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Supergravity backgrounds and symmetry superalgebras

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Abstract: We consider the bosonic sectors of supergravity theories in ten and eleven dimensions corresponding to the low energy limits of string theories and M-theory. The solutions of supergravity field equations are known as supergravity backgrounds and the number of preserved supersymmetries in those backgrounds are determined by Killing spinors. We provide some examples of supergravity backgrounds that preserve different fractions of supersymmetry. An important invariant for the characterization of supergravity backgrounds is their Killing superalgebras, which are constructed out of Killing vectors and Killing spinors of the background. After constructing Killing superalgebras of some special supergravity backgrounds, we discuss the possibilities of the extensions of these superalgebras to include the higher degree hidden symmetries of the background.

Key words: Supergravity backgrounds, Killing spinors, Killing superalgebras

1. Introduction

The unification of fundamental forces of nature is one of the biggest aims in modern theoretical physics. The most promising approaches for that aim include the ten-dimensional supersymmetric string theories and their eleven-dimensional unification called M-theory. There are five different string theories in ten dimensions: type I, type IIA and IIB, and heterotic $E_8 \times E_8$ and $SO(32)$ theories. However, some dualities called T-duality, S-duality, and U-duality between strong coupling and weak coupling limits of these theories can be defined and these dualities can give rise to one unified M-theory in eleven dimensions [1, 2, 3, 4]. The main common property of these ten and eleven dimensional theories is that their low energy limits correspond to the supergravity theories in those dimensions. This is the main reason supergravity theories attract increasing attention in recent research literature [5, 6]. Understanding supergravity theories is important for knowing the dynamics of massless fields in string theories and finding the backgrounds that strings can propagate.

Supergravity theories are extensions of general relativity to obtain an action that is invariant under supersymmetry transformations. For the consistency and invariance of the theory one has to define higher spin fermionic fields and extra bosonic fields with appropriate supersymmetry transformations. There are different consistent supergravity theories in different dimensions. The low energy limits of string and M-theories are described by the bosonic sectors of ten- and eleven-dimensional supergravity theories. Bosonic sectors of supergravity theories correspond to taking fermionic fields and their variations to be zero in the full theory. The solutions of bosonic supergravity field equations are called supergravity backgrounds. An important invariant that is used in the classification of supergravity backgrounds is the number of preserved supersymmetries in those

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backgrounds. The number of supersymmetries are described by Killing spinors of the background corresponding to the spinors that are solutions of the differential equation results from the variation of the gravitino field [7]. This differential equation also defines the spinor covariant derivative in terms of the extra bosonic fields of the theory.

The amount of supersymmetry in a supergravity background is the main tool for the classification of supersymmetric supergravity backgrounds. Constructing the Killing superalgebras of a background is an important step to achieve this classification. A Killing superalgebra is a Lie superalgebra that consists of the isometries of the background corresponding to the Killing vectors and the number of preserved supersymmetries constituting the Killing spinors of the background. These constituents of the superalgebra are even and odd parts of it, respectively. In the definition of Killing superalgebras, the spinorial Lie derivative and Dirac currents that are constructed from two Killing spinors and correspond to Killing vectors are used [8, 9]. To have a well-defined Lie superalgebra structure, the Jacobi identities of odd and even parts of the superalgebra must be satisfied. The dimension of the Killing superalgebra is related to the homogeneity structure of the background [10, 11]. By constructing the Killing superalgebras of different supergravity backgrounds, one can obtain the geometric and supersymmetric properties of those backgrounds. On the other hand, classification of supergravity backgrounds can be considered in different contexts. Besides the construction of Killing superalgebras, there are also methods that use G-structures and spinorial geometry to attack the classification problem [12, 13, 14, 15, 16]. However, we will focus on the method of constructing Killing superalgebras in the rest of the paper.

In this review paper, we consider the current status of the subject and discuss about the possible extensions of the structures, which can give hints about how to achieve some developments regarding the problem. The paper is organized as follows. In section 2, we summarize the bosonic sectors of different supergravity theories in ten and eleven dimensions that are the low energy limits of string theories and M-theory. In section 3, we provide some examples of solutions of those supergravity theories that are called supersymmetric supergravity backgrounds. Section 4 contains the construction and properties of Killing superalgebras in general supergravity backgrounds and some possible extensions of them to higher order forms. In section 5, we provide a summary and discussion before concluding the paper.

2. Bosonic sectors of supergravity theories

The low energy limits of string theories and M-theory are ten- and eleven-dimensional supergravity theories. To obtain the solutions that strings can propagate consistently, we consider the bosonic sectors of supergravities. These correspond to the backgrounds of the theories by taking the fermionic fields to be zero and supersymmetry variations of fermionic fields give rise to Killing spinor equations whose solutions give the number of supersymmetries preserved by the background. In general, a supergravity background consists of a Lorentzian spin manifold M, a metric g, and some other bosonic fields defined on M depending on the type and dimension of the supergravity theory. The metric and bosonic fields satisfy some field equations generalizing the Einstein–Maxwell equations. A supergravity background is called supersymmetric if it admits nonzero Killing spinors that are solutions of the Killing spinor equation. In this section, we will introduce the bosonic sectors of supergravity theories in ten and eleven dimensions.

