zeta' + a(u), \qquad v' = v + b(u) + \dot{a}\bar{\zeta} + \dot{\bar{a}}\zeta, \qquad u' = u \tag{39}$$

that leaves the form of the metric of Eq. (33) invariant provided that the new profile function H' is identified as $H' = H + \ddot{a}\bar{\zeta} + \ddot{\bar{a}}\zeta - \dot{a}\dot{\bar{a}} + \dot{b}$ where $\dot{} \equiv d/du$. Therefore, it is possible to choose the complex functions $a(u)$ and $b(u)$ such that the profile function has the form of Eq. (38) without any loss of generality.

In Eq. (38), the term of the form $\mu \ln \zeta$ requires a null particle source with $T_{00} \sim \delta(u)$. This impulsive pp-wave solution is known as the Aichelburg–Sexl solution [9]. It is originally obtained [29] by boosting the Schwarzchild solution to the speed of light in the limit of the Schwarzchild mass reducing to zero.

The impulsive pp-waves solutions can also be constructed by the geometrical method [10, 30] of Penrose, by cutting the Minkowski background along a null hypersurface and then reattaching the two parts with a warp. The 'Cut and Paste' method of Penrose was used to construct spherical impulsive gravitational waves [31] as well.

One can construct the pp-wave solutions with a profile function involving some other dependence on the real null coordinate u, instead of a Dirac delta function. For example, a sandwich pp-wave metric [10], which has a discontinuity that can be expressed in terms of a a particular function having nonzero values only over a finite interval $u_1 \leq u \leq u_2$, can be constructed.

The particular vacuum solution with $h = a_2(u)\zeta^2$ corresponds to gravitational plane waves (or homogeneous pp-waves) having a constant wave amplitude. The solutions with profile function of the form $b_n(u)\zeta^{-n}$ also require null particles with multipole structure [32]. The solutions with the terms of the form $a_n\zeta^n$ were recently studied in [33, 34, 35, 36, 37] and it was shown that the geodesic structure of these pp-wave spacetimes leads to chaotic motion of the test particles.

It is also possible to construct solutions corresponding to a linear superposition of two distinct null electromagnetic fields. Let us consider a four potential A in Eq. (33) expressed as a superposition of the form

$$f(u,\zeta) = f_1(u,\zeta) + f_2(u,\zeta) \tag{40}$$

where the two independent functions f_1 and f_2 are analytic in ζ. Then, for the superposed electrovacuum metric, the profile function is given by

$$H(u,\zeta) = h(u,\zeta) + \bar{h}(u,\bar{\zeta}) + \kappa^2 \left\{|f_1|^2 + |f_2|^2 + f_1\bar{f}_2 + f_2\bar{f}_1\right\}. \tag{41}$$

Although the pp-wave metric ansatz linearizes the Einstein tensor, a nonlinearity arises from the electromagnetic energy-momentum tensor [38, 39]. On the other hand, the superposition is allowed for the profile function, as well as it is valid at the level of the corresponding curvature tensor as can be observed from Eqs. (16) and (17).

The profile function of an electrovacuum solution can be combined with that of a vacuum solution defined at different regions of the transverse plane [40, 10] to have a new profile function of the form

$$H(u,\zeta,\bar{\zeta}) = \begin{cases} \alpha^2(u)(|\zeta-\zeta_0|^2 - r^2), & |\zeta-\zeta_0| \leq r \\ \alpha^2(u) r^2 \ln(|\zeta-\zeta_0|^2/r^2), & |\zeta-\zeta_0| > r. \end{cases} \tag{42}$$

The composite function H is defined to be continuous across the boundary $|\zeta-\zeta_0| = r$. Note also that the logarithmic part of the profile function is in accordance with the cylindrically symmetric metric exterior to an infinite line source that can be obtained in the linear approximation [40] and that the logarithmic singularity $|\zeta-\zeta_0| \mapsto \infty$ is a coordinate singularity as one can show by studying the geodesics of test particles. The function of Eq. (42) requires an electromagnetic ansatz of the form $f(u,\zeta) = \alpha(u)(\zeta-\zeta_0)$ for the region $|\zeta-\zeta_0| \leq r$ on the transverse planes and that the corresponding metric represent the gravitational field of an infinite uniform beam of light with circular cross-section centered at ζ_0 and having a radius a finite radius r. It is conformally flat in the region $|\zeta-\zeta_0| \leq r$, whereas $\Psi_4 = -(\alpha r)^2(\zeta-\zeta_0)^{-2}$ outside the beam. One can superpose [40, 10] any number of profile functions of the form (42) for the nonintersecting parallel light beams having distinct centers of symmetry.

The pp-waves metrics are solutions in various gravitational theories; for example, the pp-wave solutions were studied in the supergravity theories [41, 42, 43, 44, 45] and in the gravitational models relevant to the low energy limit of string theories [46, 47]. The plane wave solutions to Yang–Mills type non-Abelian gauge fields was first studied by Coleman [48]. Shortly thereafter, Güven [49] presented the extension of Coleman's solutions to curved spacetime. Dereli and Güven [50] later generalized Coleman's solutions to the non-Abelian Yang–Mills gauge fields with supersymetry. Trautmann [51] also presented the plane wave solutions to non-Abelian gauge fields and to the quadratic curvature model discussed in Section 3.5 below.

Another universal and remarkable property of the gravitational plane waves, due to Penrose, is that any spacetime has a plane wave as a limit [52]. The Penrose limit was later extended in [53] to all five of the string theories by Güven. It was also shown by Penrose [54] that there is no global Cauchy hypersurface for the plane wave metrics.

The impulsive gravitational wave solutions to some popular alternative gravitational models were studied previously in [55] by Barrabès and Hogan. The gravitational wave solutions of some modified gravity theories, such as $f(R)$ and the models involving higher curvature terms in their Lagrangian, were previously studied

[56, 57, 58] in the linearized approximation. These solutions to the linearized equations are relevant to the discussion below because it is well known that the pp-wave metrics constitute their own linearizations.

The discussion of pp-wave type solutions below is reserved to a number of modified gravitational models that allow a unified treatment in deriving the field equations that the profile functions satisfy.

3. pp-waves in modified gravity

From the point of view of a modified gravitational model, it is an interesting question to investigate whether the simple family of the pp-wave type metrics can be lifted to a set of gravitational wave solutions for that model.

3.1. pp-waves in Brans–Dicke theory

The Brans–Dicke theory of gravity [59] is one of the most popular scalar-tensor theories of gravity. Although it was proposed to incorporate Machian ideas of inertia into the general relativity theory by introducing a geometry-matter coupling via a dynamical scalar field, it has now become popular in the context of the $f(R)$-type modified gravity models. Some peculiar properties of the radiative metrics in BD theory were studied in [60, 61] and the radiative metrics in the linear approximation in BD theory were previously studied in [62] by Wagoner. More recently, the pp-wave solutions for the BD theory with Maxwell field were studied by Robinson [2], who observed that the BD-Maxwell theory admits solutions with BD scalar with propagating scalar and nonvanishing Maxwell field in the Minkowski background. Interestingly, a certain part of the results pertaining to the BD vacuum case rediscovered recently by Robinson are the pp-wave solutions presented some time ago in [1]. Similar solutions to the scalar tensor theories involving a potential term were also reported in [63]. The cylindrically symmetric gravitational wave solutions generalizing those given by Einstein and Rosen [64] were also discussed in [65] recently. As a side remark related to the paper by Einstein and Rosen, the reader is referred to references [66, 67] for interesting historical accounts on gravitational waves.

In order to be able to use the null coframe language in connection with the exterior algebra developed above, it is necessary to write the field equations relative to a null or orthonormal coframe. This can be achieved for example by using a first-order formalism where the connection and the coframe 1-forms are assumed to be the independent gravitational variables. The field equations for the pseudo-Riemannian metric (equivalently, the equations for coframe 1-forms) are then obtained by constraining the independent connection 1-form to be a Levi-Civita connection as a subcase. Such a formulation has also been worked out, for example, in [68] for the formulation of BD theory including the fermion fields. The details of the application of the first-order formalism to the BD Lagrangian is provided below for convenience.

Expressed in terms of the differential forms, the total Lagrangian 4-form for the original BD theory interacting with matter fields reads

$$L_{tot.} = L_{BD}[\phi, \theta^a, \omega^a{}_b] + \frac{8\pi}{c^4} L_m[g, \psi] \tag{43}$$

where the gravitational part in the so-called Jordan frame is

$$L_{BD} = \frac{\phi}{2}\Omega_{ab} \wedge *\theta^{ab} - \frac{\omega}{2\phi} d\phi \wedge *d\phi. \tag{44}$$

The matter fields with the Lagrangian $L_m[g, \psi]$ are assumed to couple to the metric minimally and are also assumed to be independent of the BD scalar field. The gravitational coupling constant is replaced by a dynamical scalar field ϕ^{-1} with a corresponding kinetic terms for the scalar field. ω is the free BD parameter and in general the corresponding general relativistic expression is recovered in the $\omega \mapsto \infty$ limit. However, such a correspondence is not always warranted [69, 70], as the case with matter energy-momentum tensor having a vanishing trace furnishes a well-known counter example.

In the general framework of first-order formalism for gravity, the independent gravitational variables are the set of basis coframe 1-forms $\{\theta^a\}$ and the connection 1-forms $\{\omega^a{}_b\}$. The local Lorentz invariance of a gravitational Lagrangian forbids the gravitational action to have explicit dependence on $\{\omega^a{}_b\}$ and the exterior derivatives $\{d\theta^a\}$ and $\{d\omega^a{}_b\}$. However, instead of the explicit dependence on derivatives $\{d\theta^a\}$ and $\{d\omega^a{}_b\}$, a gravitational Lagrangian can have dependence on Θ^a and $\Omega^a{}_b$, respectively. Moreover, the minimal coupling prescription for the matter fields also implies that $d\theta^a$ and $d\omega^a{}_b$ occur only in the gravitational sector in a total Lagrangian with matter fields. On the other hand, the BD scalar field ϕ couples nonminimally to the metric simply because it multiplies the scalar curvature. As a consequence of the nonminimal coupling, the BD scalar field is a dynamical field even in the absence of the kinetic term for it.

