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Bound states of the Duffin–Kemmer–Petiau equation for square potential well with position-dependent mass

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Abstract: The effective mass Duffin–Kemmer–Petiau equation for spin 0 and spin 1 with a potential well is considered, and transcendental equations are derived for the energy eigenvalues. Numerical results are reported graphically, and the variations of the energy of the bound states are computed as a function of the well width and mass.

Key words: Duffin–Kemmer–Petiau equation, position-dependent mass, bound state

1. Introduction

In recent years, many studies have focused on the problem of the mass dependent on the position in relativistic and nonrelativistic quantum systems. Therefore, position-dependent mass (PDM) formalism has been widely used in the determination of the physical properties of various microstructures [1–6], and with this approach, a variety of potentials and mass distributions have been considered by different methods [7–10]. In the nonrelativistic case, the solutions of the PDM Schrödinger equation have been considered for different potentials [11–15], and more particularly, configurations such as the step and rectangular barrier of potentials with the same shape for the masses have been used in the framework of the nonrelativistic Green function [16].

In the relativistic case, in order to explain some associated quantum effects, the Klein–Gordon (KG) particle with spin 0 and the Dirac particle with spin 1/2 within an effective mass and in different forms of potential have also been examined [17–26]. For example, the spectrum of the D-dimensional Dirac equation, where the mass is dependent on the position and within the framework of an exponential for the centrifugal term, was obtained in [27], and the N-dimensional Pöschl–Teller potential with PDM was also considered in [28], using the asymptotic iteration method. In [29], the spatially dependent mass Dirac equation for the Coulomb field plus tensor interaction was solved exactly via Laplace transformation and the effect of this tensor on the bound states was discussed. In the same way, the relativistic neutral fermions subjected to a PT-symmetric potential with PDM were investigated in [30], and the influence of this potential on the continuity equation and on the orthonormalization condition was analyzed. However, in the literature there are only a few papers related to bosons with spin 1 in comparison with relativistic spin 0 and spin 1/2 particles. Generally, to explore the relativistic problems of such particles, one has to solve either the Proca or the Duffin–Kemmer–Petiau (DKP) equations [31–33].

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The DKP equation is a direct generalization of the Dirac equation where one replaces the algebra of the gamma matrices by beta matrices, but satisfying a more complicated algebra, the so-called DKP algebra [34]. In the case where the mass is constant, some papers have been devoted to the DKP equation with a certain shape of potentials [35–39]. In the context of quantum chromodynamics, cosmology, and gravity and in many areas of physics, including particle and nuclear physics [40–44], this equation has also been examined. For the DKP equation with mass that depends on the position, there is only one paper dealing with this subject [45], and our motivation in this present work is to examine the problem of boundary conditions when the potential and the mass are not constant.

In this work, we consider the relativistic spin 0 and spin 1 bosons described by the DKP equation and subject to a square potential having the form of an asymmetric well, with a mass function m (z) similar to that used in [11]. Our aim is to obtain the bound states related to this model and to compare the results with the corresponding constant mass case.

The structure of the paper is as follows: in section 2, we use the formalism of the DKP equation with the square potential well within the PDM formalism and we analytically determine the solutions. Transcendental equations determining the energy eigenvalues via a limiting procedure for both cases of spin are obtained.

In section 3, the results are discussed and appropriate plots are presented. Finally, section 4 contains the conclusions of our work.

2. DKP equation with step mass

The (1+1)-dimensional effective mass DKP equation for the scalar and vector bosons moving in a vector potential A^{μ} (in natural units $\hbar = c = 1$) is:

$$[i\beta^{\mu}(\partial_{\mu} + ieA_{\mu}) - m(z)]\Psi(z,t) = 0, \tag{1}$$

where the matrices β^{μ} verify the algebra

$$\beta^{\mu}\beta^{\nu}\beta^{\lambda} + \beta^{\lambda}\beta^{\nu}\beta^{\mu} = g^{\mu\nu}\beta^{\lambda} + g^{\lambda\nu}\beta^{\mu}, \tag{2}$$

with μ, ν and λ being $0,1,2,3$. The tensor metric $g^{\mu\nu} =$ diag $(1, \ -1, \ -1, \ -1)$.

If we choose $eA_0 = V_0$ (z) and A $_i = 0,\ i = 1,\ 2, 3,$ Eq. (1) becomes:

$$[i\beta^{0}(\partial_{0} + iV_0(z)) + i\beta^{3}\partial_{z} - m(z)]\Psi(z\ ,t) = 0. \tag{3}$$

Since the potential does not depend on time, we need to search for the stationary states of this equation. As usual, a solution of the form $\Psi(z,t) = e^{-iEt}\Phi(z)$ reduces Eq. (3) to the following eigenvalue equation:

$$[\beta^{0}(E - V_0(z)) + i\beta^{3}\partial_{z} - m(z)]\Phi(z) = 0. \tag{4}$$

Here, V$_0$(z) and m(z) are chosen with the following form:

$$\begin{cases} V_0(z) = W_1\theta\left(-a-z\right) + W_2\theta\left(-a+z\right), \\ m(z) = (m_1 - m_2)[\theta(-a-z) + \theta(-a+z)] + m_2, \end{cases} \tag{5}$$

with θ (z) denoting the Heaviside step function:

$$\theta(z) = \begin{cases} 1 & \text{if} \quad z \geq 0 \\ 0 & \text{if} \quad z < 0 \end{cases}, \tag{6}$$

and m_1, m_2, W_1, and W_2 being positive constants such that $W_2 > W_1$ and $m_1 \neq m_2$. In what follows, we examine the problem of bound states ($0 < W_1 \leq E < W_2$).

