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# Bound-free pair production with a correction term in relativistic heavy-ion collisions

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**Abstract:** In our previous work, we calculated bound-free electron–positron pair production cross section without a correction term. In this work, bound-free electron–positron pair production with a correction term is considered to calculate the cross section for peripheral relativistic heavy ion collisions. The Dirac wave functions have been used for the leptons and first order corrections are included. It is seen that the results for the production cross sections are considerably smaller than those of the previous calculations. We applied the same method for the calculations of the antihydrogen production cross sections as well.

**Key words:** Bound-free pair production, antihydrogen production, correction term, QED, Monte Carlo method

## 1. Introduction

Bound-free pair production (BFPP) is one of the processes that restrict the luminosity of ion beams. In this process, the charge of the ion decreases and it is depleted out of the beam. For this reason, calculation of the exact bound-free electron–positron pair production cross section is important for deciding the stability of the beam [1–10].

Another important application of this calculation is for producing antihydrogen. Antihydrogen is the simplest bound state of antimatter and may be produced with the collision of antiprotons with ions. They were first produced and observed at the CERN Low Energy Antiproton Ring (LEAR) in 1995. In this process $(\bar{p} + Z \rightarrow \bar{H} + e^- + Z)$, xenon $(Z = 54)$ was used. This process was first proposed by Munger et al. and they studied the calculation of antihydrogen production cross section by equivalent photon approximation (EPA). The cross section calculation of antihydrogen production is important, because as with BFPP, the antihydrogen production mechanism also leads to beam loss [11–18].

In our previous works [19,20], we calculated the BFPP and relativistic antihydrogen production with Monte Carlo integration techniques by computing the Feynman diagrams. The wave functions for free positron, captured electron, free electron, and captured positron were given there. In this work, we calculated the bound-free electron–positron pair production cross section with correction terms. For this, we added the correction term $\Psi^{'}$ to the electron wave function known as the Sommerfeld-Maue wave function. For the bound positron, we used the Darwin wave function as in our previous work [19]. We applied the same procedure for the calculation of antihydrogen production cross section with correction terms. For the antihydrogen production mechanism

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we added the correction term $\bar{\Psi}^{'}$ to the positron wave function and for the bound electron we used the Darwin wave function as in our previous work [20].

In section 2, we derived the cross section of BFPP with the correction term. In section 3, we applied the same technique for the antihydrogen production. Finally we presented our calculations and compared them with the previous works. The obtained BFPP cross section results with the correction term are smaller than the cross section results without it.

## 2. Theoretical background

### 2.1. Electron capture with correction

To calculate the cross section for bound-free electron–positron pairs with a correction term in relativistic heavy-ion collisions, we applied the lowest-order perturbation theory in the framework of quantum electrodynamics (QED). We described the BFPP process with the correction term by the two Feynman diagrams, *direct* and *crossed* terms in lowest QED order.

For the BFPP process, free positron is described by the plane-waves

$$\Psi_q^{(+)} = N_+ \left[ e^{i\mathbf{q}\cdot\mathbf{r}} \mathbf{u}_{\sigma_q}^{(+)} + \Psi^{'} \right], \tag{1}$$

and together with the correction term $\Psi^{'}$ in order to account for the distortion due to the charge of one of the nuclei. In this work, we calculated the effect of this term on the cross section result.

While calculating the total cross section of BFPP with correction terms, we added only the Fourier transform of $\Psi^{'}$ to the calculations. The Fourier transform of $\Psi^{'}$ can be obtained from the Dirac equation of the positron in the Coulomb field of nucleus and the detailed calculations are done in [1]. The result is

$$\Psi^{'} = 4\pi Z e^2 \mathbf{u}_{\sigma_q}^{(+)} \frac{2E_q^{(+)} - \vec{\alpha}\cdot(\vec{p}-\vec{q})}{(\vec{p}-\vec{q})^2 (\vec{p}^2 - \vec{q}^2)}. \tag{2}$$

In expression (1), moreover,

$$N_+ = e^{-\frac{\pi a_+}{2}} \Gamma(1 + i a_+) \quad a_+ = \frac{Ze^2}{v_+}, \tag{3}$$

is a normalization constant that accounts for the distortion of the wave function acceptable for $Z\alpha \ll 1$ [1,3,21]. $\mathbf{u}_{\sigma_q}^{(+)}$ represents the spinor structure for the outgoing positron [19].