The vanishing torsion constraint for the independent connection 1-form can be implemented into the variational procedure by introducing Lagrange multiplier 4-form term

$$L_C = \lambda_a \wedge (d\theta^a + \omega^a{}_b \wedge \theta^b) \tag{45}$$

to the original Lagrangian form L_{BD}, where the Lagrange multiplier 2-form λ_a is a vector-valued 2-form enforcing the constraint $\Theta^a = 0$. The Lagrangian for the extended gravitational part then has the explicit form

$$L_{ext.}[\phi, \theta^a, \omega^a{}_b, \lambda_a] = L_{BD}[\phi, \theta^a, \omega^a{}_b] + L_C[\theta^a, \omega^a{}_b, \lambda^a]. \tag{46}$$

The total variational derivative of $L_{ext.}$ with respect to the independent variables can be found as

$$\begin{aligned}\delta L_{ext.} &= \delta\phi \left(\frac{1}{2}R * 1 - \frac{\omega}{2\phi} d * d\phi + \frac{\omega}{2\phi^2} d\phi \wedge *d\phi\right) + \delta\theta_a \wedge \left(\frac{\phi}{2}\Omega_{bc} \wedge *e^{abc} + D\lambda^a + \frac{\omega}{\phi} * T^a[\phi]\right) \\ &+ \delta\omega_{ab} \wedge \tfrac{1}{2}\left[D\phi * \theta^{ab} - (\theta^a \wedge \lambda^b - \theta^b \wedge \lambda^a)\right] + \delta\lambda_a \wedge \Theta^a\end{aligned} \tag{47}$$

up to an omitted exact form. The energy-momentum forms of the scalar field $*T^a[\phi] = T^a{}_b[\phi] * \theta^b$ stand for

$$*T^a[\phi] \equiv \tfrac{1}{2}((i_a d\phi) * d\phi + d\phi \wedge i_a * d\phi). \tag{48}$$

The field equations for the connection then read

$$D(\phi * \theta^{ab}) = \theta^a \wedge \lambda^b - \theta^b \wedge \lambda^a \tag{49}$$

and these equations can be considered as equations for the Lagrange multiplier 2-forms λ^a. Eq. (49) can uniquely be solved for the Lagrange multiplier 2-form by calculating its contractions and taking the constraint $\Theta^a = 0$ into account. Explicitly, by calculating two successive contractions and subsequently combining them, one finds

$$\lambda^a = *(d\phi \wedge \theta^a) \tag{50}$$

as the unique solution. Consequently, using the expression of Eq. (50) for the Lagrange multiplier form in the metric field equations induced by the coframe variational derivative $\delta L_{ext.}/\delta\theta^a \equiv *E^a$ in Eq. (47) read

$$*E^a \equiv -\phi * G^a + D * (d\phi \wedge \theta^a) + \frac{\omega}{\phi} * T^a[\phi] + \frac{8\pi}{c^4} * T^a[F] = 0 \tag{51}$$

where $*E^a = E^a{}_b * \theta^b$ is vector-valued 1-form and $*T^a[F]$ stands for the energy-momentum forms for the Faraday 2-form field F derived from the variational derivative of the Maxwell Lagrangian 4-form with respect to basis coframe 1-forms and it is defined by Eq. (31) above.

As a consequence of the diffeomorphism invariance of the BD Lagrangian, it follows from the corresponding Noether identity that $D * E^a = 0$ [71]. Consequently, from the relation $D * T^a[\psi] = 0$ one can derive the geodesic postulate for point-like test particles. Explicitly, with the help of the identities

$$D(\phi * G^a) = d\phi \wedge *G^a, \tag{52}$$

$$D^2 * (d\phi \wedge \theta^a) = d\phi \wedge *R^a, \tag{53}$$

$$D\left(\frac{\omega}{\phi} * T^a[\phi]\right) = (i^a d\phi)\left(\frac{\omega}{2\phi} d * d\phi - \frac{\omega}{2\phi^2} d\phi \wedge *d\phi\right), \tag{54}$$

one eventually arrives at

$$D * E^a = \tfrac{1}{2}(i^a d\phi)\left(\frac{\omega}{\phi} d * d\phi - \frac{\omega}{\phi^2} d\phi \wedge *d\phi + R * 1\right) \tag{55}$$

as expected. The right-hand side vanishes identically provided that the field equation for the BD scalar is satisfied since the terms on the right-hand side are proportional to the field equations for the BD scalar given below.

In addition, the field equation for the BD scalar that follows from the variational derivative $\delta L_{BD}/\delta\phi$ is given by

$$\omega d * d\phi - \frac{\omega}{\phi} d\phi \wedge *d\phi + \phi R * 1 = 0. \tag{56}$$

The BD scalar couples to the matter energy-momentum through the last term in Eq. (56). In fact, by combining it with the trace of the metric equations, the equation for the BD scalar simplifies to

$$d * d\phi = \frac{8\pi}{c^4} \frac{1}{2\omega + 3} * T[\psi] \tag{57}$$

where $T[\psi] \equiv T^a{}_a[\psi]$ is the trace of the matter energy-momentum tensor. As pointed out above, the reduced scalar field equation of Eq. (57) follows from the Bianchi identity for the BD field equations together with the trace. Since the only matter field present in the discussion is Maxwell field F, $T[F] = 0$ identically and the BD scalar satisfies a homogeneous equation corresponding to Eq. (57).

The BD-Maxwell field equations can be written by specializing the indices of the field equations of Eq. (51) to a complex null coframe. For $a = 0, 1, 2$ relative to a null coframe they can explicitly be written in the form

$$\begin{aligned}
*E^0 &= -\phi * G^0 + D * (d\phi \wedge k) + \frac{\omega}{\phi} * T^0[\phi] + \frac{8\pi}{c^4} * T^0[F] = 0,\\
*E^1 &= -\phi * G^1 + D * (d\phi \wedge l) + \frac{\omega}{\phi} * T^1[\phi] + \frac{8\pi}{c^4} * T^1[F] = 0,\\
*E^2 &= -\phi * G^2 + D * (d\phi \wedge m) + \frac{\omega}{\phi} * T^2[\phi] + \frac{8\pi}{c^4} * T^2[F] = 0,
\end{aligned} \tag{58}$$

respectively. Note that the above expressions for $*E^0$ and $*E^1$ are real, whereas the expressions for the components $*E^2$ and $*E^3$ are complex conjugates. It is worth noting that the BD field equations of Eq. (58) can be written out explicitly in terms of NP quantities with some straightforward work and further definitions identifying the components of the differential forms in terms of original NP curvature spinors and spin coefficients. On the other hand, for the modified gravitational models it is not, in general, possible to use the field equations of the form $E_{ab} = \kappa^2 T_{ab}$ to simply determine the anti-self-dual part of the curvature expression as in the case of GR in the NP formalism.

As a consequence of the simplicity of the pp-wave metrical ansatz, the Einstein field equations are quite restrictive in admitting a matter source. The pp-wave metric ansatz above admits only the scalar field ansatz of the form $\phi = \phi(u)$ and thus $d\phi = \phi_u du$. For such a scalar field there is only one nonvanishing component of the energy-momentum form of the form

$$*T^1[\phi] = -\phi_u^2 * k. \tag{59}$$

Moreover, with the assumption $\phi = \phi(u)$, the only nonvanishing term among $D * (d\phi \wedge \theta^a)$ is for $a = 1$ for which

$$D * (d\phi \wedge l) = d * (d\phi \wedge l) + \omega^1{}_2 \wedge *(d\phi \wedge m) + \bar{\omega}^1{}_2 \wedge *(d\phi \wedge \bar{m}). \tag{60}$$

The second and third terms are complex conjugates of one another, making the left-hand side a real 3-form. Moreover, one has $\omega^1{}_2 \wedge *(d\phi \wedge m) = 0$ and consequently

$$D * (d\phi \wedge l) = -\phi_{uu} * k. \tag{61}$$

Moreover, the terms involving derivatives of the scalar field can be combined to have

$$\begin{aligned} D * (d\phi \wedge l) + \frac{\omega}{\phi} T^1[\phi] &= d * (d\phi \wedge l) + \frac{\omega}{\phi} d\phi \wedge *(d\phi \wedge l) \\ &= \frac{\phi^{-\omega}}{(1+\omega)} d * \left[d(\phi^{(1+\omega)}) \wedge l \right]. \end{aligned} \tag{62}$$

By a direct calculation, it is possible to show that the expression on the right-hand side has the nonvanishing component

$$D * (d\phi \wedge l) + \frac{\omega}{\phi} T^1[\phi] = -\left(\phi_{uu} + \omega \frac{\phi_u^2}{\phi} \right) * k \tag{63}$$

for $\phi = \phi(u)$. Eventually, the only nontrivial equation $*E^1 = 0$ reduces to

$$-\phi d * dl + d * (d\phi \wedge l) + \frac{\omega}{\phi} d\phi \wedge *(d\phi \wedge l) + 2|\Phi_2|^2 * k = 0. \tag{64}$$

Consequently, $*E^1 = 0$ can compactly be rewritten as

$$-\phi d * dl + \frac{\phi^{-\omega}}{(1+\omega)} d * \left[d(\phi^{(1+\omega)}) \wedge l \right] + 2|\Phi_2|^2 * k = 0. \tag{65}$$

It is interesting to note that the BD field equations do not fully determine the profile function as in the GR case. For a profile function with a reasonable dependence on the real null coordinate u, the $\omega \mapsto \infty$ limit yields the GR equations canceling out the second term, provided that one assumes $\phi \mapsto \phi_0 = \kappa^{-2}$ in this limit.