2.1. Eleven-dimensional supergravity

The maximum dimension that a supergravity theory can be consistently constructed is eleven and eleven-dimensional supergravity is the low energy limit of M-theory [17]. The bosonic sector of eleven-dimensional supergravity consists of a Lorentzian metric g and a closed 4-form F, which is the field strength of a 3-form A, namely $F = dA$. This theory can be considered a generalization of Einstein–Maxwell theory to eleven dimensions with a generalized field strength F. The action of the theory is written as follows:

$$S = \frac{1}{12\kappa_{11}^2}\int\left(R_{ab}\wedge *e^{ab} - \frac{1}{2}F\wedge *F - \frac{1}{6}A\wedge F\wedge F\right), \tag{1}$$

where κ_{11} is the eleven-dimensional gravitational coupling constant, R_{ab} are curvature 2-forms, e^a are coframe basis constructed from the metric g with $e^{ab} = e^a \wedge e^b$, and $*$ is the Hodge star operation. Note that the second term in the action is in the form of the Maxwell action and the last term is the topological Chern–Simons term. Varying with respect to e^a and A gives rise to the following field equations:

$$Ric(X,Y) * 1 = \frac{1}{2}i_X F\wedge *i_Y F - \frac{1}{6}g(X,Y)F\wedge *F \tag{2}$$

$$d * F = \frac{1}{2}F\wedge F, \tag{3}$$

where X,Y are vector fields, Ric is the Ricci tensor, $*1$ is the volume form, and i_X is the interior derivative or contraction with respect to X. On the other hand, the spinor covariant derivative is modified by the existence of the extra bosonic fields in supergravity theories. Variation in the gravitino field gives a spinor equation whose solutions are Killing spinors

$$\mathcal{D}_X\epsilon = 0, \tag{4}$$

where ϵ is a spinor and the modified spinor covariant derivative is defined in terms of ordinary spinor covariant derivative ∇_X as

$$\mathcal{D}_X = \nabla_X + \frac{1}{6}i_X F - \frac{1}{12}\tilde{X}\wedge F \tag{5}$$

and $\tilde{X}$ is the 1-form that corresponds to the metric dual of X. The solutions of the Killing spinor equation determine the number of preserved supersymmetries in a supergravity background.

2.2. Type IIA and IIB supergravities

By compactifying eleven-dimensional supergravity on S^1 with the Kaluza–Klein reduction method, one obtains the ten-dimensional type IIA supergravity, which is the low energy limit of type IIA string theory [18]. Besides the fermionic fields gravitino and dilatino, type IIA supergravity includes several bosonic fields that are the graviton described by the metric g, a real scalar field called dilaton ϕ, a 2-form gauge potential B_2 with field strength $H_3 = dB_2$, and 1-form and 3-form Ramond–Ramond gauge potentials C_1 and C_3 with field strengths $F_2 = dC_1$ and $F_4 = dC_3$. The bosonic sector of IIA supergravity corresponds to taking the fermionic fields to be zero.

By defining a new field strength

$$\tilde{F}_4 = F_4 - C_1\wedge H_3 \tag{6}$$

with the property

$$d\tilde{F}_4 = -F_2\wedge H_3 \tag{7}$$

the bosonic action of IIA supergravity is written as follows:

$$\begin{aligned} S &= \frac{1}{2\kappa_{10}^2}\int e^{-2\phi}\left(R_{ab}\wedge *e^{ab}+4d\phi\wedge *d\phi\right) \\ &-\frac{1}{4\kappa_{10}^2}\int\left(e^{-2\phi}H_3\wedge *H_3+F_2\wedge *F_2+\tilde{F}_4\wedge *\tilde{F}_4\right) \\ &-\frac{1}{4\kappa_{10}^2}\int B_2\wedge F_4\wedge F_4. \end{aligned} \tag{8}$$

The Killing spinors in type IIA supergravity satisfy two equations since there are two fermionic fields in the theory. The differential condition comes from the variation in the gravitino and the equation corresponds to the condition of being parallel with respect to the following modified spinor covariant derivative:

$$\mathcal{D}_X=\nabla_X-\frac{1}{4}(i_XH_3)z-\frac{1}{8}e^{\phi}(i_XF_2)z+\frac{1}{8}e^{\phi}\tilde{X}\wedge F_4 \tag{9}$$

where z is the ten-dimensional volume form. The algebraic condition comes from the variation in the dilatino and reads as follows:

$$\left(-\frac{1}{3}(d\phi)z+\frac{1}{6}H_3-\frac{1}{4}e^{\phi}F_2+\frac{1}{12}e^{\phi}F_4z\right)\epsilon=0. \tag{10}$$