2.1. Spin 1

In the case of vector bosons, the β^{μ} matrices are:

$$\beta^0 = \begin{pmatrix} 0 & \bar{0} & \bar{0} & \bar{0} \\ \bar{0}^T & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \bar{0}^T & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \bar{0}^T & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{pmatrix}, \beta^i = \begin{pmatrix} 0 & \bar{0} & e_i & \bar{0} \\ \bar{0}^T & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & -iS_i \\ -e_i^T & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \bar{0}^T & -iS_i & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{pmatrix}, \tag{7}$$

with S_i matrices being 3×3 ones, $(S_i)_{jk} = -i\varepsilon_{ijk}$ where ε_{ijk} is 1, –1, 0 for an even permutation, an odd permutation, and repeated indices, respectively. e_i matrices are 1×3, $(e_i)_{1j} = \delta_{ij}$, that is, $e_1 = (1\ 0\ 0)$, $e_2 = (0\ 1\ 0)$, $e_3 = (0\ 0\ 1)$.

The matrices $\mathbf{I}_{3\times3}$ and $\mathbf{0}_{3\times3}$ represent the unit and null 3×3 matrices, respectively, and $\bar{0}$s are 1×3 ones. The wave function $\Phi(z)$ possesses ten components and can be expressed in the following form:

$$\Phi^T = (\varphi_1^{(1)}, \varphi_1^{(2)}, \varphi_1^{(3)}, \varphi_1^{(4)}, \varphi_1^{(5)}, \varphi_1^{(6)}, \varphi_1^{(7)}, \varphi_1^{(8)}, \varphi_1^{(9)}, \varphi_1^{(10)}). \tag{8}$$

Since the DKP equation, as a relativistic equation, is essentially related to the KG, then we can convert the form of the problem to that of the KG by partitioning the wave function $\Phi(z)^T$ as follows:

$$\psi^T = (\varphi_1^{(2)}\ \varphi_1^{(3)}\ \varphi_1^{(7)}), \phi^T = (\varphi_1^{(5)}\ \varphi_1^{(6)}\ \varphi_1^{(4)}), \Theta^T = (\varphi_1^{(9)}\ \varphi_1^{(8)}\ \varphi_1^{(1)}) \text{ and } \varphi_1^{(10)} \tag{9}$$

under these appropriate notations, it is easy to see that only ψ components are independent and obey the following KG-type equation:

$$\mathbf{0}_{KG}\psi = 0. \tag{10}$$

where the scalar differential operator $\mathbf{0}_{KG}$ is the corresponding KG operator given by:

$$\mathbf{0}_{KG} = \frac{d^2}{dz^2} + [(E - V_0(z))^2 - m^2(z)]. \tag{11}$$

From Eq. (10), we can ensure that the three components satisfy the same differential equation. Indeed, if we solve this equation with the component $\varphi_1^{(2)}$, then we can easily deduce the other components, i.e. $\varphi_1^{(3)}$ and $\varphi_1^{(7)}$. By inserting $\varphi_1^{(2)}$ into Eq. (10), we have:

$$\frac{d^2\varphi_1^{(2)}}{dz^2} - \frac{m'(z)}{m(z)}\frac{d\varphi_1^{(2)}}{dz} + [(E - V_0(z))^2 - m^2(z)]\varphi_1^{(2)}(z) = 0, \tag{12}$$

and we can rewrite Eq. (12) in the following form:

$$\frac{1}{m(z)}[(E - V_0(z))^2 - m^2(z)]\varphi_1^{(2)}(z) + \frac{d}{dz}\left(\frac{1}{m}\frac{d\varphi_1^{(2)}(z)}{dz}\right) = 0. \tag{13}$$

Now we are going to determine the matching conditions for $\varphi_1^{(2)}$ at $z=-a$. As $\varphi_1^{(2)}$(z) obeys the KG-type modified equation in Eq. (13), we must impose on it and on its derivative for the continuity conditions. By integrating Eq. (13) around the discontinuity $z=-a$, one gets:

$$\frac{1}{m\left(-a+\varepsilon\right)}{\varphi_1}'^{(2)}\left(-a+\varepsilon\right)-\frac{1}{m\left(-a-\varepsilon\right)}{\varphi_1}'^{(2)}\left(-a-\varepsilon\right)=\int_{-a-\varepsilon}^{-a+\varepsilon}\frac{1}{m(z)}\left[m^2\left(z\right)-\left(E-V_0\left(z\right)\right)^2\right]\varphi_1^{(2)}(z)dz. \quad (14)$$