For the BD theory, the following pp-wave solutions can be constructed:

(1) For the vacuum case with $\Phi_2 = 0$, there are two subcases depending on the numerical value of the BD parameter ω, one with $\omega = -1$ and the other with $\omega + 1 \neq 0$. In this case, the first and the second terms in Eq. (65) are set equal to zero separately: $d * dl = -2H_{\zeta\bar{\zeta}} * k = 0$ and

$$d * \left[d(\phi^{(1+\omega)}) \wedge l\right] = 0. \tag{66}$$

Now calculate $d * [df(u) \wedge l]$ for an arbitrary function of $f(u)$ and the basis coframe l. One finds

$$d * [df(u) \wedge l] = -f'' * k - if'(dm \wedge \bar{m} - m \wedge d\bar{m}) \tag{67}$$

where a prime denotes an ordinary derivative with respect to the coordinate u. The second term on the right-hand side vanishes by the definition of the coframe for the pp-wave metric: $dm = dd\zeta \equiv 0$. The first term, on the other hand, vanishes if and only if $f'' = 0$. In other words, $d * [df(u) \wedge l] = 0$ is satisfied iff $f(u) \sim u$ up to a constant. Thus, with the assumption $\omega \neq 1$, one finds

$$\phi(u) = \phi_0 u^{1/(1+\omega)}. \tag{68}$$

The case $\omega = -1$ has to be treated separately starting from Eq. (62). For $\omega = -1$, Eq. (62) explicitly becomes

$$D * (d\phi \wedge l) - \frac{1}{\phi} * T^1[\phi] = \phi d * [(d \ln \phi) \wedge l] \tag{69}$$

where $d \ln \phi \equiv \phi^{-1} d\phi$, and thus in this case the right-hand side vanishes identically iff $\phi(u) = \phi_0 e^u$. These solutions were reported in [1] and then they were rediscovered in [2] recently, including the electromagnetic field into the discussion. The flat spacetime requires a vanishing Riemann tensor (or equivalently the vanishing curvature 2-form) according to the well-known Riemann theorem and that the flat spacetime be a trivial solution of $G_{ab} = 0$. The BD field equations $E_{ab} = 0$ with the pp-wave metric ansatz, on the other hand, also admit the flat spacetime solution together with a propagating scalar field in the flat background.

(2) For the electrovacuum case in the BD theory, it is possible to construct the following general expression

$$H(u, \zeta) = h(u, \zeta) + \bar{h}(u, \bar{\zeta}) - |\zeta|^2 \frac{(\phi^{1+\omega})_{uu}}{2(1+\omega)\phi^{1+\omega}} - \frac{8\pi}{c^4 \phi} f\bar{f} = 0 \tag{70}$$

for profile function. This particular solution was, somewhat surprisingly, reported in [2] recently. The electrovacuum solutions of BD in common with the vacuum solutions of GR now follow if

$$|\zeta|^2 \frac{(\phi^{\omega+1})_{uu}}{(\omega+1)\phi^{(1+\omega)}} + \frac{16\pi}{c^4} f\bar{f} = 0 \tag{71}$$

is satisfied separately from the vanishing of the Einstein tensor. Note that, as in the previous case, the scalar field equation of Eq. (71) is satisfied for the flat spacetime as well. This particular case corresponds to the propagating Maxwell and scalar fields in the flat background and the fields are referred to as nongravitating waves in [2] by Robinson.

In the manner of solving BD field equations with pp-wave metric ansatz as in cases (1) and (2) above, the scalar field equations satisfy the same field equations irrespective of the assumptions either $H_{\zeta\bar{\zeta}} = 0$ or $H = 0$ because the BD field equations in this ansatz can be decoupled into two separate equations, one for the metric and one for the BD scalar field, together with the energy momentum component coming from the Maxwell field. Consequently, the BD field equations admit flat spacetime solutions with propagating BD scalar.

Evidently, the source of the nongravitating propagating scalar field can be traced back to the constraint term, namely the presence of the term $D * (d\phi \wedge \theta^a)$, which arises from the nonminimal coupling of the BD scalar field to the curvature [63]. In the flat spacetime, the term $D * (d\phi \wedge \theta^a)$ becomes

$$d * (d\phi \wedge dx^a) = (P^a{}_b \phi) * dx^b \tag{72}$$

where $*$ now stands for the Hodge dual for Minkowski spacetime, whereas the second-order differential operator P_{ab} is explicitly given by

$$P_{ab} = \partial_a \partial_b - \eta_{ab} \Box, \tag{73}$$

which is a projection operator in Minkowski spacetime with $\Box \equiv \eta^{ab}\partial_a\partial_b$. P_{ab} is also a transverse differential operator: $\partial^a P_{ab} \equiv 0$. Its trace is given by $P^a{}_a = -3\Box$.

By "switching off" the gravitational interaction in the theory, the field equations allow one to have a propagating scalar solution in the Minkowski background. Explicitly, by introducing a potential term $V(\phi) * 1$ into the original BD Lagrangian, the field equations in this case reduce to

$$P_{ab}\phi + \frac{\omega}{\phi} T_{ab}[\phi] + \eta_{ab} V(\phi) = 0. \tag{74}$$

By requiring the existence of nontrivial solutions to these equations, one determines the form of the self-interaction potential term and these solutions were studied in [63] recently.

3.2. pp-waves in a metric $f(R)$ gravity

The simplest modification of the general theory of relativity encompassing sufficient generality involves the modification of the Einstein–Hilbert Lagrangian to a general function of the scalar curvature of the form $f(R)$. Although the field equations for $f(R)$ models were worked out long time ago in [72], these fourth-order models are studied intensively by the current motivations arising mainly from the recent cosmological observations. See, for example, [73, 74] for a thorough review on various aspects of $f(R)$ gravity models.

It is well known that a generic $f(R)$ gravitational model with the Lagrangian

$$L = \tfrac{1}{2} f(R) * 1 \tag{75}$$

has a dynamically equivalent scalar-tensor model [74, 75]. By introducing an auxiliary Lagrangian,

$$L_{aux.} = \{f(\chi) + f'(\chi)(R - \chi)\} * 1 \tag{76}$$

in terms an auxiliary field χ with the prime denoting a derivative with respect to χ and assuming that $f'' \neq 0$, and by using the field equations $\delta L_{aux.}/\delta\chi = 0$ that follow from Eq. (76) to eliminate the auxiliary scalar field χ, one ends up with the equivalent Lagrangian of the form

$$L_{ST} = \frac{\phi}{2} \Omega_{ab} \wedge *\theta^{ab} - \tfrac{1}{2} V(\phi) * 1, \tag{77}$$

similar to the original BD Lagrangian of Eq. (43) with the BD parameter $\omega = 0$ while having an additional potential term for the nonminimally coupled BD-type scalar field. However, the equivalent BD-type scalar-tensor model has no kinetic term for the scalar field ϕ. The potential term in Eq. (77) is obtained by the Legendre transform of the function $f(R)$:

$$V(\phi) = R(\phi)f'(R(\phi)) - f(R(\phi)) \tag{78}$$

and consequently, the potential term $V(\phi)$ with $\phi \equiv f'(R) = df/dR$ in the resulting scalar-tensor equivalent Lagrangian is defined by the Legendre transformation. Eq. (78) also implies $dV/d\phi = R$ and consequently

$$f(R(\phi)) = \phi\frac{dV}{d\phi} - V(\phi). \tag{79}$$

The contribution of $\frac{1}{2}V(\phi) * 1$ to the metric field equations that follow from Eq. (77) is of the form of a variable cosmological term $\frac{1}{2}V(\phi) * \theta^a$. However, such a term is incompatible with the pp-wave metric ansatz of Eq. (10). Thus, it is more convenient to discuss pp-wave solutions by making use of the Lagrangian of Eq. (75) as has previously been discussed, for example, in [3] recently by Mohseni.

The derivation of the field equations for the $f(R)$ models with minimally coupled matter fields ψ can be obtained along the lines of the BD equations that are derived in some detail above (see also Ref. [76]) and it is straightforward to show that the metric field equations that follow from the coframe variation of the Lagrangian 4-form of Eq. (75) are

$$-f' * G^a + D * (df' \wedge \theta^a) + \tfrac{1}{2}(f - Rf') * \theta^a + \kappa^2 * T^a[\psi] = 0 \tag{80}$$

in the form similar to the BD field equations of Eq. (51). Here, $T^a[\psi]$ stands for the energy-momentum 1-form for a matter field ψ. Because the pp-wave metric ansatz is incompatible with a cosmological constant, the form of a generic function $f(R)$ has to be restricted for the pp-wave ansatz to solve the corresponding field equations of Eq. (80).

The fourth-order terms can explicitly be written out in the form

$$D * (df' \wedge \theta^a) = f'' D * (dR \wedge \theta^a) + f''' dR \wedge i^a * dR \tag{81}$$

and this term vanishes identically for the pp-wave ansatz for which $R = 0$ identically. Consequently, for the ansatz of Eq. (10) the metric field equations boil down to the form

$$-f'(0) * G^a + \tfrac{1}{2}f(0) * \theta^a + \kappa^2 * T^a[\psi] = 0. \tag{82}$$

The cosmological-like term for the scalar-tensor equivalent Lagrangian persists in Eq. (82) as well and the only way for the pp-wave ansatz to satisfy these equations is now transformed to the condition that $f(0) = 0$. In this particular case, the $f(R)$ theory field equations reduce to Einstein field equations with a new gravitational coupling constant $\kappa^2/f'(0)$. Hence, the electrovacuum solutions to the Einstein field equations with the effective coupling constant $\kappa^2/f'(0)$ are also solutions to $f(R)$ models with the function $f(R)$ satisfying the condition $f(0) = 0$ and $f'(0) > 0$. Explicit forms for some $f(R)$ models relevant to the cosmological applications satisfying this condition were reported in [3], in a study of Aichelburg–Sexl type solutions in various modified gravity models. It is also interesting to note that, introducing a cosmological term $\Lambda * \theta^a$ to the field equations of Eq. (80), the compatibility condition $f(0) = 0$ then becomes $f(0) + 2\Lambda = 0$. As a consequence, one can conclude that in general the $f(R)$ models admit a particular pp-wave solution with a cosmological constant term.

The linearized field equations of the $f(R)$ model were recently studied in [56] by making use of the scalar-tensor equivalent models of such theories and it was showed explicitly that there is a massive scalar mode of gravitational radiation in addition to the transverse modes.