There is another ten-dimensional supergravity theory, which is called type IIB supergravity and cannot be obtained from the eleven-dimensional theory [19]. Fermionic content of IIB theory is the same as in the IIA case; however, the bosonic sector differs from it. Besides the metric g, dilaton ϕ, and the 2-form B_2 with field strength $H_3=dB_2$, it also contains 0-form, 2-form, and 4-form Ramond–Ramond gauge potentials C_0, C_2, and C_4 with field strengths $F_1=dC_0$, $F_3=dC_2$, and self-dual $F_5=dC_4=*F_5$. By defining the new combinations of field strengths

$$\begin{aligned} \tilde{F}_3 &= F_3-C_0\wedge H_3 \\ \tilde{F}_5 &= F_5-\frac{1}{2}C_2\wedge H_3+\frac{1}{2}B_2\wedge F_3, \end{aligned} \tag{11}$$

the action of the bosonic sector of the type IIB supergravity can be written as

$$\begin{aligned} S &= \frac{1}{2\kappa_{10}^2}\int e^{-2\phi}\left(R_{ab}\wedge *e^{ab}+4d\phi\wedge *d\phi\right) \\ &-\frac{1}{4\kappa_{10}^2}\int\left(e^{-2\phi}H_3\wedge *H_3+F_1\wedge *F_1+\tilde{F}_3\wedge *\tilde{F}_3+\tilde{F}_5\wedge *\tilde{F}_5\right) \\ &-\frac{1}{4\kappa_{10}^2}\int C_4\wedge H_3\wedge F_3. \end{aligned} \tag{12}$$

In fact, it is only a pseudo-action since only after externally imposing the self-duality condition $F_5=*F_5$ can the field equations of the theory be obtained from it.

The differential Killing spinor equation in type IIB supergravity is written from the following modified spinor covariant derivative [20]:

$$\mathcal{D}_X=\nabla_X+\frac{1}{4}i_XH_3\otimes\sigma_3-\frac{1}{16}e^{\phi}\tilde{X}\wedge\left(F_1\otimes i\sigma_2-\tilde{F}_3\otimes\sigma_1+\tilde{F}_5\otimes i\sigma_2\right), \tag{13}$$

where σ_i are Pauli matrices and the algebraic Killing spinor equation is

$$\left(\frac{1}{2}(d\phi - ie^{\phi}dC_0) + \frac{1}{4}(ie^{\phi}\tilde{F}_3 - H_3)\right)\epsilon = 0. \tag{14}$$

2.3. Type I and heterotic supergravities

The low energy limit of type I superstring theory is a ten-dimensional supergravity theory coupled with a super Yang–Mills theory with gauge group $SO(32)$ in ten dimensions [21]. This low energy limit is called type I supergravity theory and it contains the metric g, the dilaton ϕ, a Ramond–Ramond 2-form C_2, and the non-abelian Yang–Mills $SO(32)$ gauge connection A_1 as bosonic fields. The field strengths are defined as

$$\begin{aligned} F_2 &= dA_1 + A_1 \wedge A_1 \\ \tilde{F_3} &= dC_2 + \frac{1}{4}\omega_3 \end{aligned}$$

where $\omega_3 = A_1 \wedge dA_1 + \frac{2}{3}A_1 \wedge A_1 \wedge A_1$ is the Chern–Simons form of the gauge connection. The bosonic action of type I supergravity is defined in terms of these field strengths as follows:

$$S = \int \left[e^{-2\phi}\left(R_{ab} \wedge *e^{ab} + 4d\phi \wedge *d\phi\right) - \frac{1}{2}\tilde{F}_3 \wedge *\tilde{F}_3 - e^{-\phi}F_2 \wedge *F_2\right]. \tag{15}$$

The variation in the gravitino gives rise to the following spinor covariant derivative used in the definition of the Killing spinors:

$$\mathcal{D}_X = \nabla_X - \frac{1}{8}e^{\phi}\tilde{F}_3\tilde{X}. \tag{16}$$

Besides the gravitino, there are two more fermionic fields, dilatino and gaugino, and hence we have the following two algebraic Killing spinor equations:

$$\begin{aligned} \left(d\phi + \frac{1}{2}e^{\phi}\tilde{F}_3\right)\epsilon &= 0 \\ F_2\epsilon &= 0. \end{aligned} \tag{17}$$