In the domain $(-a-\varepsilon,-a+\varepsilon)$, the functions m(z) and $V_0(z)$ have finite discontinuities at $z=-a$. Therefore, when ε tends to zero, the integral at the second side of (14) goes to zero. Consequently, one can write:

$$\begin{cases} \varphi_1^{(2)}(-a^-)=\varphi_1^{(2)}(-a^+) \\ \frac{1}{m(-a^-)}\frac{d}{dz}\varphi_1^{(2)}\left(-a^-\right)=\frac{1}{m(-a^+)}\frac{d}{dz}\varphi_1^{(2)}\left(-a^+\right). \end{cases} \quad (15)$$

Following the same procedure, we obtain:

$$\begin{cases} \varphi_1^{(2)}(a^-)=\varphi_1^{(2)}(a^+) \\ \frac{1}{m(a^-)}\frac{d}{dz}\varphi_1^{(2)}\left(a^-\right)=\frac{1}{m(a^+)}\frac{d}{dz}\varphi_1^{(2)}\left(a^+\right). \end{cases} \quad (16)$$

In summary, we need to solve Eq. (12) with the boundary conditions

$$\begin{cases} \frac{d^2\varphi_1^{(2)}(z)}{dz^2}-\frac{m'(z)}{m(z)}\frac{d\varphi_1^{(2)}(z)}{dz}+\left[\left(E-V_0\left(z\right)\right)^2-m^2\left(z\right)\right]\varphi_1^{(2)}(z)=0 \\ \frac{1}{m(-a^+)}\frac{d}{dz}\varphi_1^{(2)}\left(-a^+\right)=\frac{1}{m(-a^-)}\frac{d}{dz}\varphi_1^{(2)}\left(-a^-\right) \\ \frac{1}{m(a^+)}\frac{d}{dz}\varphi_1^{(2)}\left(a^+\right)=\frac{1}{m(a^-)}\frac{d}{dz}\varphi_1^{(2)}\left(a^-\right). \end{cases} \quad (17)$$

KG-type Eq. (12) for the component $\varphi_1^{(2)}(z)$ has the following form in each region:

$$[(E-W_1)^2-m_1^2]\varphi_1^{(2)}(z)+\frac{d^2\varphi_1^{(2)}(z)}{dz^2}=0 \;\text{ for }\; z<-a, \quad (18)$$

$$[E^2-m_2^2]\varphi_1^{(2)}(z)+\frac{d^2\varphi_1^{(2)}(z)}{dz^2}=0 \;\text{ for }\; -a<z<a, \quad (19)$$

$$[(E-W_2)^2-m_1^2]\varphi_1^{(2)}(z)+\frac{d^2\varphi_1^{(2)}(z)}{dz^2}=0 \;\text{ for }\; z>a. \quad (20)$$

Let us choose the physical solutions:

$$\varphi_1^{(2)}(z)=\begin{cases} a_1e^{k_1z} \;\text{ for }\; z<-a, \\ a_2e^{ik_2z}+a_3e^{-ik_2z} \;\text{ for }\; -a<z<a, \\ a_4e^{-k_3z} \;\text{ for }\; z>a, \end{cases} \quad (21)$$

where a_1, a_2, a_3, and a_4 are arbitrary constants and k_1, k_2, and k_3 are:

$$k_1=\sqrt{m_1^2-(E-W_1)^2},\; k_2=\sqrt{E^2-m_2^2},\; k_3=\sqrt{m_1^2-(E-W_2)^2}. \quad (22)$$

The continuity of $\varphi_1^{(2)}(z)$ at z = ± a and the application of Eqs. (15) and (16) lead to the following four equations:

$$\begin{cases} a_1 e^{-k_1 a} = a_2 e^{-ik_2 a} + a_3 e^{ik_2 a} \\ a_4 e^{-k_3 a} = a_2 e^{ik_2 a} + a_3 e^{-ik_2 a} \\ \frac{k_1 a_1}{m_1} e^{-k_1 a} = \frac{ik_2}{m_2}(a_2 e^{-ik_2 a} - a_3 e^{ik_2 a}) \\ \frac{-k_3 a_4}{m_1} e^{-k_3 a} = \frac{ik_2}{m_2}(a_2 e^{-ik_2 a} - a_3 e^{ik_2 a}) \end{cases}. \tag{23}$$

Using the following constraints equations, we deduce the other components:

$$\begin{pmatrix} \phi \\ \Theta \end{pmatrix} = \begin{pmatrix} \frac{E - V_0(z)}{m(z)} \\ \frac{i}{m(z)} \frac{d}{dz} \end{pmatrix} \otimes \psi, \tag{24}$$

that is,

$$\frac{(E - V_0(z))}{m(z)} \varphi_1^{(2)}(z) = \phi_1^{(5)}(z), \tag{25}$$

$$\frac{(E - V_0(z))}{m(z)} \varphi_1^{(3)}(z) = \phi_1^{(6)}(z), \tag{26}$$

$$\frac{(E - V_0(z))}{m(z)} \varphi_1^{(7)}(z) = \phi_1^{(4)}(z), \tag{27}$$

$$\frac{i}{m(z)} \frac{d\varphi_1^{(2)}(z)}{dz} = \varphi_1^{(9)}(z) \tag{28}$$

$$\frac{-i}{m(z)} \frac{d\varphi_1^{(3)}(z)}{dz} = \varphi_1^{(8)}(z), \tag{29}$$

$$\frac{i}{m(z)} \frac{d\varphi_1^{(7)}(z)}{dz} = \varphi_1^{(1)}(z), \tag{30}$$

and $\varphi_1^{(10)}$ automatically vanishes ($\varphi_1^{(10)} = 0$).