3.3. pp-waves in a nonminimal $f(R)$ gravity

Another popular $f(R)$ model that allows one to discuss pp-waves in a manner in line with the discussion above is the model that was recently introduced in [77], and it involves two analytical functions $f_1(R)$ and $f_2(R)$. The Lagrangian 4-form of the model can be written in the form

$$L_{n.m.} = \frac{1}{2} f_1(R) * 1 + [1 + \lambda f_2(R)] L_m \tag{83}$$

with a new parameter λ giving the strength of the nonminimal coupling of matter to the modified gravitational Lagrangian function $f_2(R)$. As before, the matter Lagrangian 4-form $L_m \equiv \mathcal{L}_m * 1$ is assumed to depend on the metric tensor but not on the connection 1-forms. One can show that the metric field equations $*E^a = 0$ that follow from $\delta L_{n.m.}/\delta\theta_a \equiv *E^a$ can be written compactly as

$$\begin{aligned} &-(f_1' + 2\lambda f_2' \mathcal{L}_m) * G^a + D * [d(f_1' + 2\lambda f_2' \mathcal{L}_m) \wedge \theta^a] \\ &\quad + \tfrac{1}{2}[(f_1 - R f_1') - 2\lambda R f_2' \mathcal{L}_m] * \theta^a + (1 + \lambda f_2) * T^a[\psi] = 0 \end{aligned} \tag{84}$$

in the same way as the $f(R)$ field equations of Eq. (80) by using the constrained first-order formalism [78]. The derivation of the field equations of Eq. (84) proceeds first by extending the Lagrangian 4-form Eq. (83) by the constraint term Eq. (45), and then solving the connection equations for the Lagrange multiplier and then subsequently using it to obtain the total variational derivative with respect to the coframe as the metric field equations of Eq. (84) as in the BD case. The matter energy-momentum 3-form $*T^a[\psi]$ is defined as the variational derivative $\delta L_m/\delta\theta_a$ as in the GR case.

An important feature of the model that follow from the Lagrangian of Eq. (83) can be explained briefly in the present notation as follows. The Lagrangian 4-form of Eq. (83) is apparently invariant under an arbitrary coordinate transformation and thus leads to the Noether identity $D * E^a = 0$. (see [79] for general conservation laws derived from Lagrange–Noether methods for the models with nonminimal couplings). One can show, by a direct computation of the covariant exterior derivative of the field equations (83), that

$$D * T^a[\psi] = \frac{\lambda f_2'}{1 + \lambda f_2} dR \wedge (i^a L_m - *T^a[\psi]) \tag{85}$$

as a result of the nonminimal coupling of matter fields. Note that for $\lambda = 0$ the above expression reduces to the previous case and implies the usual covariant expression $D * T^a[\psi] = 0$ for the matter energy-momentum forms as one should expect on consistency grounds.

The modified covariant expression of Eq. (85) implies that the massive test particles do not follow geodesic curves, which invalidates the principle of equivalence. It is argued [77] that the term on the right-hand side can be interpreted as an extra force arising from the particular nonminimal matter coupling defined by Eq. (83). Accordingly, as will be exemplified below, such a nonminimal coupling is naturally bound to modify the matter field equations as well.

Let us consider, for example, the electromagnetic field as the matter Lagrangian present in the above model. Let us assume further that $dF = 0$, which can also be imposed to the field equations, by introducing an appropriate Lagrange multiplier term. Explicitly, for the Maxwell Lagrangian 4-form of the form

$$L_m = L_m[g, F] = -\tfrac{1}{2} F \wedge *F, \tag{86}$$

the electromagnetic field equation given by the variational derivative $\delta L_{n.m}/\delta F = 0$ is modified to the form [80, 81, 82, 83]

$$d[(1 + \lambda f_2) * F] = 0. \tag{87}$$

Note that the modified electromagnetic field equation can also be rewritten in the alternate form

$$d * F + \frac{\lambda f_2'}{1 + \lambda f_2} dR \wedge *F = 0. \tag{88}$$

It is possible to show that Eq. (88) can also be derived, in a somewhat indirect manner, by making use of the general formula of Eq. (85). For the null electromagnetic field ansatz of Eq. (33), the modified field equation (88) simplifies to the familiar source-free Maxwell equation $d * F = 0$.

Now returning to discussion of the gravitational waves, by inserting the pp-wave metric ansatz into the field equations of Eq. (84), and assuming that the only matter Lagrangian present is the electromagnetic Lagrangian and noting the fact that for the null fields $L_m = F \wedge *F = 0$, Eq. (84) reduces to

$$-f_1'(0) * G^1 + \tfrac{1}{2} f_1(0) * l + (1 + \lambda f_2(0)) * T^1[F] = 0. \tag{89}$$

As in the previous case, with the further assumption $f_1(0) = 0$, which is now required for the satisfaction of both the electromagnetic and the metric field equations, one reobtains Einstein field equations in the form

$$- * G^1 + \left(\frac{1 + \lambda f_2(0)}{f_1'(0)} \right) * T^1[F] = 0 \tag{90}$$

with the gravitational coupling constant κ^2 now replaced by the constant factor in parentheses on the right-hand side. Consequently, one can apply the formula given in the previous section to write down the pp-wave solutions of the form given in Eq. (37) to the gravitational model with the Lagrangian of Eq. (83).

3.4. pp-waves in a gravitational model with a nonminimal Maxwell coupling

The nonminimal coupling of the general matter Lagrangian studied in the preceding section discussed the particular nonminimal coupling of type $f(R) F \wedge *F$. On the other hand, one can formulate a more general nonminimally coupled Einstein–Maxwell system by considering all mathematically admissible coupling terms involving curvature and the square of the Faraday tensors. In particular, there are many mathematically admissible interaction terms of the general form RF^2 that one can consider [84]. Such coupling terms can explicitly be written in the following forms: $F \wedge F_{ab} * \Omega^{ab}$, $F \wedge F_{ab} \Omega^{ab}$, $F^a \wedge R_a \wedge *F$, or $F^a \wedge R_a \wedge F$ [4, 85], where R_a is the Ricci 1-form $R_a = R_{ab}\theta^b$ that can be defined in terms of the contraction $R_a \equiv i_b \Omega^b{}_a$.

The pp-wave solution to the nonminimal coupling involving the term of the particular form $F \wedge F_{ab} * \Omega^{ab}$ was recently studied in [4] based on the Lagrangian of the form

$$L = \frac{1}{2\kappa^2}\Omega_{ab} \wedge *\theta^{ab} - \frac{1}{2}F \wedge *F + \frac{\gamma}{2}F \wedge F_{ab} * \Omega^{ab}, \tag{91}$$

where γ is a coupling constant and F_{ab} denotes the components of the Faraday 2-form $F = \frac{1}{2}F_{ab}\theta^a \wedge \theta^b$ relative to an orthonormal coframe. Note that the nonminimal coupling term involving the Riemann tensor can explicitly be written out in the form

$$F \wedge F_{ab} * \Omega^{ab} = \tfrac{1}{2}F_{ab}R^{ab}{}_{cd}F^{cd} * 1. \tag{92}$$

The expression of Eq. (92) has the same form relative to a coordinate basis as well and the nonminimal RF^2 coupling of this particular type was first considered by Prasanna [86] some time ago.

By using the constrained first-order formalism it is possible to obtain the metric field equations that follow from Eq. (91), which can be written in the form [4]

$$-\frac{1}{\kappa^2} * G_a + *T_a[F] + D\lambda_a + \gamma F_{ac}(i_b F) \wedge *\Omega^{bc} + \gamma * T_a[F, \Omega] = 0 \tag{93}$$

by using the auxiliary 3-form definition

$$*T_a[F, \Omega] \equiv -\tfrac{1}{4}F_{bc}\left(i_a F \wedge *\Omega^{bc} + i_a\Omega^{bc} \wedge *F - F \wedge i_a * \Omega^{bc} - \Omega^{bc} \wedge i_a * F\right), \tag{94}$$

and likewise the 3-form $*T_a[F]$ denotes the electromagnetic energy-momentum 3-form defined in Eq. (31) above. The vector-valued Lagrange multiplier 2-form λ^a is obtained by solving the connection equations and it has the explicit expression given by

$$\lambda^a = \gamma i_b D(F^{ba} * F) + \frac{\gamma}{4}\theta^a \wedge i_b i_c D(F^{bc} * F). \tag{95}$$

In addition, the modified field equations for the Faraday 2-form then take the form $dF = 0$ and

$$d * (F - \gamma F_{ab}\Omega^{ab}) = 0, \tag{96}$$

which involves the third-order partial derivatives of the metric variable in general.

For the electrovacuum pp-wave metric ansatz of Eqs. (10) and (33) above, the nonminimal interaction term vanishes identically and consequently the field equations for the Faraday 2-form reduce to Maxwell's equations $dF = d * F = 0$. Moreover, another consequence of the vanishing of the nonminimal coupling term is that $*T_a[F, \Omega] \equiv 0$. The term of the form $\gamma F_{ac}(i_b F) \wedge *\Omega^{bc}$ vanishes for the electrovacuum pp-waves identically as well.

For the pp-wave metric ansatz of Eq. (10), the only nontrivial contribution of the nonminimal coupling terms arises from the Lagrange multiplier 2-forms. Explicitly, the λ^1 component can be expressed in the form

$$\lambda^1 = \gamma\left[f_{\zeta\zeta}f_{\bar{\zeta}} * (k \wedge m) + f_{\bar{\zeta}\bar{\zeta}}f_{\zeta} * (k \wedge \bar{m})\right], \tag{97}$$

whereas one has $\lambda^0 = \lambda^2 = 0$. Using the result of Eq. (97) in the expression for the covariant exterior derivative of the Lagrange multiplier 2-form, one finds

$$D\lambda^1 = 2\gamma f_{\zeta\zeta}f_{\bar{\zeta}\bar{\zeta}} * k. \tag{98}$$

Now taking all these results into account, the metric equation of Eq. (93) for $a = 1$ eventually leads to the following second-order partial differential equation:

$$H_{\zeta\bar{\zeta}} = \kappa^2 f_\zeta \bar{f}_{\bar{\zeta}} - \gamma\kappa^2 f_{\zeta\zeta} \bar{f}_{\bar{\zeta}\bar{\zeta}}. \tag{99}$$

The general solution to Eq. (99) can be written in the form

$$H(u,\zeta,\bar{\zeta}) = h(u,\zeta) + \bar{h}(u,\bar{\zeta}) + \kappa^2 f\bar{f} - \gamma\kappa^2 f_\zeta \bar{f}_{\bar{\zeta}} \tag{100}$$

in terms of the complex functions $h(u,\zeta)$ and $f(u,\zeta)$ having arbitrary u-dependence that are analytic in the variable ζ as in the previous cases.