The heterotic supergravity differs from type I supergravity in its fermionic sector. Since we consider only the bosonic sectors of supergravities, we can write the bosonic action of the heteroric supergravity in terms of the metric g, the dilaton ϕ, a 2-form B_2 with field strength $H_3 = dB_2$ and 1-form non-abelian $SO(32)$, or $E_8 \times E_8$ gauge potential A_1 with the field strength F_2 as in the type I case:

$$S = \int e^{-2\phi}\left(R_{ab} \wedge *e^{ab} + 4d\phi \wedge *d\phi - \frac{1}{2}H_3 \wedge *H_3 - \frac{1}{2}F_2 \wedge *F_2\right). \tag{18}$$

The differential Killing spinor equation is determined by the following spin connection:

$$\mathcal{D}_X = \nabla_X - \frac{1}{4}i_X H_3. \tag{19}$$

Although the fermionic sector of heterotic and type I supergravities are not the same, the algebraic Killing spinor equations of heterotic supergravity have similar forms as in the type I case:

$$\begin{aligned} \left(d\phi + \frac{1}{2}H_3\right)\epsilon &= 0 \\ F_2\epsilon &= 0. \end{aligned} \tag{20}$$

3. Supergravity backgrounds

There are various special solutions for eleven-dimensional and ten-dimensional supergravity theories. One of the methods that can give hints about obtaining all solutions is to find a way of classification for them. However, the complete classification of all supergravity backgrounds has not been achieved yet. On the other hand, for some special cases such as for backgrounds that have maximal supersymmetry and for symmetric space backgrounds, the classification problem can be solved. In this section, we consider some special solutions of the supergravity theories and summarize the classification results for symmetric and maximally supersymmetric backgrounds.

In eleven dimensions, a spinor space is a 32-dimensional real space and hence the space of the solutions of Killing spinor equation can be at most 32-dimensional. An important invariant for a supersymmetric supergravity background is the fraction of preserved supersymmetries in that background, which is denoted as $\nu = \frac{1}{32}n$, and n is the dimension of the Killing spinor space. There are various numbers of known supergravity backgrounds that have different fractions of preserved supersymmetries. Indeed, the known solutions correspond to $n = 0, 1, 2, 3, 4, 5, 6, ..., 8, ..., 12, ..., 16, ..., 18, ..., 20, ..., 22, ..., 24, ..., 26, ..., 32$ [8]. The numbers that do not appear in the list correspond to the cases of the fraction of preserved supersymmetries whose exact forms are not known. Moreover, for $n = 30$ and 31, it is known that there are no supersymmetric supergravity backgrounds [22, 23, 24, 25]. It is also proved that for the cases of $\nu > \frac{1}{2}$, those supergravity backgrounds are locally homogeneous [10, 11]. For $\nu = 1$, the solutions are called maximally supersymmetric or BPS solutions and examples for this case include eleven-dimensional flat Minkowski spacetime and Freund–Rubin backgrounds such as $AdS_7 \times S^4$ and $AdS_4 \times S^7$. There are also half-BPS, namely $\nu = \frac{1}{2}$, solutions of the eleven-dimensional supergravity. Examples for this case are M-wave, Kaluza–Klein monopole, and M2- and M5-brane solutions [26].

3.1. Half-BPS solutions

A gravitational *pp*-wave is defined as a spacetime with a parallel null vector. A supersymmetric gravitational *pp*-wave, which is called the M-wave, is a solution of eleven-dimensional supergravity and its metric is given as follows [27]:

$$ds^2 = 2dx^+dx^- + a(dx^+)^2 + (dx^9)^2 + g_{ij}dx^i dx^j, \tag{21}$$

where $i, j = 1$ to 8, x^+, x^- are light cone coordinates, a is an arbitrary function with $\partial_- a = 0$, and g_{ij} is a family of metrics dependent on x^+. The holonomy group of the manifold on which g_{ij} is defined is contained in $Spin(7)$ and the following property is satisfied:

$$\partial_+\Omega = \lambda\Omega + \Psi, \tag{22}$$

where Ω is the self-dual $Spin(7)$-invariant Cayley 4-form, λ a smooth function of (x^+, x^-), and Ψ is an anti self-dual 4-form. The closed 4-form F of eleven-dimensional supergravity vanishes in this solution. The metric

(21) is a supersymmetric solution of eleven-dimensional supergravity if and only if it is Ricci-flat. By dropping the x^9 coordinate, one can also obtain a solution of ten-dimensional supergravity written in the following form:

$$ds^2 = 2dx^+dx^- + a(dx^+)^2 + g_{ij}dx^i dx^j. \tag{23}$$

There is also a brane solution of eleven-dimensional supergravity with the following metric that describes a number of parallel M2-branes [28]:

$$g = H^{-2/3}g_{2+1} + H^{1/3}g_8 \tag{24}$$

where g_{2+1} is the metric on the three-dimensional Minkowskian worldvolume $E^{2,1}$ of the branes and g_8 is the metric on the eight-dimensional Euclidean space E^8 transverse to the branes. H is a harmonic function on E^8 and can be chosen as