The corresponding total wave function in each region is then:

$$\Psi(z, t) = \begin{pmatrix} \frac{i\rho_1 k_1}{m_1} \\ a_1 \\ b_1 \\ \frac{(E - W_1)\rho_1}{m_1} \\ \frac{(E - W_1)a_1}{m_1} \\ \frac{(E - W_1)b_1}{m_1} \\ \rho_1 \\ -\frac{ib_1 k_1}{m_1} \\ \frac{ia_1 k_1}{m_1} \\ 0 \end{pmatrix} e^{(k_1 z - iEt)} \quad \text{for} \quad z < -a, \tag{31}$$

$$\Psi(z,\ t)=\begin{pmatrix} \frac{-k_2}{m_2}(\rho_2 e^{ik_2z}-\rho_3 e^{-ik_2z}) \\ (a_2 e^{ik_2z}+a_3 e^{-ik_2z}) \\ (b_2 e^{ik_2z}+b_3 e^{-ik_2z}) \\ \frac{E}{m_2}(\rho_2 e^{ik_2z}+\rho_3 e^{-ik_2z}) \\ \frac{E}{m_2}(a_2 e^{ik_2z}+a_3 e^{-ik_2z}) \\ \frac{E}{m_2}(b_2 e^{ik_2z}+b_3 e^{-ik_2z}) \\ (\rho_2 e^{ik_2z}+\rho_3 e^{-ik_2z}) \\ \frac{k_2}{m_2}(b_2 e^{ik_2z}-b_3 e^{-ik_2z}) \\ \frac{-k_2}{m_2}(a_2 e^{ik_2z}-a_3 e^{-ik_2z}) \\ 0 \end{pmatrix} e^{-iEt} \quad \text{for} \quad -a<z<a, \tag{32}$$

$$\Psi(z\ ,t)=\begin{pmatrix} -i\rho_4\frac{k_3}{m_1} \\ a_4 \\ b_4 \\ \frac{\rho_4}{m_1}(E-W_2) \\ \frac{a_4}{m_1}(E-W_2) \\ \frac{b_4}{m_1}(E-W_2) \\ \rho_4 \\ ib_4\frac{k_3}{m_1} \\ -ia_4\frac{k_3}{m_1} \\ 0 \end{pmatrix} e^{-(k_3z+iEt)} \quad \text{for} \quad z>a\ , \tag{33}$$

where b_1, b_2, b_3, b_4, and ρ_1, ρ_2, ρ_3, ρ_4, analogous to a_1, a_2, a_3, a_4, are constants to be determined by the boundary and matching conditions.

Let us now pass on to the determination of the bound energy condition. For that, let us impose the continuity of the $\Phi(z)$ wave function at $z = \pm a$, and let us apply the conditions of Eqs. (15) and (16). Thus, after tedious calculations, we find the relation that gives energy eigenvalues for the bound states:

$$e^{4iak_2}=\frac{\left(1+i\frac{k_2m_1}{k_1m_2}\right)\left(1+i\frac{k_2m_1}{k_3m_2}\right)}{\left(1-i\frac{k_2m_1}{k_1m_2}\right)\left(1-i\frac{k_2m_1}{k_3m_2}\right)}. \tag{34}$$

By using the formula $\arctan(z) = \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right)$, we obtain the relativistic transcendental equation

$$2ak_2=n\pi-\arctan\frac{k_2m_1}{k_1m_2}-\arctan\frac{k_2m_1}{k_3m_2}, \tag{35}$$

where inside the well the momentum k_2 of the bound states is deduced from the values n = 1, 2, 3... and the range of the arctan is taken between 0 and $\frac{\pi}{2}$. We notice that the explicit solutions of Eq. (35) showing the dependence of the energy E and the number of the bound states on the width of the well 2**a** can be readily determined numerically when the other parameters are fixed. However, in order to analyze the variation of the energy on the mass function, we must apply an approximate method because of the nonlinear terms in the energy eigenvalues equation, Eq. (35). When the $E \leq (W_1 + m_1)$ approximation is used, Eq. (35) becomes an inequality for the number of bound states:

$$2a\sqrt{(W_1+m_1)^2-m_2^2} > \pi\left(n-\frac{1}{2}\right) - \arctan\left(\frac{m_1}{m_2}\right)\sqrt{\frac{(W_1+m_1)^2-m_2^2}{2m_1\Delta W(\Delta W)^2}}, \tag{36}$$

where $\Delta W = W_2 - W_1 \rangle\, 0$, and this signifies that the total number of bound states N will be the highest n satisfying this inequality when we treat Eq. (36) as an equation for N = 1, 2,. . .

$$2a\sqrt{(W_1+m_1)^2-m_2^2} = \left(N-\frac{1}{2}\right) - \arctan\left(\frac{m_1}{m_2}\right)\sqrt{\frac{(W_1+m_1)^2-m_2^2}{2m_1\Delta W(\Delta W)^2}}, \tag{37}$$