A family of solutions to the field equation of Eq. (99) was reported recently in [4] associated with a partially massless spin-2 photon and a partially massive spin-2 graviton that were introduced in [87] by Deser and Waldron previously. For the particular pp-wave solution reported in Eq. (99), the electromagnetic function f introduced into the electrovacuum ansatz explicitly has the form

$$f(u,\zeta) = f_1(u)\zeta + f_2(u)\zeta^2 \tag{101}$$

corresponding to a superposition of two null electromagnetic fields.

3.5. pp-waves in a general quadratic curvature gravity

Considered as a low-energy limit of some theory of quantum gravity, general relativity is expected to receive correction terms involving higher powers of curvature tensor to the Einstein–Hilbert action. In particular, the quadratic curvature terms in the effective action are essential for the power-counting renormalizability [88], although they are not free from the problem of the ghosts. The quadratic curvature terms also appear in the low-energy effective action in string theory [89]. More recently, the modified gravity models involving the quadratic curvature terms following from the Lagrangian $f(R, R_{ab}R^{ab}, R^{abcd}R_{abcd})$ in a cosmological context were discussed in [90]. The complicated modified gravitational models of this type are usually discussed to investigate the effect of terms that gain significance in the case where the spacetime has a small curvature.

The general quadratic curvature gravity leads to a set of fourth-order field equations and, naturally, there are not as many exact solutions as in the general theory of relativity. The gravitational wave solutions to the quadratic curvature gravity were studied in [91, 92, 93, 94] and recently in [95, 96] in the more general setting of metric-affine gravity. The quadratic curvature gravity has a long history [97] initiated shortly after the introduction of GR. One of the earliest examples of the quadratic curvature gravity is the conformally invariant gravity following from the square of Weyl's conformal tensor introduced by Bach [98] in 1921.

The particular quadratic curvature model discussed in this section was introduced as a Yang–Mills type action in [99, 100, 101, 102] and, following the terminology introduced in [103, 104], the Lagrangian will be denoted by L_{SKY} below with the acronym "SKY" standing for Stephenson–Kilmister–Yang. The scale invariant gravitational Lagrangian 4-form, depending on both the metric (owing to the presence of the Hodge dual) and the connection expressed in terms of curvature 2-form, explicitly reads

$$L_{SKY} = \tfrac{1}{2}\Omega_{ab} \wedge *\Omega^{ab}. \tag{102}$$

The vacuum field equations for the pseudo-Riemannian metric that follow from Eq. (102) by using the constrained coframe variational derivative in the first-order formalism can be written in the form

$$*E^a = D\lambda^a + *T^a[\Omega] = 0 \tag{103}$$

with the Lagrange multiplier 2-form having the explicit expression

$$\lambda^a = i_b D * \Omega^{ba} + \tfrac{1}{4}\theta^a \wedge i_b i_c D * \Omega^{bc}. \tag{104}$$

Consequently, Eq. (102) leads to the equations that are fourth-order in the partial derivatives of the metric components as indicated by the Lagrange multiplier term in Eq. (103). The quadratic curvature term, namely the $*T^a[\Omega]$ term in Eq. (103), has the explicit form

$$*T_a[\Omega] \equiv -\tfrac{1}{2}\left(i_a\Omega^{bc} \wedge *\Omega_{bc} - \Omega_{bc} \wedge i_a * \Omega^{bc}\right), \tag{105}$$

which arises from commuting the variational derivative with the Hodge dual operator. The energy-momentum-like 3-form term of Eq. (105) can be obtained in a way similar to a matter energy-momentum 3-form calculated by a coframe variation.

As in the previous cases, the pp-wave metric ansatz of Eq. (33) simplifies the metric equations of Eq. (103) considerably. It is straightforward to show that all the components of $*T_a[\Omega]$ vanishes identically in this case. The only nonvanishing contribution of the Lagrange multiplier term then arises from the covariant exterior derivative $D * \Omega^1{}_2$ and its complex conjugate $D * \Omega^1{}_3$. By using the general expression

$$D * \Omega^{ab} = d * \Omega^{ab} + \omega^a{}_c \wedge *\Omega^{cb} + \omega^b{}_c \wedge *\Omega^{ac} \tag{106}$$

for the covariant exterior derivative, it is straightforward to show that

$$D * \Omega^1{}_2 = -2H_{\zeta\zeta\bar{\zeta}} * k \tag{107}$$

and, consequently, using the fact that $dk = 0$ identically, one can show that the Lagrange multiplier can be written in the form

$$\lambda^1 = -4 * d(H_{\zeta\bar{\zeta}}k). \tag{108}$$

Using this result in Eq. (103), and also noting that $D\lambda^1 = d\lambda^1$, one eventually ends up with

$$*E^1 = -4d * d(H_{\zeta\bar{\zeta}}k) = 0. \tag{109}$$

It is interesting to note that the field equation of Eq. (103) can be rewritten in the following compact forms

$$*E^1 = -2d * d * d * dl = -4H_{\zeta\zeta\bar{\zeta}\bar{\zeta}} * k = 0 \tag{110}$$

for the profile function.

A comparison of the form of the quadratic curvature field equation of Eq. (110) to those of GR given in Eq. (29) suggests, among other things, that the linearized form of the metric field equations for the quadratic curvature theory [105, 106, 107, 108] formally has a structure similar to those of GR.

Another observation related to the expression of Eq. (110) is that it is probably the simplest subcase of a remarkable theorem due to Gürses et al. [109]. The theorem allows one to find the exact solution to a wide class of modified gravitational models governed by the Lagrangian depending on a function of the Riemann tensor $f(R^{ab}{}_{cd})$. For the Kundt class of Petrov type N metrics with all the scalar invariants being constant, the theorem states that any symmetric, second-rank tensor constructed from the Riemann tensor and its covariant derivatives can be expressed as a linear combination of the metric components, the traceless Ricci tensor components, and the higher covariant derivatives of the traceless Ricci tensor components.

Now consider the quadratic curvature gravity model obtained by adding the Einstein–Maxwell Lagrangian

$$L_{E-M} = \frac{1}{2\kappa^2}\Omega_{ab} \wedge *\theta^{ab} - \frac{1}{2}F \wedge *F \tag{111}$$

to the quadratic curvature Lagrangian of Eq. (102). One can show that the field equation following from the total Lagrangian can be written in the form

$$d * d(\ell^2 + *d * d)l = 2\kappa^2 f_\zeta \bar{f}_{\bar{\zeta}} * k \tag{112}$$

for the pp-wave metric ansatz. In Eq. (111), ℓ^2 stands for an appropriate coupling constant for the quadratic curvature terms in the total Lagrangian. The fourth-order equation of Eq. (112) for 3-forms can be written as a second-order equation in the form

$$d * d\sigma = 2\kappa^2 f_\zeta \bar{f}_{\bar{\zeta}} * k \tag{113}$$

in terms of the 1-form σ as an equation analogous to Eq. (36) by defining the auxiliary 1-form σ as

$$\sigma \equiv (*d * d + \ell^2)l. \tag{114}$$

Now for a given electromagnetic potential $f(\zeta, \bar{\zeta})$, Eq. (113) is to be solved for 1-form σ and subsequently the resulting expression for it is to be used to solve Eq. (114) for the 1-form $l = dv + Hdu$, or equivalently the profile function H to obtain a solution.

In four spacetime dimensions, the general quadratic curvature gravity Lagrangian can be written in a preliminary form as

$$L = \gamma\Omega_{ab} \wedge *\Omega^{ab} + \alpha R_a \wedge *R^a + \beta R^2 * 1 \tag{115}$$

involving the quadratic curvature terms built out of the contractions of the curvature 2-form, namely the Ricci-squared and scalar curvature-squared terms with respective coupling constants α, β and γ being another coupling constant.

At this point, it is convenient to recall Lovelock's theorem [110, 111], stating that the most general gravitational actions that generalize Einstein–Hilbert action leading to second-order field equations in metric components involve the Einstein–Hilbert action complemented with a cosmological constant term and terms with curvature polynomials, the dimensionally continued Euler–Poincaré (EP) forms (these forms are also commonly known as the Gauss–Bonnet terms and the famous Chinese geometer Chern was the first to use dimensionally continued Euler–Poincaré forms to generalize the Gauss–Bonnet theorem to higher dimensions [112]). In four spacetime dimensions, the EP term involving quadratic curvature expression does not contribute to the gravitational field equations. Thus, one can exploit the EP forms that are quadratic in curvature components to remove the redundancy in the general Lagrangian of Eq. (115) without loss of generality. Explicitly, the dimensionally continued Euler–Poincare term

$$L_{EP} = \tfrac{1}{4}\Omega_{ab} \wedge \Omega_{cd} * \theta^{abcd} \tag{116}$$

can be expressed in a particular linear combination of quadratic terms as

$$L_{EP} = \tfrac{1}{2}\Omega_{ab} \wedge *\Omega^{ab} - R_a \wedge *R^a + \tfrac{1}{4}R^2 * 1 \tag{117}$$

by a straightforward computation in four dimensions. After some tedious computation in exterior algebra starting from the expression in Eq. (116), it is possible to show that the L_{EP} term can be written as an exact form as

$$L_{EP} = d\left(\omega_{ab} \wedge \Omega^{cd} - \tfrac{1}{3}\omega_{ae} \wedge \omega^{e}{}_{b} \wedge \omega^{cd}\right)\epsilon^{ab}{}_{cd} \tag{118}$$

in a form remarkably similar to the gravitational Chern–Simons term arising from another topological term, the Pontryagin form, namely the 4-form $\Omega_{ab} \wedge \Omega^{ba}$ [103, 104], cf. Eq. (136) below. Consequently, because L_{EP} is an exact form that will not contribute to the field equations in four dimensions, one can add L_{EP} to the general quadratic curvature action of Eq. (115) to eliminate one of the terms in Eq. (115) in favor of the remaining two. Following this custom, one can write the most general quadratic curvature Lagrangian in the form

$$L = \alpha R_a \wedge *R^a + \beta R^2 * 1 \tag{119}$$

in *four spacetime dimensions*. The pp-wave solutions to the model that follows from Eq. (119) have been previously been studied in, for example in [113, 114]. With the coupling constant satisfying the relation $\alpha = -3\beta$, the most general Lagrangian of Eq. (119) corresponds to the conformally invariant quadratic curvature model introduced by Bach mentioned above and the corresponding field equations is expressed in terms of the so-called Bach tensor [115].