$$H(r) = 1 + \frac{a^6}{r^6}, \tag{25}$$

where r is the radial coordinate and $a^6 = 2^5\pi^2 N {l_p}^6$. Here N is the number of coincident membranes at $r = 0$ and l_p is the eleven-dimensional Planck length. The harmonic function has the property $\lim_{r\to\infty} H(r) = 1$. The closed 4-form field in this background corresponds to

$$F = \pm z_{2+1} \wedge dH^{-1}, \tag{26}$$

where z_{2+1} is the volume form of the brane worldvolume. The M2-brane solution preserves half of the supersymmetries, namely $\nu = \frac{1}{2}$. However, it interpolates between two maximally supersymmetric solutions; near the brane horizon $r \ll a$, it corresponds to the maximally supersymmetric Freund–Rubin background $AdS_4 \times S^7$ and for infinitely far away from the brane $r \to \infty$, it corresponds to the flat Minkowski space $E^{10,1}$.

There is a similar brane solution describing a number of parallel M5-branes with metric and 4-form given as follows [29]:

$$\begin{aligned} g &= H^{-1/3}g_{5+1} + H^{2/3}g_5 && (27)\\ F &= \pm 3 *_5 H, && (28)\end{aligned}$$

where g_{5+1} is the metric on the six-dimensional Minkowskian worldvolume $E^{5,1}$ of the branes, g_5, and $*_5$ is the metric and Hodge dual on the five-dimensional Euclidean space E^5 transverse to the branes. The harmonic function H is defined as

$$H(r) = 1 + \frac{a^3}{r^3} \tag{29}$$

where $a^3 = \pi N {l_p}^3$, N is the number of coincident fivebranes at $r = 0$, and $\lim_{r\to\infty} H(r) = 1$. The M5-brane preserves half of the supersymmetries and it interpolates between the maximally supersymmetric Freund–Rubin background $AdS_7 \times S^4$ and flat Minkowski space $E^{10,1}$.

The ten-dimensional type IIB supergravity also has a brane solution, which is called D3-brane, with the metric given by [30]

$$g = H^{-1/2}g_{3+1} + H^{1/2}g_6, \tag{30}$$

where g_{3+1} is the metric on the worldvolume $E^{3,1}$ of the brane and g_6 is the Euclidean metric on the transverse space E^6 to the brane. The self-dual 5-form F_5 is defined on $S^5 \subset E^6$ and has quantized flux and the dilaton

ϕ is constant. The harmonic function H is defined as

$$H(r) = 1 + \frac{a^4}{r^4} \tag{31}$$

with $a^4 = 4\pi g N {l_s}^4$ and g is the string coupling constant and l_s is the string length. N is the number of parallel D3-branes at $r = 0$. This solution interpolates between flat Minkowski space at infinity and $AdS_5 \times S^5$ at the near horizon limit.

3.2. Maximally supersymmetric backgrounds

The solutions of supergravity theories that have maximum number of Killing vector fields and maximum number of Killing spinors are called maximally supersymmetric supergravity backgrounds. In this case the dimension of the space of Killing spinors is 32 and we have $\nu = 1$.

In eleven dimensions, there are four classes of solutions that have maximal supersymmetry. The first one is the flat Minkowski spacetime $M^{1,10}$ with $F = 0$. Two of the other nontrivial classes include the Freund–Rubin backgrounds of which we have a four-dimensional and seven-dimensional split of spacetime $M = M^4 \times M^7$ and the total metric is written as the sum of the metrics of the split spacetimes:

$$ds^2 = {ds_4}^2 + {ds_7}^2. \tag{32}$$

One of the split spacetimes has positive and the other one has negative constant curvatures. The first case of this type of solution corresponds to the following background:

$$\begin{aligned} &AdS_7(-7R) \times S^4(8R) \\ &F = \sqrt{6R}\, z_{S^4}, \end{aligned} \tag{33}$$

where the numbers in parentheses correspond to the constant scalar curvatures of the relevant backgrounds with $R > 0$ and z_{S^4} is the volume form of S^4. The second case is the following spacetime:

$$\begin{aligned} &AdS_4(8R) \times S^7(-7R) \\ &F = \sqrt{-6R}\, z_{AdS^4} \end{aligned} \tag{34}$$

with $R < 0$ and z_{AdS^4} is the volume form of AdS_4. The fourth class of maximally supersymmetric solutions is a one-parameter family of symmetric plane waves with the metric and the 4-form

$$\begin{aligned} g &= 2dx^+dx^- - \frac{1}{36}\mu^2 \left(4\sum_{i=1}^{3}(x^i)^2 + \sum_{i=4}^{9}(x^i)^2 \right)(dx^-)^2 + \sum_{i=1}^{9}(dx^i)^2 \\ F &= \mu dx^- \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \tag{35}$$

where μ is a real number.