We affirm that the parameters of the well are critical, in the sense that by varying their values slightly we will have one bound state less or one bound state more. Putting N = 1, Eq. (37) becomes:

$$\arctan\left(\frac{m_1}{m_2}\right)\sqrt{\frac{(W_1+m_1)^2-m_2^2}{2m_1\Delta W-(\Delta W)^2}} = \frac{\pi}{2} - 2a\sqrt{(W_1+m_1)^2-m_2^2}, \tag{38}$$

Eq. (38) expresses the appearance condition of the first bound state of the asymmetric square well. When $\Delta W \neq 0$, it is possible to fix some values of the parameters of the well that do not permit bound states, as it is also the case for the conventional constant mass problem and the PDM nonrelativistic case [11]. The conventional constant mass case is obtained by putting $m_1 = m_2 = m$ in Eq. (35):

$$2ak_2 = n\pi - \arctan\frac{k_2}{\sqrt{2m^2-k_2^2-W_1^2+2W_1\sqrt{k_2^2-m^2}}} - \arctan\frac{k_2}{\sqrt{2m^2-k_2^2-W_2^2+2W_2\sqrt{k_2^2-m^2}}}, \tag{39}$$

where n = 1, 2, 3,... The inequality for the number of bound states is obtained now from Eq. (36) if we take $m_1 = m_2 = m$:

$$2a\sqrt{W_1\left(W_1+2m\right)} > \pi\left(n-\frac{1}{2}\right) - \arctan\sqrt{\frac{W_1(W_1+2m)}{2m\Delta W-(\Delta W)^2}}. \tag{40}$$

It is worth noting that in the PDM case the critical values given by Eq. (36) for the number of bound states depend on both m_1 and m_2.

In particular, as $m_2 \to 0$, there will at least one possible energy level.

2.2. Spin 0

In this case, the wave function Φ can be written as :

$$\Phi = \begin{pmatrix} \varphi_0^{(0)} & \varphi_0^{(1)} & \varphi_0^{(2)} & \varphi_0^{(3)} & \varphi_0^{(4)} \end{pmatrix}^T. \tag{41}$$

If we insert Eq. (41) into Eq. (4), we obtain a differential equation for $\varphi_0^{(0)}$, which is:

$$\frac{d^2\varphi_0^{(0)}(z)}{dz^2} - \frac{m'(z)}{m(z)}\frac{d\varphi_0^{(0)}(z)}{dz} + [(E-V_0(z))^2 - m^2(z)]\varphi_0^{(0)}(z) = 0. \tag{42}$$

The matrices β^0, β^3 have been chosen according to [46]. The other components can be expressed as:

$$\varphi_0^{(1)}(z) = \frac{(E - V_0(z))}{m(z)} \varphi_0^{(0)}(z), \varphi_0^{(4)}(z) = \frac{i}{m(z)} \frac{d\varphi_0^{(0)}}{dz}(z), \varphi_0^{(2)}(z) = \varphi_0^{(3)}(z) = 0. \tag{43}$$

Following the same steps as in the preceding case, we arrive at the result:

$$\Psi(z,t) = \begin{pmatrix} 1 \\ \frac{E-W_1}{m_1} \\ 0 \\ 0 \\ \frac{ik_1}{m_1} \end{pmatrix} Ae^{(k_1 z - iEt)} \quad \text{for} \quad z < -a, \tag{44}$$

$$\Psi(z,t) = \begin{pmatrix} Fe^{ik_2 z} + Ge^{-ik_2 z} \\ \frac{E}{m_2}(Fe^{ik_2 z} + Ge^{-ik_2 z}) \\ 0 \\ 0 \\ \frac{-k_2}{m_2}(Fe^{ik_2 z} - Ge^{-ik_2 z}) \end{pmatrix} e^{-iEt)} \quad \text{for} \quad -a < z < a, \tag{45}$$

$$\Psi(z,t) = \begin{pmatrix} 1 \\ \frac{E-W_2}{m_1} \\ 0 \\ 0 \\ -\frac{ik_3}{m_1} \end{pmatrix} De^{(k_3 z - iEt)} \quad \text{for} \quad z > a, \tag{46}$$

where A, F, G, and D are constants to be determined by the boundary and matching conditions. Using the same method as for the spin 1 case, we find another form of the transcendental equation, which is:

$$2ak_2 = n\pi + \arctan\frac{k_1 m_2}{k_2 m_1} + \arctan\frac{k_3 m_2}{k_2 m_1}, \tag{47}$$

where n = 1, 2, 3,... and the range of the inverse tan is always taken between 0 and $\frac{\pi}{2}$. To treat the variation of the energy eigenvalues against m_2, for $m_1 = 1$, $m_1 = 10$, we follow the same stages as in the preceding section: taking $E \leq (W1 + m_1)$ into account in Eq. (47), the number of bound states N is given by the highest n verifying the following inequality:

$$2a\sqrt{(W_1 + m_1)^2 - m_2^2} > n\pi + \arctan\left(\frac{m_2}{m_1}\right)\sqrt{\frac{2m_1\Delta W - (\Delta W)^2}{(W_1 + m_1)^2 - m_2^2}}, \tag{48}$$

and as in the spin 1 case, Eq. (48) means that if $W1 + m_1 > m_2$, there are always bound states for the asymmetric square well. Here also, contrary to what was found for the symmetric well [11], a dependence on m_1 and m_2 is observed for the number of bound states.