In the present notation, the general form of the field equations that follow from Eq. (119) were reported in [116] and they explicitly read

$$*E^a = D * D\left\{2\beta R^a + \left(2\alpha + \tfrac{1}{2}\beta\right) R\theta^a\right\} + *T^a[\alpha, \beta] = 0 \tag{120}$$

where the quadratic term $*T_a[\alpha, \beta]$ has the explicit form

$$*T_a[\alpha, \beta] \equiv \Omega_{bc} \wedge i_a * X^{bc} - \tfrac{1}{2} i_a(\Omega_{bc} \wedge *X^{bc}) \tag{121}$$

and the auxiliary tensor-valued 2-form X^{ab} standing for

$$X^{ab} = \alpha(\theta^a \wedge R^b - \theta^b \wedge R^a) + 2\beta R\theta^{ab}. \tag{122}$$

With these formulas at hand, it is now straightforward to show by inspection that the most general quadratic curvature vacuum equations of Eq. (120) reduce to $d * dR^1 = 0$, leading to the result given in Eq. (110). The result is also in harmony with the results previously reported, for example in [113], by using the tensorial methods.

One of the novel effects arising from the quadratic curvature model is that there is now additional transverse massive scalar and the massive spin-2 (ghost) modes. See, for example, [58, 117] for further details on this issue.

3.6. pp-waves in a tensor-tensor gravity with a torsion

The quadratic curvature gravity model studied in this section involves a symmetric second-rank tensor $\Phi = \Phi_{ab}\theta^a \otimes \theta^b$ [5]. The Lagrangian of such a tensor-tensor model involves an interaction term of the form $\Phi^{ab}\Omega_{ac} \wedge *\Omega^{c}{}_{b}$ and the particularly interesting feature of this model is that the field equations yield the Bell tensor [118, 119, 120, 121].

Following closely the definition given in [5], the fourth-rank Bell tensor B can be written in the form

$$B \equiv T_{abc} \otimes \theta^a \otimes \theta^b \otimes \theta^c \tag{123}$$

where three-indexed 1-form T_{abc} can be defined in terms of the following expressions of the curvature 2-forms

$$*T_{abc} \equiv \tfrac{1}{2}\left(i_a\Omega_{bd} \wedge *\Omega^d{}_c - \Omega_{bd} \wedge i_a * \Omega^d{}_c\right) \tag{124}$$

in the current notation. For the curvature 2-forms corresponding to a Levi-Civita connection, the Bell tensor defined by Eq. (124) has some mathematical properties in common with the energy-momentum 3-form $*T^a[F]$ of the Maxwell field defined in Eq. (31) above. Remarkably, for the null coframe defined by Eq. (12), the nonvanishing component of the Bell tensor is

$$*T^1{}_{00} = -2\left(H_{\zeta\bar{\zeta}}H_{\zeta\bar{\zeta}} + H_{\zeta\zeta}H_{\bar{\zeta}\bar{\zeta}}\right) * k. \tag{125}$$

The Lagrangian 4-form introduced by Dereli and Tucker [5] can explicitly be written in the form

$$L = \frac{1}{2\kappa^2}\Omega_{ab} \wedge *\theta^{ab} + \tfrac{1}{2}\Phi_{ab}\Omega^a{}_c \wedge *\Omega^{cb} + \tfrac{1}{2}D\Phi_{ab} \wedge *D\Phi^{ab} - \tfrac{1}{2}F \wedge *F \tag{126}$$

with the help of a symmetrical second rank tensor Φ_{ab} coupling to a particular quadratic curvature term. The issue of energy and momentum carried by a gravitational wave was also discussed in [92] in the context of teleparallel gravity by making use of the Bell tensor.

Assuming that the connection and the coframe 1-forms are independent gravitational variables, the field equations that follow from the total variational derivative of the Lagrangian 4-form of Eq. (126) yield the gravitational field equations [5]

$$-\frac{1}{\kappa^2} * G^a + *T^a[F] + \lambda * T^a[D\Phi] + \Phi^{bc} * T^a{}_{bc} = 0 \tag{127}$$

for the coframe 1-forms where the energy-momentum 3-forms $*T^a[D\Phi]$ is defined as

$$*T^a[D\Phi] = \tfrac{1}{2}\left(i_a D\Phi_{bc} \wedge *D\Phi^{bc} + D\Phi_{bc} \wedge i_a * D\Phi^{bc}\right). \tag{128}$$

The equations of motion for the independent connection are

$$-\frac{1}{2\kappa^2}\Theta_c \wedge *\theta^{abc} + \tfrac{1}{2}D\left(\Phi^a{}_c * \Omega^{cb} - \Phi^b{}_c * \Omega^{ca}\right) + \lambda\left(\Phi^a{}_c D * \Phi^{cb} - \Phi^b{}_c * D\Phi^{ca}\right) = 0. \tag{129}$$

The above gravitational field equations are defined in a Riemann–Cartan spacetime in general. The Ricci tensor in this case is defined as in the pseudo-Riemannian case, and likewise the Einstein 3-form is constructed from the curvature 2-form of a general connection with a nonvanishing torsion as in the pseudo-Riemannian case. However, the Einstein tensor is not symmetrical in general in the non-Riemannian case with a nonvanishing torsion. Accordingly, the covariant exterior derivative is defined with a more general connection with a nonvanishing torsion as well.

These equations are to be supplemented with the equations

$$\lambda D * D\Phi^{ab} - \tfrac{1}{2}\Omega^a{}_c \wedge *\Omega^{cb} = 0 \tag{130}$$

for the tensor field Φ_{ab} and Maxwell's equations $d*F = dF = 0$. In contrast to the pseudo-Riemannian case, the field equations are obtained by constraining the independent connection to be a Levi-Civita connection and are more complicated then the field equations above in the non-Riemann geometry with torsion.

Keeping in mind that the only contribution of the Bell tensor to the coframe equations is of the form given in Eq. (125), it is natural to assume that a compatible ansatz for the tensor Φ is of the form $\Phi = \Phi_{ab}\theta^a \otimes \theta^b = \Phi_{11} l \otimes l$ with the only nonvanishing components Φ_{11} and the choice $\Phi_{11} = const.$ renders Φ a covariantly constant tensor. (To avoid confusion, the nonvanishing constant component is denoted by Φ_c below.) As a consequence of the judicious choice for Φ_{ab}, all $*T^a[D\Phi]$ vanish identically as well. Furthermore, one also obtains that $\Theta^a = 0$ from the independent connection equation and Eq. (130) are satisfied identically. Eventually, it is consistent to write the coframe equations of Eq. (127) in terms of the pseudo-Riemannian quantities. Fot the pp-wave ansatz of Eq. (33), this then yields a nonlinear partial differential equation of the form

$$-H_{\zeta\bar{\zeta}} + \kappa^2 \Phi_c \left(H_{\zeta\bar{\zeta}} H_{\zeta\bar{\zeta}} + H_{\zeta\zeta} H_{\bar{\zeta}\bar{\zeta}} \right) + \kappa^2 f_\zeta \bar{f}_{\bar{\zeta}} = 0. \tag{131}$$

A particularly simple solution to Eq. (131) can be constructed with a homogeneous profile function of the form

$$H(u,\zeta,\bar{\zeta}) = h_1(u)\zeta^2 + h_1(u)\bar{\zeta}^2 + h_2(u)\zeta\bar{\zeta} \tag{132}$$

with $f(u,\zeta) = \alpha(u)\zeta$ corresponding to the electromagnetic part of the ansatz. $h_1(u), h_2(u)$, and $\alpha(u)$ are real functions of the variable u satisfying

$$h_2 = \kappa^2 \Phi_c (h_2^2 + h_1^2) + \kappa^2 \alpha^2. \tag{133}$$

As one can observe from Eq. (125) that, for a positive Φ_c, the solution of the form of Eq. (132) then leads to a positive definite expression for T_{000} [5] admitting finite values only.

3.7. pp-waves in the Chern–Simons modified GR

Chern–Simons modified gravity is a parity violating extension of GR introduced by Jackiw and Pi [122]. It is also motivated by string theory [123]. In this modified gravity model, the three-dimensional CS-topological current is embedded into four spacetime dimensions. It has subsequently found diverse applications, for example in the context of inflationary models and in the study of primordial gravitational waves, as well as in many other topics in cosmology.

The Lagrangian 4-form for the CS-modified GR employs the Pontryagin topological term in addition to the familiar Einstein–Hilbert term. In particular, the CS term is favored by string theory predicting the Pontryagin topological correction term in the low-energy limit. A derivation of the CS-modified GR field equations from a truncation of a low-energy effective heterotic string theory models involving the Kalb–Ramond field and a dilaton field was recently given in [124], and a similar derivation was also presented in [125] at around the same time.

The field equations for the CS-modified GR model follow from the Lagrangian 4-form

$$L_{CS} = \tfrac{1}{2}\Omega_{ab} \wedge *\theta^{ab} - \tfrac{1}{8}\theta(x)\Omega_{ab} \wedge \Omega^{ba}, \tag{134}$$

where the first term is the familiar Einstein–Hilbert Lagrangian 4-form and the second term, which is known as the Pontryagin term, is an exact form

$$\Omega_{ab} \wedge \Omega^{ba} = dK \tag{135}$$

with the Chern–Simons 3-form K defined as

$$K = \omega_{ab} \wedge \Omega^{ba} - \tfrac{1}{3}\omega^{a}{}_{b} \wedge \omega^{b}{}_{c} \wedge \omega^{c}{}_{a}, \tag{136}$$

and therefore the term in Eq. (135) does not contribute to the field equations for a constant θ. Written in this form, the Pontryagin term depends on the connection 1-form and it contributes to the coframe equations only through the Lagrange multiplier term L_C of the form of Eq. (45) introduced to impose the vanishing torsion constraint for the independent connection in the first-order formalism.