After the free pair production, the captured electron is described as a bound state with the Darwin wave function. In a semirelativistic approximation, these electron states are often represented by [22, 23]

$$\Psi^{(-)} = \left(1 - \frac{i}{2m}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\right) \mathbf{u}\Psi_{non-rel}(r), \tag{4}$$

i.e. in terms of the nonrelativistic (ground) state function

$$\Psi_{non-rel}(r) = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_H}\right)^{3/2} e^{-Zr/a_H}, \tag{5}$$

of the hydrogen-like ion, and where $\mathbf{u}$ represents the spinor part of the captured electron and $a_H = 1/e^2$ the Bohr radius of atomic hydrogen.

Using the positron wave function with the correction term and the captured electron wave function from above, the direct term for the BFPP can be written as

$$\begin{aligned}
\left\langle \Psi^{(-)} \left| S_{ab} \right| \Psi_q^{(+)} \right\rangle &= i\sum_p \sum_s \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\left\langle \Psi^{(-)} \left| V_a\left(\omega - E^{(-)}\right) \right| \chi_p^{(s)} \right\rangle \left\langle \chi_p^{(s)} \left| V_b\left(E_q^{(+)} - \omega\right) \right| \Psi_q^{(+)} \right\rangle}{\left(E_p^{(s)} - \omega\right)} \\
&= i\sum_p \sum_s \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} d^3\mathbf{r} \left(1 + \frac{i}{2m}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\right) \Psi_{non-rel}(r) e^{i\mathbf{p}\cdot\mathbf{r}} A_a(\mathbf{r}; \omega - E^{(-)}) \\
&\times \int_{-\infty}^{\infty} d^3\mathbf{r}' N_+ e^{-i\mathbf{p}\cdot\mathbf{r}'} \left(e^{i\mathbf{q}\cdot\mathbf{r}'} + \Psi'\right) A_b\left(\mathbf{r}'; E_q^{(+)} - \omega\right) \frac{\left\langle \mathbf{u} \left| (1-\beta\alpha_z) \right| \mathbf{u}_{\sigma_p}^{(s)} \right\rangle \left\langle \mathbf{u}_{\sigma_p}^{(s)} \left| (1+\beta\alpha_z) \right| \mathbf{u}_{\sigma_q}^{(+)} \right\rangle}{\left(E_p^{(s)} - \omega\right)} \\
&= i\sum_p \sum_s \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} d^3\mathbf{r} \left(1 + \frac{i}{2m}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\right) \Psi_{non-rel}(r) e^{i\mathbf{p}\cdot\mathbf{r}} A_a(\mathbf{r}; \omega - E^{(-)}) \\
&\times \int_{-\infty}^{\infty} d^3\mathbf{r}' N_+ e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}'} A_b\left(\mathbf{r}'; E_q^{(+)} - \omega\right) P(\mathbf{u}) \\
&+ i\sum_p \sum_s \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} d^3\mathbf{r} \Psi_{non-rel}(r) e^{i\mathbf{p}\cdot\mathbf{r}} A_a(\mathbf{r}; \omega - E^{(-)}) \\
&\times \int_{-\infty}^{\infty} d^3\mathbf{r}' N_+ e^{-i\mathbf{p}\cdot\mathbf{r}'} \Psi' A_b(\mathbf{r}'; E_q^{(+)} - \omega) P(\mathbf{u}) \qquad (6)
\end{aligned}$$

Detailed information about the vector potential terms for ( $A_a^\mu$ and $A_b^\mu$) can be found in our previous BFPP calculations without a correction term [19].

Here the spinor part of above equation can be expressed as

$$P(\mathbf{u}) = \frac{\left\langle \mathbf{u} \left| (1-\beta\alpha_z) \right| \mathbf{u}_{\sigma_p}^{(s)} \right\rangle \left\langle \mathbf{u}_{\sigma_p}^{(s)} \left| (1+\beta\alpha_z) \right| \mathbf{u}_{\sigma_q}^{(+)} \right\rangle}{(E_p^{(s)} - \omega)}. \qquad (7)$$