In ten dimensions, the only maximally supersymmetric solution of type I, heterotic, and IIA supergravities is the flat Minkowski spacetime. However, in the type IIB case we have two nontrivial classes. The first one is the Freund–Rubin background

$$\begin{aligned} &AdS_5(-R) \times S^5(R) \\ &F_5 = \sqrt{\frac{4R}{5}}\,(z_{AdS_5} - z_{S^5}), \end{aligned} \tag{36}$$

where F_5 is the self-dual 5-form and $R > 0$. The second one is a family of symmetric plane waves with the following metric and self-dual 5-form F_5:

$$\begin{aligned} g &= 2dx^+dx^- - \frac{1}{4}\mu^2\sum_{i=1}^{8}(x^i)^2(dx^-)^2 + \sum_{i=1}^{8}(dx^i)^2 \\ F_5 &= \frac{1}{2}\mu dx^- \wedge \left(dx^1\wedge dx^2\wedge dx^3\wedge dx^4 + dx^5\wedge dx^6\wedge dx^7\wedge dx^8\right). \end{aligned} \tag{37}$$

The classification of all supergravity backgrounds is one of the main goals of the study of supergravity theories. The homogeneity theorem for supergravity backgrounds says that a supergravity background that preserves more than half of the supersymmetries must be homogeneous [10, 11]. A homogeneous spacetime can be characterized as a spacetime of which the tangent space at any point is spanned by Killing vectors that are constructed by squaring the Killing spinors of the background. If a homogeneous background also corresponds to a symmetric space, then we call it a symmetric supergravity background. The classification of symmetric supergravity backgrounds in eleven-dimensional and ten-dimensional type IIB cases are achieved in [31]. However, the full classification of all supergravity backgrounds is still an open problem.

4. Killing superalgebras

An important invariant that characterizes the supersymmetric supergravity backgrounds is the Killing superalgebra of that background [8, 32]. A Killing superalgebra $\mathfrak{g}$ has a Lie superalgebra structure that consists of a Z_2-graded algebra that is a direct sum of two components $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The first component $\mathfrak{g}_0$ of the Lie superalgebra is called the even part and has a Lie algebra structure; the second one $\mathfrak{g}_1$ is called the odd part and corresponds to a module of $\mathfrak{g}_0$. A Lie bracket $[,]$ on a Lie superalgebra is defined as a bilinear multiplication

$$[,] : \mathfrak{g}_i \times \mathfrak{g}_j \longrightarrow \mathfrak{g}_{i+j} \tag{38}$$

where $i, j = 0, 1$ and satisfies the following skew-supersymmetry, and super-Jacobi identities for a, b, c are elements of $\mathfrak{g}$, and $|a|$ denotes the degree of a, which corresponds to 0 or 1 depending on that a is in $\mathfrak{g}_0$ or $\mathfrak{g}_1$, respectively

$$\begin{aligned} [a,b] &= -(-1)^{|a||b|}[b,a] \\ [a,[b,c]] &= [[a,b],c] + (-1)^{|a||b|}[b,[a,c]]. \end{aligned} \tag{39}$$

For a Killing superalgebra, the even and odd parts of it are defined as bosonic and fermionic parts of the superalgebra. The bosonic part of the Killing superalgebras of supergravity backgrounds corresponds to the Lie algebra of Killing vector fields generated by Killing spinors. A Killing vector field K is an isometry of the background, which means that the Lie derivative of the metric g with respect to K is zero

$$\mathcal{L}_K g = 0. \tag{40}$$

Killing vector fields in the Killing superalgebra also preserve the other bosonic fields in the corresponding supergravity background. The fermionic part of the Killing superalgebra consists of the Killing spinors of the background. The supersymmetric properties of a supergravity background can be found from the Killing superalgebra, which consists of the supersymmetry generators and isometries of the background.

Three operations corresponding to the Lie brackets in the superalgebra can be defined for the even (bosonic) and odd (fermionic) parts of a Killing superalgebra. Since the even part corresponds to the algebra of Killing vector fields, the Lie bracket defined on it is the ordinary Lie bracket for vector fields

$$[\,,\,] : \mathfrak{g}_0 \times \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0. \tag{41}$$

The action of the even part to the odd part is defined as the Lie derivative of spinor fields with respect to a Killing vector:

$$\mathcal{L} : \mathfrak{g}_0 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_1. \tag{42}$$