The critical values of the well that indicate when we have one bound state more are obtained from:

$$2a\sqrt{(W_1+m_1)^2-m_2^2} = N\pi + \arctan\left(\frac{m_2}{m_1}\right)\sqrt{\frac{2m_1\Delta W-(\Delta W)^2}{(W_1+m_1)^2-m_2^2}}, \tag{49}$$

Moreover, we observe that for $(W_1 + m_1) \leq$ m2 there is no bound state in both cases of spin 1 and 0. For the constant mass case, we have:

$$2ak_2 = n\pi + arctan\frac{\sqrt{2m^2-k_2^2-W_1^2+2W_1\sqrt{k_2^2-m^2}}}{k_2} + arctan\frac{\sqrt{2m^2-k_2^2-W_2^2+2W_2\sqrt{k_2^2-m^2}}}{k_2}. \tag{50}$$

3. Dependence of the relativistic asymmetric well spectra on some parameters

This section is devoted to the discussion of our numerical results and plots (Tables 1–4; Figures 1–8). As mentioned above, we treat the problem of spin 1 and spin 0 bosons subject to a square well potential and investigate the effect of the PDM on the energy spectra. We note that the number of bound states depends linearly on width 2**a** of the well (see Eqs. (36) and (48)). Figures 1 and 2 show that, analogous to what was observed for the PDM Schrödinger particle by a square potential [11], as the value of **a** increases, the energy value decreases. On the other hand, the mass inside the well (m_2) is an increasing function of parameter **a** (see Figures 3–6). The effect of mass outside the well (m_1) on the number of bound states is slight. Regarding the variation of the energy on the m_2 mass, we proceed approximately like in the classical PDM system. Thus, the energy eigenvalues increase with increasing m_2 as shown in Figures 7 and 8. This behavior is different from that observed in the above reference. We also investigate the effect of the value of m_2 on the first critical values of the **a** parameter. The results are reported in Tables 1 and 3. It is readily seen that these first critical values of the **a** parameter decrease with decreasing m_2.

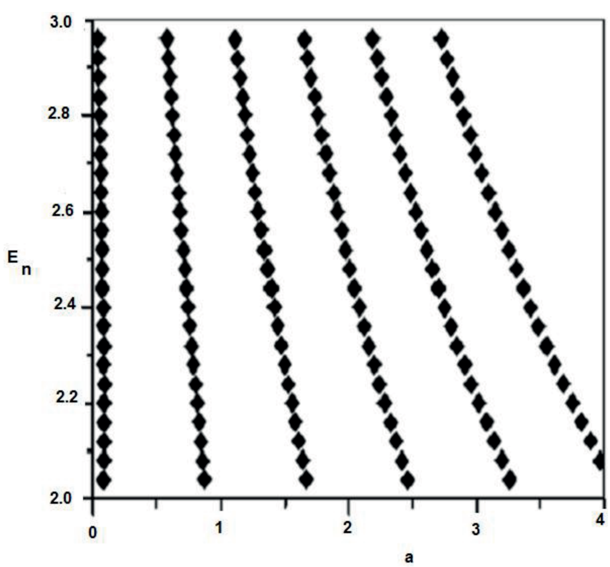

Figure 1. The curves of the energy spectrum against **a** for some excited states in the case of spin 1 ($m_1 = 1$, $m_2 = 0.5$, $W_1 = 2$, $W_2 = 3$).

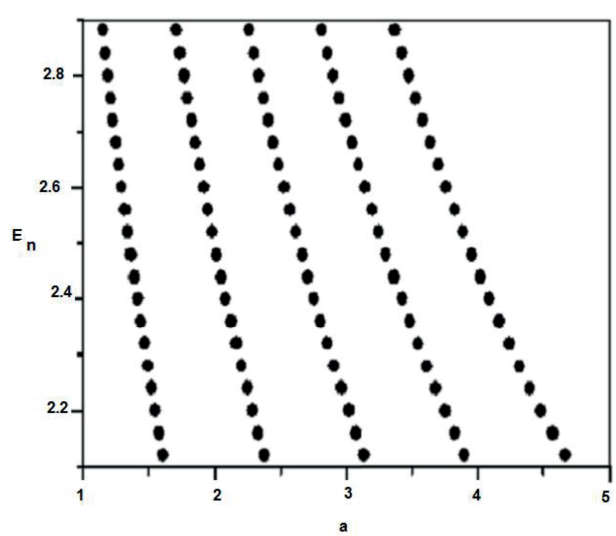

Figure 2. A reproduction of Figure 1 for spin 0.