For the calculation of the overall integral, we need the results of the first part over $\mathbf{r}$ integral. In this work, we just give the result of the integral over $\mathbf{r}$ without details,

$$\begin{aligned}
&\int_{-\infty}^{\infty} d^3\mathbf{r} \left(1 + \frac{i}{2m}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\right) \Psi_{non-rel}(r) e^{i\mathbf{p}\cdot\mathbf{r}} A_a\left(\mathbf{r}; \omega - E^{(-)}\right) \\
&= -\left[1 + \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{2m}\right] 8\pi^2 Ze \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_H}\right)^{3/2} \frac{\delta(\omega - E^{(-)} - \beta p_z)}{\left(\frac{Z^2}{a_H^2} + \frac{p_z^2}{\gamma^2} + \mathbf{p}_\perp^2\right)} e^{\left[i\mathbf{p}_\perp\cdot\frac{\mathbf{b}}{2}\right]}, \qquad (8)
\end{aligned}$$

where $E^{(-)}$ is the energy of the captured electron. Again, we just give the result of the first part of integral over $\mathbf{r}'$:

$$\int_{-\infty}^{\infty} d^3\mathbf{r}' N_+ e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}'} A_b\left(\mathbf{r}'; E_q^{(+)} - \omega\right) = -N_+ 8\pi^2 Ze\gamma^2 \frac{\delta(E_q^{(+)} - \omega - \beta(p_z - q_z))}{(p_z - q_z)^2 + \gamma^2(\mathbf{p}_\perp - \mathbf{q}_\perp)^2} e^{i(\mathbf{p}_\perp - \mathbf{q}_\perp)\cdot\frac{\mathbf{b}}{2}}, \qquad (9)$$

where $E_q^{(+)}$ is the energy of the positron.

Up to now, we included the correction term in the BFPP expressions and when we compare these results with the previous calculations [19], it is clear that there are differences between them. In Eq. (6) for the third term of the integral, we used the explicit form of $\Psi_{non-rel}(r)$ given in Eq. (5) and the vector potential $A_a(\mathbf{r};\omega^{'}-E^{(-)})$. The result of this integration can be found as

$$\int_{-\infty}^{\infty} d^3\mathbf{r}\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_H}\right)^{3/2} e^{-i\mathbf{p}\cdot\mathbf{r}}A_a\left(\mathbf{r};\omega^{'}-E^{(-)}\right)=-8\pi^2 Ze\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_H}\right)^{\frac{3}{2}}\frac{\delta\left(\omega-E^{(-)}-\beta p_z\right)}{\left(\frac{p_z^2}{\gamma^2}+\mathbf{p}_\perp^2\right)}e^{i\mathbf{p}_\perp\cdot\frac{\mathbf{b}}{2}} \tag{10}$$

Since the corrections are to first order in $Z\alpha$, we replaced $\Psi_{non-rel}(r)$ with its constant value $\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_H}\right)^{3/2}$ in the third term of Eq. (6) [1]. By using the analogue steps, for the calculation of the overall integral of the fourth part over $\mathbf{r}^{'}$ in Eq. (6)

$$\int_{-\infty}^{\infty} d^3\mathbf{r}^{'} N_+ e^{-i\mathbf{p}\cdot\mathbf{r}^{'}}\Psi^{'}A_b\left(\mathbf{r}^{'};E_q^{(+)}-\omega^{'}\right)=4\pi Ze^2\mathbf{u}_{\sigma_q}^{(+)}\frac{2E_q^{(+)}-\vec{\alpha}\cdot(\vec{p}-\vec{q})}{(\vec{p}-\vec{q})^2(\vec{p}^2-\vec{q}^2)} \tag{11}$$

Finally, by combining all parts of integral components given in Eqs. (8)–(11), we can obtain the explicit expression for the direct BFPP amplitude with the correction term:

$$\begin{aligned}\left\langle\Psi^{(-)}\left|S_{ab}\right|\Psi_q^{(+)}\right\rangle=&iN_+\sum_s\sum_{\sigma_p}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\int\frac{d\omega}{2\pi}e^{i(\mathbf{p}_\perp-\frac{\mathbf{q}_\perp}{2})\cdot\mathbf{b}}8\pi^2 Ze\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_H}\right)^{3/2}\frac{\delta(\omega-E^{(-)}-\beta p_z)}{\left(\frac{Z^2}{a_H^2}+\frac{p_z^2}{\gamma^2}+\mathbf{p}_\perp^2\right)}\left[1+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{2m}\right]\\&\times 8\pi^2 Ze\gamma^2\frac{\delta\left(E_q^{(+)}-\omega+\beta\left(p_z-q_z\right)\right)}{\left(p_z-q_z\right)^2+\gamma^2\left(\mathbf{p}_\perp-\mathbf{q}_\perp\right)^2}P(\mathbf{u})\\&-iN_+\sum_s\sum_{\sigma_p}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\int\frac{d\omega}{2\pi}e^{i\mathbf{p}_\perp\cdot\frac{\mathbf{b}}{2}}8\pi^2 Ze\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_H}\right)^{3/2}\frac{\delta(\omega-E^{(-)}-\beta p_z)}{\left(\frac{p_z^2}{\gamma^2}+\mathbf{p}_\perp^2\right)}\\&\times 4\pi Ze^2\frac{2E_q^{(+)}-\vec{\alpha}\cdot(\vec{p}-\vec{q})}{(\vec{p}-\vec{q})^2+(\vec{p}^2-\vec{q}^2)}P(\mathbf{u})\end{aligned} \tag{12}$$

After integrating Eq. (12) over $\omega$, $p_z$, the transition matrix element for a fixed spin and momentum state of the positron as well as for a given intermediate state can be expressed as

$$\begin{aligned}\left\langle\Psi^{(-)}\left|S_{ab}\right|\Psi_q^{(+)}\right\rangle=&\frac{iN_+}{2\beta}\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_H}\right)^{\frac{3}{2}}\int\frac{d^2\mathbf{p}_\perp}{(2\pi)^2}e^{i\left(\mathbf{p}_\perp-\frac{\mathbf{q}_\perp}{2}\right)\cdot\mathbf{b}}F(-\mathbf{p}_\perp:\omega_a)F(\mathbf{p}_\perp-\mathbf{q}_\perp:\omega_b)T_q(\mathbf{p}_\perp:+\beta)\\&-iN_+\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_H}\right)^{\frac{3}{2}}\int\frac{d^3\mathbf{p}}{(2\pi)^4}e^{i\mathbf{p}_\perp\cdot\frac{\mathbf{b}}{2}}\mathbf{F}\left(\mathbf{p},\mathbf{q}:\omega_a\right)T_q^{'}\left(\mathbf{p}_\perp:+\beta\right)\end{aligned} \tag{13}$$

where $\mathbf{b}$ is again the impact parameter of the ion–ion collision, and the function $F(q,\ \omega)$ can be described as the scalar parts of the field associated with the ions $a$ and $b$ in momentum space.

The explicit form of the scalar field and frequencies of the virtual photons can be investigated in [19]. The scalar field that represents the correction term also can be written as

$$\mathbf{F}\left(\mathbf{p},\mathbf{q}:\omega_a\right)=\frac{8\pi^2 Ze\gamma^2\beta^2}{\left(\omega_a^2+\gamma^2\beta^2\mathbf{p}_\perp^2\right)}\left[4\pi Ze^2\frac{2E_q^{(+)}-\vec{\alpha}\cdot\left(\vec{p}-\vec{q}\right)}{\left(\vec{p}-\vec{q}\right)^2\left(\vec{p}^2-\vec{q}^2\right)}\right]. \tag{14}$$

Apart from the scalar field of each ion, Eq. (13) also contains the transition amplitude $T$, which relates the intermediate photon lines to the outgoing electron–positron lines. This amplitude depends explicitly on the (relative) velocity of the ions $\beta$, the transverse momentum $\mathbf{p}_\perp$, and the momentum of the positron $q$, and it is given by

$$T_q^{'}\left(\mathbf{p}_\perp:+\beta\right)=\sum_s\sum_{\sigma_p}\frac{\left\langle \mathbf{u}\left|(1-\beta\alpha_z)\right|\mathbf{u}_{\sigma_p}^{(s)}\right\rangle\left\langle \mathbf{u}_{\sigma_p}^{(s)}\left|(1+\beta\alpha_z)\right|\mathbf{u}_{\sigma_q}^{(+)}\right\rangle}{\left(E_p^{(s)}-\left(\frac{E^{(-)}+E_q^{(+)}}{2}\right)-\beta\frac{q_z}{2}\right)} \tag{15}$$

and $T_q(\mathbf{p}_\perp:+\beta)$ can be seen in [19].