The Lie derivative of a Killing spinor will again be a Killing spinor. The spinor Lie derivative is defined only with respect to Killing vectors and it is induced from the Lie derivative on differential forms as sections of the Clifford bundle of the manifold [33, 34]. It is written in terms of the spinor covariant derivative and a 2-form constructed from a Killing 1-form as follows:

$$\mathcal{L}_K\psi = \nabla_K\psi + \frac{1}{4}(d\tilde{K})\psi, \tag{43}$$

where $\tilde{K}$ is the Killing 1-form that is the metric dual of the Killing vector K. The third operation, which takes two odd elements to obtain an even element, is described by the squaring map of spinors

$$\mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0. \tag{44}$$

This map is defined in terms of the spin invariant inner product on spinors [35]. The Clifford product of a spinor ψ with a dual spinor $\bar{\phi}$ can be written as a sum of differential forms by using the spinor inner product $(,)$. This decomposition is known as the Fierz identity:

$$\psi\bar{\phi} = (\psi,\phi) + (\psi, e_a\phi)e^a + (\psi, e_{a_2a_1}\phi)e^{a_1a_2} + ... + (-1)^{\lfloor\frac{n}{2}\rfloor}(\psi, z\phi)z \tag{45}$$

where $\lfloor\rfloor$ is the floor function and z is the volume form. The squaring map corresponds to the projecting the Fierz identity onto the 1-form component. If ψ and ϕ are Killing spinors, then the dual of the resulting 1-form is a Killing vector and for $\psi = \phi$ it is called the Dirac current V_ψ of ψ.

In this way, all the needed brackets are defined for the Killing superalgebra. However, to obtain a Lie superalgebra structure, the Jacobi identities of the algebra must be satisfied. There are four Jacobi identities corresponding to the $[\mathfrak{g}_0, \mathfrak{g}_0, \mathfrak{g}_0]$, $[\mathfrak{g}_0, \mathfrak{g}_0, \mathfrak{g}_1]$, $[\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_1]$, and $[\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1]$ components. The first one is the ordinary Jacobi identity for the Lie algebra of Killing vector fields. The second and third ones are equivalent to the following properties of the spinor Lie derivative defined on spinors:

$$[\mathcal{L}_{K_1}, \mathcal{L}_{K_2}]\psi = \mathcal{L}_{[K_1,K_2]}\psi \tag{46}$$

$$\mathcal{L}_K(\psi\bar{\phi}) = (\mathcal{L}_K\psi)\bar{\phi} + \psi\overline{\mathcal{L}_K\phi}. \tag{47}$$

The fourth Jacobi identity is the vanishing of the Lie derivative of a spinor with respect to the Dirac current of itself:

$$\mathcal{L}_{V_\psi}\psi = 0. \tag{48}$$

This is not a trivial result and it has been proved for different cases in [8, 9].

To construct the Killing superalgebra of a supergravity background, one needs to know the isometry algebra of the background, which consists of the Killing vectors on that background and the algebra consisting of the space of Killing spinors. However, by using the cone construction [36], which states that there is a one to one correspondence between the Killing spinors of a background and the parallel spinors of a background corresponding to the metric cone over the first one, one can also construct the Killing superalgebra without knowing the exact form of Killing spinors. For some special examples such as Freund–Rubin backgrounds in eleven dimensions, this kind of construction can be achieved. For the $AdS_4 \times S^7$ spacetime, the isometry algebra consists of the direct product of isometry algebras of the component spaces. The isometry algebra of AdS_4 is $\mathfrak{so}(3,2)$ and of S^7 is $\mathfrak{so}(8)$, and so the isometry algebra of $AdS_4 \times S^7$ is $\mathfrak{so}(3,2) \times \mathfrak{so}(8)$ and by using the cone construction procedures the Killing spinor space can also be obtained [32]. As a result, the Killing superalgebra of this background is the following orthosymplectic Lie superalgebra:

$$\mathfrak{osp}(8|4). \tag{49}$$

As another example, the isometry algebra of $AdS_7 \times S^4$ is $\mathfrak{so}(6,2) \times \mathfrak{so}(5)$ and the Killing superalgebra is

$$\mathfrak{osp}(6,2|4). \tag{50}$$

In an eleven-dimensional Minkowski background, the isometries generated by Killing spinors are translational Killing vector fields and the Killing superalgebra correspond to the supertranslation ideal of the Poincare superalgebra. However, one can also consider all isometries of the background that also contain rotational Killing vector fields and extend the Killing superalgebra to a symmetry superalgebra corresponding to the Poincare superalgebra. The Killing superalgebras of the fourth class of eleven-dimensional maximally supersymmetric backgrounds denoted in (35) can be obtained by taking contractions of the Killing superalgebras of Freund–Rubin backgrounds.

Killing superalgebras play an important role in the classification of supergravity backgrounds. Maximally supersymmetric backgrounds that have $\nu = 1$ and the minimally supersymmetric ones can be classified completely. However, the classification of less than maximal and more than minimal supersymmetric backgrounds is related to the Killing superalgebras of them. In eleven dimensions, the backgrounds preserving the $\nu > \frac{1}{2}$ fraction of supersymmetry are locally homogeneous and this result is achieved by first constructing the Killing superalgebras and then obtaining the dimension of the translational component of the squaring map from Killing spinors to Killing vectors. In ten dimensions, local homogeneity is guaranteed for $\frac{1}{2} \leq \nu \leq \frac{3}{4}$ and this can be proved from the construction of the Killing superalgebras [9].