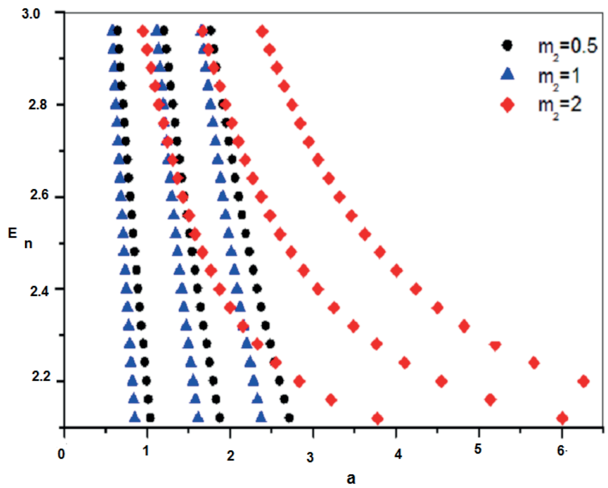

Figure 3. The energy degeneracy with respect to n = 1, 2, 3 against **a** in the case of spin 0 for different values of m_2, $W_1 = 2$, $W_2 = 3$.

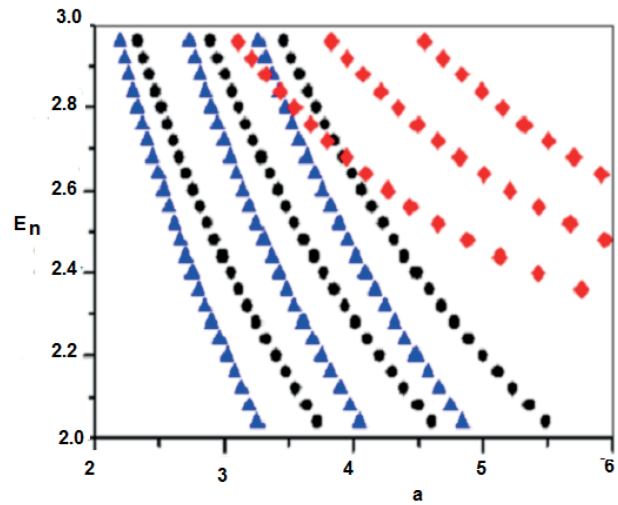

Figure 4. Similar to Figure 3, but with n = 4, 5, 6.

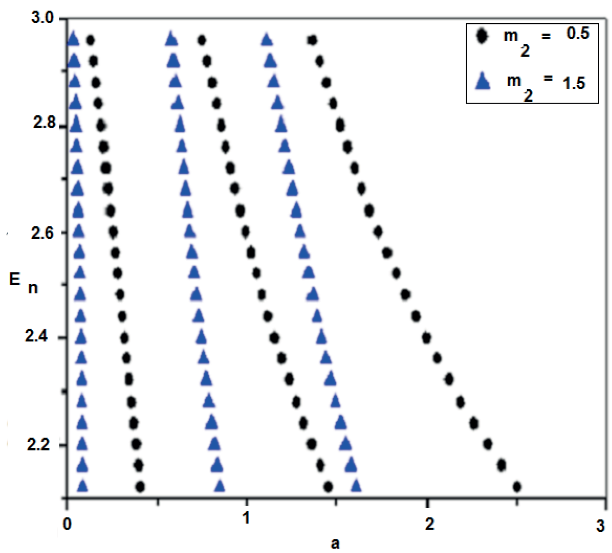

Figure 5. Absence of the energy degeneracy with respect to n = 1, 2, 3 against **a** in the case of spin 1 and for $m_1 = 1$, $W_1 = 2$, $W_2 = 3$.

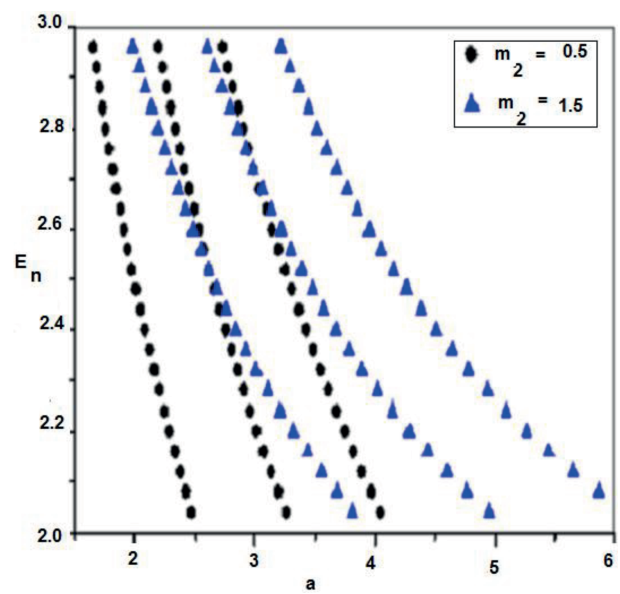

Figure 6. The energy degeneracy with respect to n = 4, 5, 6 against **a** in the case of spin 1 and for ($m_1 = 1$, $W_1 = 2$, $W_2 = 3$).