Finally, we must note that the integration over the impact parameter $b$ Eq. (13) can be carried out also analytically. Following very similar lines, it is possible also to evaluate the *crossed*-term amplitude that is described $\left\langle\Psi^{(-)}\left|S_{ba}\right|\Psi_q^{(+)}\right\rangle$.

After having the amplitudes for the *direct* and *crossed* diagram, we are now prepared to write down the cross section for the generation of a free-bound electron–positron pair in collisions of two heavy ions with correction term

$$\sigma=\int d^2b\sum_{q<0}\left|\left\langle\Psi^{(-)}\left|S\right|\Psi_q^{(+)}\right\rangle\right|^2, \tag{16}$$

where $S=S_{ab}+S_{ba}$ denotes the sum of the *direct* and *crossed* terms.

As a result, after making use of all the simplifications from above, these cross sections for the BFPP with the correction term can be expressed as

$$\begin{aligned}
\sigma&=\int d^2b\sum_{q<0}\left|\left\langle\Psi^{(-)}\left|S_{ab}\right|\Psi_q^{(+)}\right\rangle+\left\langle\Psi^{(-)}\left|S_{ba}\right|\Psi_q^{(+)}\right\rangle\right|^2\\
&=\left|N_+\right|^2\frac{1}{\pi}\left(\frac{Z}{a_H}\right)^3\sum_{q<0}\int d^2b\left(\left(\int\frac{d^2\mathbf{p}_\perp}{(2\pi)^2}e^{i\left(\mathbf{p}_\perp-\frac{\mathbf{q}_\perp}{2}\right)\cdot\mathbf{b}}F\left(-\mathbf{p}_\perp:\omega_a\right)F\left(\mathbf{p}_\perp-\mathbf{q}_\perp:\omega_b\right)T_q\left(\mathbf{p}_\perp:+\beta\right)\right.\right.\\
&\quad\left.-\int\frac{d^3\mathbf{p}}{(2\pi)^4}e^{i\mathbf{p}_\perp\cdot\frac{\mathbf{b}}{2}}\mathbf{F}(\mathbf{p},\mathbf{q}:\omega_a)T_q^{'}(\mathbf{p}_\perp:+\beta)\right)\\
&\quad+\left(\int\frac{d^2\mathbf{p}_\perp}{(2\pi)^2}e^{-i\left(\mathbf{p}_\perp-\frac{\mathbf{q}_\perp}{2}\right)\cdot\mathbf{b}}F\left(-\mathbf{p}_\perp:\omega_b\right)F\left(\mathbf{p}_\perp-\mathbf{q}_\perp:\omega_a\right)T_q\left(\mathbf{p}_\perp:-\beta\right)\right.\\
&\quad\left.\left.-\int\frac{d^3\mathbf{p}}{(2\pi)^4}e^{-i\mathbf{p}_\perp\cdot\frac{\mathbf{b}}{2}}\mathbf{F}(\mathbf{p},\mathbf{q}:\omega_a)T_q^{'}(\mathbf{p}_\perp:-\beta)\right)\right)^2.
\end{aligned} \tag{17}$$

## 3. Antihydrogen production with correction

In our previous work [20], we calculated the cross section of the $\bar{H}$ production by pair production with positron capture in relativistic collisions of ions. In these antihydrogen cross section calculations, we did not add the effect of the distortion term due to the wave function of the free electron. Conclusions we reached from our previous work showed that correction terms play important roles for large charges. Because of this reason, we added the correction term to the Sommerfeld-Maue (plane-wave) wave function for the electron,

$$\Psi_k^{(-)} = N_+ \left[ e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{u}_{\sigma_k}^{(-)} + \bar{\Psi}' \right] \tag{18}$$

In expression (18), $N_+$ and $a_+$ are described in (3).

The Fourier transform of $\bar{\Psi}'$ can be obtained from the Dirac equation of the electron in the Coulomb field of an antinucleus. The detailed calculations can be found in [17]. The correction term for the free electron state is given as

$$\bar{\Psi}' = \Psi'^{*}\gamma^0 = 4\pi Z e^2 \bar{\mathbf{u}}_{\sigma_k}^{(-)} \frac{2E_k^{(-)}\gamma^0 + \vec{\gamma}\cdot(\vec{p}-\vec{k})}{\left(\vec{p}-\vec{k}\right)^2 (\vec{p}^2 - \vec{k}^2)} \gamma^0 \tag{19}$$

and after the pair production, the positron is captured by antiprotons and it is described as a bound state. In a semirelativistic approximation, these positron states are represented by the Darwin wave function

$$\Psi^{(+)} = \left( 1 + \frac{i}{2m} \boldsymbol{\alpha}\cdot\boldsymbol{\nabla} \right) \mathbf{u}\Psi_{non-rel}(r), \tag{20}$$

nonrelativistic (ground) state function is described in (5).