The total variational derivative of the extended action $L_{ext.} = L_{CS} + L_C$ with respect to the variables can be found in a straightforward manner as

$$\begin{aligned} \delta L_{ext.} = {} & \delta\theta_a \wedge (-*G^a + D\lambda^a) - \delta\theta \tfrac{1}{8}\Omega_{ab} \wedge \Omega^{ba} + \delta\lambda_a \wedge \Theta^a \\ & + \delta\omega_{ab} \wedge \left\{ \tfrac{1}{2} D * \theta^{ab} - \tfrac{1}{4} D(\theta\Omega^{ab}) - \tfrac{1}{2}\left(\theta^a \wedge \lambda^b - \theta^b \wedge \lambda^a\right) \right\} \end{aligned} \tag{137}$$

up to an omitted exact form. Because the contribution of the topological terms to the coframe equations results from the Lagrange multiplier term, as before, one first solves the independent connection equations $\delta L_{ext.}/\delta\omega_{ab} = 0$ for the Lagrange multiplier 2-form.

The connection equations, which explicitly read

$$D * \theta^{ab} - \tfrac{1}{2} D(\theta\Omega^{ab}) - \left(\theta^a \wedge \lambda^b - \theta^b \wedge \lambda^a\right) = 0, \tag{138}$$

can be solved for the Lagrange multiplier 2-form to have

$$\lambda^a = -i_b(\Omega^{ba} \wedge d\theta) - \tfrac{1}{4}\theta^a \wedge i_b i_c(\Omega^{bc} \wedge d\theta) \tag{139}$$

as the unique solution. At this point, it is convenient to define auxiliary vector-valued 2-form P^a as

$$P^a \equiv i_b(\Omega^{ba} \wedge d\theta) \tag{140}$$

and its contraction $P \equiv i_a P^a$. In terms of the vector-valued auxiliary form P^a and its contraction, the Lagrange multiplier 2-form can be written conveniently as

$$\lambda^a = -(P^a - \tfrac{1}{4}\theta^a \wedge P), \tag{141}$$

formally resembling the expression for the Schouten 1-form defined in 2+1 dimensions in the context of topologically massive gravity [126]. In 2+1 dimensions, the Cotton 2-form is derived from the Schouten 1-form [115], $L^a = R^a - \frac{1}{4}R\theta^a$. In general, the Cotton tensor is defined in any spacetime dimension $D \geq 3$, but the definition depends explicitly on the spacetime dimensions.

In the present geometrical framework, it is convenient to define the vector-valued 3-form $C^a = \frac{1}{6}C^{a}{}_{bc}\theta^{abc}$ by

$$C^a \equiv D\lambda^a = D(P^a - \tfrac{1}{4}\theta^a \wedge P) \tag{142}$$

as well. Note that C^a defined in this way can be expressed in terms of a symmetric traceless tensor C_{ab} by using the relation

$$C^{ab} = -\tfrac{1}{2} * (\theta^a \wedge C^b + \theta^b \wedge C^a), \tag{143}$$

where the tensor C^{ab} represents the orthonormal components of the tensor, which is often called the C-tensor in the literature. See, for example, the review article by Alexander and Yunes [123] and the references therein for an extensive discussion. The relation of Eq. (143) can be used to derive a more familiar expression for the C-tensor relative to a coordinate expression. In the presentation below, the vector-valued 3-form C^a will be called "C-form".

Eventually, after taking the matter energy-momentum forms $*T_a[\psi] \equiv \delta L_m/\delta\theta^a$ coming from the matter Lagrangian $L_m[g,\psi]$ into account, the coframe equations for the CS-modified gravity then take the form

$$*G^a + C^a = \kappa^2 * T^a[\psi] \tag{144}$$

with κ^2 denoting the coupling constant in GR. As in the previous cases, only the vacuum and the electrovacuum solutions to these field equations will be discussed in this subsection.

The variational derivative of the extended Lagrangian with respect to the CS scalar field, which can be considered as a Lagrange multiplier 0-form for the model, leads to the constraint

$$\Omega_{ab} \wedge \Omega^{ba} = 0. \tag{145}$$

Eqs. (144) and (145) constitute the field equations for the CS-modified gravity model expressed in terms of the exterior forms relative to a null coframe.

It is worth noting at this point that the use of NP formalism in a study of CS-modified gravity is also favored by the constraint of Eq. (145) because Eq. (145) can be rewritten in terms of Weyl 2-forms in the form

$$C_{ab} \wedge C^{ba} = 0 \tag{146}$$

by making use of the expansion of Eq. (8) and the first Bianchi identity satisfied by the curvature 2-form. Consequently, because Eq. (146) involves only the Weyl spinor scalars Ψ_k, the Pontryagin constraint can be considered as a constraint on the Petrov type. It is possible to show that Eq. (146) explicitly reads

$$\tfrac{1}{8} C_{ab} \wedge C^{ba} = i\left\{3(\Psi_2^2 - \bar{\Psi}_2^2) - 4(\Psi_1\Psi_3 - \bar{\Psi}_1\bar{\Psi}_3) + \Psi_0\Psi_4 - \bar{\Psi}_0\bar{\Psi}_4\right\} * 1 \tag{147}$$

by using Eq. (9) after some straightforward algebra. The constraint of Eq. (145), or equivalently Eq. (146), imposing the vanishing of the Pontryagin term, is essential to have the diffeomorphism invariance of the model. Note that the Pontryagin constraint of Eq. (145) is satisfied identically for the ansatz of Eq. (33) because the only nonvanishing Weyl spinor scalar is Ψ_4. Hence, it suffices to consider the CS-modified equations of Eq. (144) for a type N metric in general.

As a consequence of the constraint of Eq. (145), the C-form is covariantly constant and therefore the matter coupling to the CS-modified gravity requires a covariantly constant matter energy-momentum tensor as in the GR. Explicitly, by making use of the first Bianchi identity satisfied by the curvature 2-form, it is possible to show that

$$DC^a = -\tfrac{1}{4}(i^a d\theta)\Omega_{bc} \wedge \Omega^{cb}. \tag{148}$$

The C-form is a traceless vector-valued 3-form by definition and it is covariantly constant provided that the Pontryagin constraint is satisfied. It is important to note that these properties are in common with the Cotton 2-form in 2+1 dimensions. However, the C-form has some other properties that are not in common with those of the Cotton 2-form defined in three dimensions.

In order to construct the solutions to the CS-modified gravity, one has to make additional assumptions for the CS scalar field θ along with the metric ansatz of Eq. (33). It is convenient to start with a general CS scalar such that $\theta = \theta(u, v, \zeta, \bar{\zeta})$ and then subsequently restrict it to a convenient form as one proceeds.

By making use of curvature expressions of Eq. (17) for the pp-wave metric ansatz of Eq. (10), one finds the following expressions

$$\begin{aligned} P^1 &= -2\theta_v H_{\zeta\bar{\zeta}} k \wedge l + \left(\theta_{\bar{\zeta}} H_{\zeta\zeta} - \theta_\zeta H_{\zeta\bar{\zeta}}\right) k \wedge m + \left(\theta_\zeta H_{\bar{\zeta}\bar{\zeta}} - \theta_{\bar{\zeta}} H_{\zeta\bar{\zeta}}\right) k \wedge \bar{m}, \\ P^2 &= -\theta_v H_{\bar{\zeta}\bar{\zeta}} k \wedge \bar{m} - \theta_v H_{\zeta\bar{\zeta}} k \wedge m, \end{aligned} \tag{149}$$

for the nonvanishing components of the auxiliary form P^a, and also note that P^3 can be obtained by the complex conjugation relation $P^3 = \bar{P}^2$. Moreover, using the expressions of Eq. (149), one can find the contraction of the 2-form P^a as

$$P = 4\theta_v H_{\zeta\bar{\zeta}} k. \tag{150}$$

By combining the above results, the nonvanishing Lagrange multiplier 2-forms can then be expressed in the form

$$\begin{aligned} \lambda^1 &= -\theta_v H_{\zeta\bar{\zeta}} k \wedge l + \left(\theta_{\bar{\zeta}} H_{\zeta\zeta} - \theta_\zeta H_{\zeta\bar{\zeta}}\right) k \wedge m + \left(\theta_\zeta H_{\bar{\zeta}\bar{\zeta}} - \theta_{\bar{\zeta}} H_{\zeta\bar{\zeta}}\right) k \wedge \bar{m}, \\ \lambda^2 &= +\theta_v H_{\zeta\bar{\zeta}} k \wedge m - \theta_v H_{\bar{\zeta}\bar{\zeta}} k \wedge \bar{m} - \theta_v H_{\zeta\bar{\zeta}} k \wedge m, \end{aligned} \tag{151}$$

where $\lambda^3 = \bar{\lambda}^2$. Finally, by using the fact that the C-tensor is given by the covariant exterior derivative as $C^a = D\lambda^a$, one ends up with the following nonvanishing components of the C-form:

$$\begin{aligned} C^1 = &-i\left(\theta_{\bar{\zeta}\bar{\zeta}} H_{\zeta\zeta} - \theta_{\zeta\zeta} H_{\bar{\zeta}\bar{\zeta}} - 2\theta_\zeta H_{\zeta\bar{\zeta}\bar{\zeta}} + 2\theta_{\bar{\zeta}} H_{\zeta\zeta\bar{\zeta}}\right) * k \\ &- i\left(\theta_v H_{\zeta\zeta}\right)_{\bar{\zeta}} * m + i\left(\theta_v H_{\bar{\zeta}\bar{\zeta}}\right)_\zeta * \bar{m}, \end{aligned} \tag{152}$$

$$C^2 = -i\left(\theta_v H_{\bar{\zeta}\bar{\zeta}}\right)_\zeta * k - i\theta_{vv} H_{\bar{\zeta}\bar{\zeta}} * \bar{m}, \tag{153}$$

where $C^3 = \bar{C}^2$ by definition.

An immediate observation about the general equations of motion for the CS-modified gravity equations is that one of the vacuum equations, namely $\theta_{vv} H_{\zeta\zeta} = 0$, decouples from the rest of the field equations. Moreover, because this equation involves the only nonvanishing Weyl component $\Psi_4 = H_{\zeta\zeta}$, one has to assume that $\theta_{vv} = 0$ in order to maintain the Petrov type of the metric.

A classification scheme of the solutions to the CS-modified GR was introduced in [6], and it is convenient to discuss the pp-waves solution in this regard as well. It is possible to construct $\mathcal{P}$ Class and $\mathcal{CS}$ Class solutions to the CS-modified GR as follows.