Table 1. The calculated critical values of **a** from Eq. (37), fixing the width where a new bound state arises in the case of spin 1.

m_1	m_2	$a^{(1)}$	$a^{(2)}$	$a^{(3)}$	$a^{(4)}$	$a^{(5)}$	$a^{(6)}$
1	2	0.1629	0.8653	1.5677	2.2701	2.9725	3.6749
1	1	0.0599	0.6152	1.1705	1.7258	2.2811	2.8364
1	$\frac{1}{2}$	0.0281	0.5591	1.0901	1.6211	2.1521	2.6831

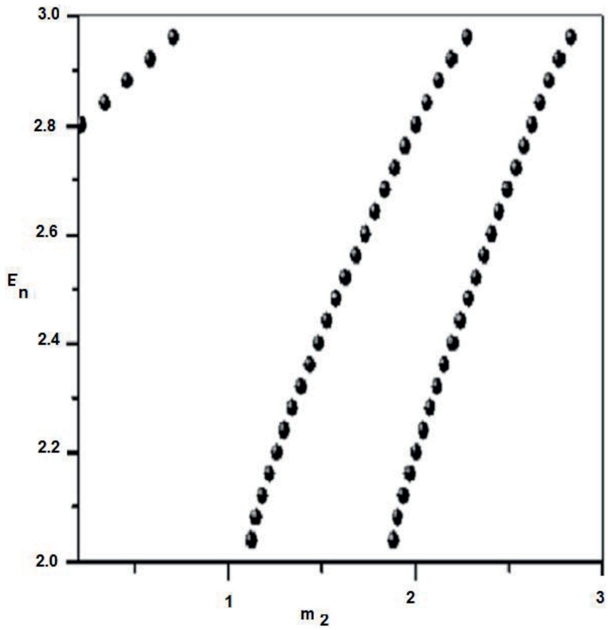

Figure 7. Energy eigenvalues versus $\mathbf{m_2}$ in the case of $m_1 = 1$, $W_1 = 2$, $W_2 = 3$ and for spin 1.

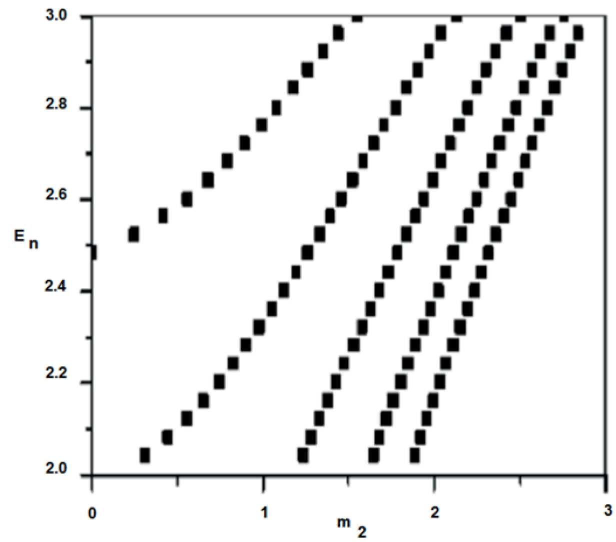

Figure 8. A reproduction of Figure 7 for spin 0.

Table 2. The calculated critical values of $\mathbf{m_2}$ from Eq. (37), fixing the mass where a new bound state arises in the case of spin 1.

m_1	$m_2^{(1)}$	$m_2^{(2)}$	$m_2^{(3)}$
1	2.9400	2.4168	0.8799
10	11.8937	10.9510	8.5540

Table 3. The calculated critical values of **a** from Eq. (49), fixing the width where a new bound state arises in the case of spin 0.

m_1	m_2	$a^{(1)}$	$a^{(2)}$	$a^{(3)}$	$a^{(4)}$	$a^{(5)}$	$a^{(6)}$
1	2	0.7965	1.4982	2.2014	2.9038	3.6063	4.3088
1	1	0.6154	1.1707	1.7261	2.2814	2.8368	3.3921
1	$\frac{1}{2}$	0.5861	1.1171	1.6481	2.1791	2.7101	3.2412

Table 4. The calculated critical values of $\mathbf{m_2}$ from Eq. (49), fixing the mass where a new bound state arises in the case of spin 0.

m_1	$m_2^{(1)}$	$m_2^{(2)}$	$m_2^{(3)}$	$m_2^{(4)}$	$m_2^{(5)}$
1	2.9156	2.7619	2.5098	2.1350	1.5568
10	11.7555	11.2999	10.5611	9.4659	7.8699

In addition, we note that the DKP square potential energy levels for some adjacent numbers intersect at some values of **a**, thus resulting in degeneracy (see Figures 3, 4, and 6). However, an absence of energy degeneracy is illustrated in Figure 5. It is consistent with the results obtained in the literature. As for Table 2 (and respectively Table 4), it proves the effect of m_1 on the first critical energy values of m_2. For the case of spin 0, we also note a degeneracy of energy eigenvalues with respect to the numbers of the bound states. These results obtained are in good agreement with previous works [27].

4. Conclusion

In this work, the (1+1)-dimensional effective mass DKP equation (spin 0 and spin 1) with potential well having an asymmetric form has been considered.

From the generalized boundary conditions, we have shown that the bound states in both cases of spin 0 and spin 1 are solutions (numerically) of transcendental equations and this was afterwards shown to have reduced the DKP equation to the KG equation. A difference of sign in the transcendental equations related to spin 0 and spin 1 should be noted. For some critical values of the well, i.e. the values of the characteristic potential width, and the mass inside the well, we have calculated the bound states. Thus, we have noted that there is a difference in the mass configuration when it is not constant.

Finally, our results could be a starting point for the analysis of the scattering of boson particles by a square potential well in the DKP equation when the mass is dependent on the position. This problem will be discussed elsewhere.

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