Using the Sommerfeld-Maue wave function for the free electron and the Darwin wave function for the captured positron, the direct Feynman diagram can be written as

$$\begin{aligned}
\left\langle \Psi^{(+)} \left| S_{ab} \right| \Psi_k^{(-)} \right\rangle =& iN_+ \sum_s \sum_{\sigma_p} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{i(\mathbf{p}_\perp - \frac{\mathbf{k}_\perp}{2})\cdot\mathbf{b}} 8\pi^2 1e \frac{1}{\sqrt{\pi}} \left(\frac{1}{a_H}\right)^{3/2} \frac{\delta(\omega - E^{(+)} - \beta p_z)}{\left(\frac{1^2}{a_H^2} + \frac{p_z^2}{\gamma^2} + \mathbf{p}_\perp^2\right)} \left[1 - \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{2m}\right] \\
& \times 8\pi^2 Z e \gamma^2 \frac{\delta\left(E_k^{(-)} - \omega - \beta\left(p_z - k_z\right)\right)}{\left(p_z - k_z\right)^2 + \gamma^2\left(\mathbf{p}_\perp - \mathbf{k}_\perp\right)^2} P\left(\mathbf{u}\right) \\
& - iN_+ \sum_s \sum_{\sigma_p} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{i\mathbf{p}_\perp\cdot\frac{\mathbf{b}}{2}} 8\pi^2 1e \frac{1}{\sqrt{\pi}} \left(\frac{1}{a_H}\right)^{3/2} \frac{\delta(\omega - E^{(+)} - \beta p_z)}{\left(\frac{p_z^2}{\gamma^2} + \mathbf{p}_\perp^2\right)} \\
& \times 4\pi Z e^2 \frac{2E_k^{(-)}\gamma^0 + \vec{\gamma}\cdot\left(\vec{p}-\vec{k}\right)}{\left(\vec{p}-\vec{k}\right)^2\left(\vec{p}^2 - \vec{k}^2\right)} \gamma^0 P(\mathbf{u}).
\end{aligned} \tag{21}$$

After integrating the above equation over $\omega$ and $p_z$, the transition matrix element for a fixed spin and momentum state of the positron as well as for a given intermediate state can be expressed as;

$$\begin{aligned}
\left\langle \Psi^{(+)} \left| S_{ab} \right| \Psi_k^{(-)} \right\rangle =& \frac{iN_+}{2\beta} \frac{1}{\sqrt{\pi}} \left(\frac{1}{a_H}\right)^{\frac{3}{2}} \int \frac{d^2\mathbf{p}_\perp}{(2\pi)^2} e^{i\left(\mathbf{p}_\perp - \frac{\mathbf{k}_\perp}{2}\right)\cdot\mathbf{b}} F\left(-\mathbf{p}_\perp : \omega_a\right) F\left(\mathbf{p}_\perp - \mathbf{k}_\perp : \omega_b\right) T_k\left(\mathbf{p}_\perp : +\beta\right) \\
& - iN_+ \frac{1}{\sqrt{\pi}} \left(\frac{1}{a_H}\right)^{\frac{3}{2}} \int \frac{d^3\mathbf{p}}{(2\pi)^4} e^{i\mathbf{p}_\perp\cdot\frac{\mathbf{b}}{2}} E(\mathbf{p},\mathbf{k} : \omega_a) T_k'(\mathbf{p}_\perp : +\beta),
\end{aligned} \tag{22}$$

where $\mathbf{b}$ is the impact parameter of the ion–ion collision, the function $F(-\mathbf{p}_\perp : \omega_a)$, $F(\mathbf{p}_\perp - \mathbf{k}_\perp : \omega_b)$, and $E(\mathbf{p}, \mathbf{k} : \omega_a)$ are the scalar part of the field associated with the ions $a$ and $b$ in momentum space:

$$F\left(-\mathbf{p}_\perp : \omega_a\right) = \frac{4\pi 1e}{\left(\frac{1^2}{a_H^2} + \frac{\omega_a^2}{\gamma^2\beta^2} + \mathbf{p}_\perp^2\right)}, \tag{23}$$

$$F\left(\mathbf{p}_\perp - \mathbf{k}_\perp : \omega_b\right) = \frac{4\pi Ze\gamma^2\beta^2}{\left(\omega_b^2 + \gamma^2\beta^2(\mathbf{p}_\perp - \mathbf{k}_\perp)^2\right)}, \tag{24}$$

and

$$E\left(\mathbf{p}, \mathbf{k} : \omega_a\right) = \frac{8\pi^2 1e\gamma^2\beta^2}{(\omega_a^2 + \gamma^2\beta^2\mathbf{p}_\perp^2)} \left[4\pi Ze^2 \frac{2E_k^{(-)}\gamma^0 + \vec{\gamma}\cdot(\vec{p} - \vec{k})}{\left(\vec{p} - \vec{k}\right)^2 (\vec{p}^2 - \vec{k}^2)}\gamma^0\right]. \tag{25}$$

The transition amplitudes $T_k(\mathbf{p}_\perp : +\beta)$ and $T_k^{'}(\mathbf{p}_\perp : +\beta)$ can be expressed explicitly for antihydrogen production as

$$T_k\left(\mathbf{p}_\perp : +\beta\right) = \sum_s \sum_{\sigma_p} \left[1 - \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{2m}\right] \frac{\left\langle \mathbf{u} \left|(1 - \beta\alpha_z)\right| \mathbf{u}_{\sigma_p}^{(s)} \right\rangle \left\langle \mathbf{u}_{\sigma_p}^{(s)} \left|(1 + \beta\alpha_z)\right| \mathbf{u}_{\sigma_k}^{(-)} \right\rangle}{\left(E_p^{(s)} - \left(\frac{E^{(+)} + E_k^{(-)}}{2}\right) - \beta\frac{k_z}{2}\right)}, \tag{26}$$

and

$$T_k^{'}\left(\mathbf{p}_\perp : +\beta\right) = \sum_s \sum_{\sigma_p} \frac{\left\langle \mathbf{u} \left|(1 - \beta\alpha_z)\right| \mathbf{u}_{\sigma_p}^{(s)} \right\rangle \left\langle \mathbf{u}_{\sigma_p}^{(s)} \left|(1 + \beta\alpha_z)\right| \mathbf{u}_{\sigma_k}^{(-)} \right\rangle}{\left(E_p^{(s)} - \left(\frac{E^{(+)} + E_k^{(-)}}{2}\right) - \beta\frac{k_z}{2}\right)} \tag{27}$$

After calculating the amplitude for the *direct* diagram, we calculate the amplitude for the *crossed* diagram and repeat the same procedure that we did for BFPP to write down the cross section for the generation of relativistic antihydrogen production with the correction term

$$\begin{aligned}
\sigma &= \int d^2b \sum_{k>0} \left|\left\langle \Psi^{(+)} \left|S\right| \Psi_k^{(-)} \right\rangle\right|^2 = \int d^2b \sum_{k>0} \left|\left\langle \Psi^{(+)} \left|S_{ab}\right| \Psi_k^{(-)} \right\rangle + \left\langle \Psi^{(+)} \left|S_{ba}\right| \Psi_k^{(-)} \right\rangle\right|^2 \\
&= \left|N_+\right|^2 \frac{1}{\pi} \left(\frac{1}{a_H}\right)^3 \sum_{k>0} \int d^2b \left(\left(\int \frac{d^2\mathbf{p}_\perp}{(2\pi)^2} e^{i\left(\mathbf{p}_\perp - \frac{\mathbf{k}_\perp}{2}\right)\cdot\mathbf{b}} F\left(-\mathbf{p}_\perp : \omega_a\right) F\left(\mathbf{p}_\perp - \mathbf{k}_\perp : \omega_b\right) T_k\left(\mathbf{p}_\perp : +\beta\right)\right.\right. \\
&\quad \left. - \int \frac{d^3\mathbf{p}}{(2\pi)^4} e^{i\mathbf{p}_\perp\cdot\frac{\mathbf{b}}{2}} E(\mathbf{p}, \mathbf{k} : \omega_a) T_k^{'}(\mathbf{p}_\perp : +\beta)\right) \\
&\quad + \left(\int \frac{d^2\mathbf{p}_\perp}{(2\pi)^2} e^{-i\left(\mathbf{p}_\perp - \frac{\mathbf{k}_\perp}{2}\right)\cdot\mathbf{b}} F\left(-\mathbf{p}_\perp : \omega_b\right) F\left(\mathbf{p}_\perp - \mathbf{k}_\perp : \omega_a\right) T_k\left(\mathbf{p}_\perp : -\beta\right)\right. \\
&\quad \left.\left. - \int \frac{d^3\mathbf{p}}{(2\pi)^4} e^{-i\mathbf{p}_\perp\cdot\frac{\mathbf{b}}{2}} E(\mathbf{p}, \mathbf{k} : \omega_a) T_k^{'}(\mathbf{p}_\perp : -\beta)\right)\right)^2 .
\end{aligned} \tag{28}$$