4.1. Extension to higher order superalgebras

Isometries of a background that are generated by Killing spinors constitute the even part of the Killing superalgebras. If one considers all isometries of the background (not necessarily generated by Killing spinors), then one can define more general symmetry superalgebras of the background. Besides this fact, the squaring map of spinors can be extended to define higher order spinor bilinears. Although there are some attempts to include these higher order objects in the symmetry superalgebras, the construction of the so-called maximal superalgebras is still an open problem [37, 38]. It is known that the Killing vector fields have higher order antisymmetric generalizations to hidden symmetries and these hidden symmetries are called Killing–Yano (KY) forms. A KY p-form ω is defined as the solution of the following equation:

$$\nabla_X \omega = \frac{1}{p+1} i_X d\omega. \tag{51}$$

Moreover, one can also define the generalizations of Dirac currents to higher-degree forms. From Fierz identity (45), one can define projection operators $\wp_p$ that project onto the p-form component of the Clifford product of a spinor with its dual. The generalized currents are called p-form Dirac currents and defined as follows:

$$\wp_p(\psi\bar{\psi}) = (\psi, e_{a_p \dots a_2 a_1}\psi)e^{a_1 a_2 \dots a_p}. \tag{52}$$

It is shown in [39] that these p-form Dirac currents of twistor spinors correspond to conformal KY forms and for the Killing spinors case they correspond to the KY forms. It is also known that KY forms satisfy a graded Lie algebra structure in constant curvature spacetimes [40]. The following Schouten–Nijenhuis bracket is defined for a p-form α and a q-form β:

$$[\alpha, \beta]_{SN} = i_{X_a}\alpha \wedge \nabla_{X^a}\beta + (-1)^{pq} i_{X_a}\beta \wedge \nabla_{X^a}\alpha \tag{53}$$

and corresponds to a Lie bracket for KY forms in constant curvature spacetimes. This means that the Killing superalgebras can be extended to include KY forms in some constant curvature supergravity backgrounds. On the other hand, a generalized Lie derivative on spinor fields with respect to KY forms has to be defined and the Jacobi identities also have to be satisfied. The spinorial Lie derivatives with respect to Killing vector fields also correspond to the symmetry operators of the Dirac equation in curved backgrounds. Symmetry operators of the Dirac equation that contain higher degree forms are also constructed in curved backgrounds by using KY forms [41, 42, 43, 44]. Hence, these operators are natural candidates for the generalized Lie derivatives of extended superalgebras. The construction of these extended Killing superalgebras can give rise to new hints about the classification problem of supergravity backgrounds. As a side remark, KY forms are also used in relation to G-structures in the supergravity context [45].

5. Discussion

Ten- and eleven-dimensional supergravity backgrounds correspond to the spacetimes that strings can propagate in a well-defined manner. Therefore, the complete classification of supergravity backgrounds can give hints about the possible string backgrounds and the unification of string theories. There are many known solutions of supergravity theories that correspond to the backgrounds that have different fractions of preserved supersymmetries. Finding the common geometrical properties of these backgrounds and the complete classification of them according to the preserved supersymmetries is one of the main problems in supergravity and string theory. In some special cases, the classification problem is generally understood, but in most cases it is not completely yet. To achieve this aim, finding the invariants of these backgrounds is an important step. One of the main invariants of supersymmetric supergravity backgrounds is their Killing superalgebras and they are constructed out of the isometries and Killing spinors of the background.

To construct a Killing superalgebra in a special supergravity background, one needs to know the Lie algebras of Killing vectors and Killing spinors. However, without knowing the Killing spinors of the background, one can also find the odd part of the superalgebra by using the cone construction and the knowledge about the parallel spinors. In this construction, the Lie derivatives of spinor fields and the Dirac currents of spinors are used and satisfying the Jacobi identities of the superalgebra completes the construction procedure. In this way, the Killing superalgebras of some supersymmetric backgrounds are obtained in the literature. Structures of these Killing superalgebras in different backgrounds and relations between them give a way to classify these supersymmetric backgrounds. For example, by using the properties of Killing superalgebras, the local homogeneity of a supergravity background that preserves more than half of supersymmetries is proved [10, 11].

Obtaining more hints about the classification problem may be possible by extending the Killing superalgebras to higher order geometric objects. Killing vector fields have natural antisymmetric generalizations to higher-degree forms that correspond to KY forms. Dirac currents also have generalizations to higher-degree components that are called p-form Dirac currents. The correspondence between the p-form Dirac currents and KY forms is proved in [39]. Therefore, extending the Killing superalgebras to include KY forms and p-form Dirac currents with a new definition of the generalized spinor Lie derivative may be possible. The properties of these extended superalgebras may give new insights into the classification of supergravity backgrounds.

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