(1) $\mathcal{P}$ Class solutions (GR solutions lifted to the CS-modified GR): The solutions in this class satisfy $*G^a = \kappa^2 * T^a[F]$ and also by demanding $C^a = 0$ separately. For the $\mathcal{P}$ Class solutions for the pp-wave ansatz, the field equations reduce to the following third-order coupled partial differential equations:

$$H_{\zeta\bar{\zeta}} = \kappa^2 f_\zeta \bar{f}_{\bar{\zeta}}, \tag{154}$$

$$\theta_{\bar{\zeta}\bar{\zeta}} H_{\zeta\zeta} - \theta_{\zeta\zeta} H_{\bar{\zeta}\bar{\zeta}} - 2\theta_\zeta H_{\zeta\bar{\zeta}\bar{\zeta}} + 2\theta_{\bar{\zeta}} H_{\zeta\zeta\bar{\zeta}} = 0, \tag{155}$$

$$(\theta_v H_{\zeta\zeta})_{\bar{\zeta}} = 0, \qquad \theta_{vv} H_{\bar{\zeta}\bar{\zeta}} = 0. \tag{156}$$

First note that the last of the above equations implies that $\theta_{vv} = 0$ for a nonvanishing Ψ_4. For the vacuum case, these equations simplify considerably and take the form

$$H_{\zeta\bar{\zeta}} = 0, \tag{157}$$

$$\theta_{\bar{\zeta}\bar{\zeta}} H_{\zeta\zeta} - \theta_{\zeta\zeta} H_{\bar{\zeta}\bar{\zeta}} = 0, \tag{158}$$

$$\theta_{v\bar{\zeta}} H_{\zeta\zeta} = 0. \tag{159}$$

Furthermore, the equations in Eq. (159) imply that $\theta_{v\zeta} = \theta_{v\bar{\zeta}} = 0$. As a consequence, these relations involving only the partial derivative of the CS scalar field, one can conclude that the CS scalar must be of the form

$$\theta(u, v, \zeta, \bar{\zeta}) = vA(u) + B(u, \zeta, \bar{\zeta}) \tag{160}$$

with two undetermined functions $A = A(u)$ and $B = B(u, \zeta, \bar{\zeta})$. After determining the general solution Eq. (157), the function B then can be determined by the equations

$$B_{\bar{\zeta}\bar{\zeta}} H_{\zeta\zeta} - B_{\zeta\zeta} H_{\bar{\zeta}\bar{\zeta}} = 0 \tag{161}$$

by using Eq. (158). The CS scalar field of the form given in Eq. (160) with the function $B(u, \zeta, \bar{\zeta})$ satisfying Eq. (161) then leads to the most general vacuum pp-wave solution that the CS-modified GR model has in common with the GR solutions.

For the vacuum solutions of the Einstein field equations of the form $H(u, \zeta, \bar{\zeta}) = h(u, \zeta) + \bar{h}(u, \bar{\zeta})$ where the function h is an analytical function of ζ with arbitrary u-dependence, Eq. (161) is solved by the function B having the same form as H and thus one has

$$B(u, \zeta, \bar{\zeta}) = h(u, \zeta) + \bar{h}(u, \bar{\zeta}). \tag{162}$$

It is also possible to construct another $\mathcal{P}$-class solution by considering the Aichelburg–Sexl solution [6]. For the Aichelburg–Sexl solution with $h(u, \zeta) \sim \delta(u) \ln \zeta$, which requires a null particle source term in the Einstein field equations, Eq. (161) now becomes

$$\zeta^2 B_{\zeta\zeta} - \bar{\zeta}^2 B_{\bar{\zeta}\bar{\zeta}} = 0 \tag{163}$$

and the resulting equation implies that B is an arbitrary function of the real variable $|\zeta|^4$ leaving the u dependence undetermined.

(2) $\mathcal{CS}$ Class solutions (non-GR solutions): This class of solutions can be found by solving the general equations of Eq. (144). They are, in general, third-order partial differential equations in the metric components, and for the pp-wave ansatz, they can explicitly be written as

$$(\theta_v H_{\zeta\zeta})_{\bar{\zeta}} = 0, \tag{164}$$

$$2H_{\zeta\bar{\zeta}} - i\left(\theta_{\bar{\zeta}\bar{\zeta}} H_{\zeta\zeta} - \theta_{\zeta\zeta} H_{\bar{\zeta}\bar{\zeta}} - 2\theta_{\zeta} H_{\zeta\bar{\zeta}\bar{\zeta}} + 2\theta_{\bar{\zeta}} H_{\zeta\zeta\bar{\zeta}}\right) = 2\kappa^2 f_{\zeta} \bar{f}_{\bar{\zeta}}. \tag{165}$$

In addition, it is also assumed in this case that the CS scalar satisfies the equation $\theta_{vv} = 0$ as before. With the further simplifying assumption $\theta_v = 0$, the field equations of Eq. (164) are satisfied identically and one is left with Eq. (165) with the expression in Eq. (160) reducing to $\theta = \theta(u, \zeta, \bar{\zeta})$. In this case, the functions $\theta = \theta(u, \zeta, \bar{\zeta})$ and $H = H(u, \zeta, \bar{\zeta})$ are to be determined from Eq. (165) alone.

The vacuum solution presented in [6] by Grumiller and Yunes is constructed under the additional assumption that $\theta_{\zeta\zeta} = 0$. The CS scalar then simplifies to the linear form in the complex coordinates ζ and $\bar{\zeta}$, which can be written as

$$\theta = a(u)\zeta + \bar{a}(u)\bar{\zeta} + b(u) \tag{166}$$

with a a complex function of the real null coordinate u, whereas $b = b(u)$ is a real function. Consequently, Eq. (165) reduces to

$$H_{\zeta\bar{\zeta}} + i\left(\theta_{\zeta} H_{\zeta\bar{\zeta}\bar{\zeta}} - \theta_{\bar{\zeta}} H_{\zeta\zeta\bar{\zeta}}\right) = 0. \tag{167}$$

In order to put this equation into the form of a Poisson equation, it is convenient to introduce [6] the following field redefinition:

$$H_{\zeta\bar{\zeta}} \equiv q(u, \zeta, \bar{\zeta}). \tag{168}$$

In terms of the new function q, Eq. (167) can now be rewritten as a first-order partial differential equation of the form

$$q - i(aq_{\bar{\zeta}} - \bar{a}q_{\zeta}) = 0. \tag{169}$$

One can verify that this equation has the general solution of the form

$$q(u, \zeta, \bar{\zeta}) = e^{(\zeta+\bar{\zeta})/i(\bar{a}-a)}\phi(\bar{a}\zeta + a\bar{\zeta} + b) \tag{170}$$

with ϕ being an arbitrary function of the argument $\bar{a}\zeta + a\bar{\zeta} + b$. Subsequently, this solution can be inserted back into the Eq. (168) to construct a $\mathcal{CS}$-class solution by solving the resulting Poisson equation [6], for example, by the method of Green's functions by introducing some appropriate boundary conditions on the transverse planes spanned by the complex coordinates for different values of the coordinate u.

4. Concluding comments

Although there is convincing but indirect evidence [127, 128] for the gravitational waves confirming the accuracy of the theoretical prediction obtained from the linearized Einstein field equations, there are ongoing efforts in various projects [129, 130, 131, 132] to observe them directly. In the near future, the advances on the observational front will be a powerful tool for testing the viable theoretical models of gravity [133] and some popular modified gravity models will certainly be ruled out by the prospective observations. Furthermore, considering the puzzling observational data evidencing a current accelerated expansion phase of the universe contrary to the former expectations, the direct detection of gravitational waves will probably have an impact on the theoretical efforts as well.

Contrary to the remarks in [1], implying that the use of the Newman–Penrose null tetrad formalism is somewhat cumbersome in deriving the pp-wave type solutions to the BD theory, the NP formalism provides probably the most convenient and efficient mathematical framework in any topic involving the gravitational radiation and it is also certainly well suited to the discussions of these issues in the context of a variety of modified gravity models. In particular, with some further appropriate development of the technical presentation [134], the discussion of the exact solutions above can be extended to more complicated family of metrics, at the same time also taking the algebraic character of such solutions into account.

It is well known that the family of pp-wave metrics belongs to a more general family of metrics, known as the Kundt waves [18, 19, 20]. The family of Kundt waves is also described by a null geodesic with vanishing optical scalars; however, the assumption that the null vector is covariantly constant is dropped. Consequently, for the gravitational waves metrics in this family, the transverse planes are not flat. In addition, the metrics in the Kundt family can have Petrov types II, D, III, and N [10]. The discussion of such algebraically special solutions of the modified gravity models will be work for the future research.

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Appendix

As an alternative to the set of basis coframe 1-forms of Eq. (12), it is also possible to adopt the null coframe

$$k = Hdu + dv, \qquad l = du, \qquad m = d\bar{\zeta} \tag{171}$$

for the pp-wave ansatz of Eq. (10). The set of coframes of Eq. (171) is related to the set of coframes of Eq. (12) by the interchanges $k \leftrightarrow l$ and $m \leftrightarrow \bar{m}$. Under these interchanges of the basis 1-forms, the Cartan structure equations, Eqs. (4) and (7), are mapped onto themselves. This symmetry, originally called the prime symmetry, is a computationally useful symmetry in the NP formalism. The tensorial objects or scalars related by prime symmetry are said to be prime companions. In terms of the null coframe indices, the prime symmetry corresponds to symmetry of the structure equations under the interchanges $0 \leftrightarrow 1$ and $2 \leftrightarrow 3$.

The connection and the curvature forms belonging to the null coframe of Eq. (171), which are the prime companions to those of Eq. (12), are given by

$$\omega^0{}_3 = H_{\bar{\zeta}} l, \tag{172}$$

$$\Omega^0{}_3 = d\omega^0{}_3 = -H_{\bar{\zeta}\bar{\zeta}} l \wedge \bar{m} - H_{\zeta\bar{\zeta}} l \wedge m, \tag{173}$$

respectively. It follows from Eq. (173) that $*G^0 = -2H_{\zeta\bar{\zeta}} * l$ and the corresponding curvature scalars are $\Phi_{00} = H_{\zeta\bar{\zeta}}$, and $\Psi_0 = H_{\bar{\zeta}\bar{\zeta}}$.