## 4. Results and concluding remarks

In this work, we have performed our calculations for the peripheral collisions of relativistic heavy ions for the total cross section of bound-free electron–positron and antihydrogen production with a correction term.

In order to compare our new cross section results (with correction term) with our previous work (without correction term) [19], we performed total BFPP cross section results for selected collision energies in Table 1. It is clear that correction term reduces the cross section quite substantially.

**Table 1.** BFPP cross sections $\sigma_{BFPP}$ (in barn) for selected collision systems and cross sections at RHIC and LHC collider facilities without and with the correction term added to the positron wave function, respectively.

| | | Ref. [19] | This work (cor) |
|---|---|---|---|
| RHIC | Au+Au | 94.5 b | 58.3 b |
| LHC | Pb+Pb | 202 b | 138.8 b |

In some previous works, the expression for pair production with electron capture in the nucleus with target charge is obtained via first order perturbation theory [1,24,25]. We compared our BFPP cross section correction term added results given in [25]. They found $45\,barn$ for $Au+Au$ collisions at RHIC and $102\,barn$ for $Pb+Pb$ collisions at LHC. In this work, they also added the effect of higher shells. These results are about 30% lower compared with our work.

Our BFPP result with the correction term for $Cu+Cu$ ions at RHIC is $0.238\,barn$. The obtained result for the beam of $63Cu29+$ ions at 100 GeV/nucleon at the BNL Relativistic Heavy Ion Collider (RHIC) given in [26] is approximately $0.2\,barn$, which is very close to our result with the correction term.

Similarly, in Table 2, we compared the new antihydrogen production cross section with our previous work [20] for $Au-\bar{p}$ and $Pb-\bar{p}$ collisions at RHIC and LHC energies. In $Au-\bar{p}$ and $Pb-\bar{p}$ collisions, our new cross section results for RHIC and LHC energies are smaller than those in our previous work. The reason for this reduction at cross section is the effect of the corrections of order $Z\alpha$ added to the free particle wave function for the capture processes.

**Table 2.** Antihydrogen production cross sections results for $Au-\bar{p}$ and $Pb-\bar{p}$ collisions at RHIC and LHC collider facilities without and with the correction term added to the electron wave function, respectively.

| | RHIC [20] | RHIC (cor) | LHC [20] | LHC (cor) |
|---|---|---|---|---|
| Au-$\bar{p}$ | 15.3 mb | 9.3 mb | 31.1 mb | 19.2 mb |
| Pb-$\bar{p}$ | 14.9 mb | 10 mb | 30.2 mb | 20.7 mb |

In Figure 1 the BFPP cross sections for symmetric collisions of ions are shown as a function of the nuclear charge $Z$. Cross section calculations are done for the two collision energies with and without the correction term for $E = 100$ GeV/nucleon and for 3400 GeV/nucleon. As seen from the figure, correction term that is added to the positron wave function lowers the cross section results especially for small values of the colliding ions. Figure 2 displays the total BFPP cross sections for two different systems as functions of the Lorentz contraction factor $\gamma$. Results are displayed with and without the correction term for $Au+Au$ and $Pb+Pb$ collisions. As expected, the correction term lowers the cross section results.

Figure 3 displays the relativistic antihydrogen production cross sections as functions of the nuclear charge for RHIC and LHC energies with and without the correction term added to the electron wave function. Here collisions of antiprotons and heavy ions are considered. As seen in this figure, without the correction term the

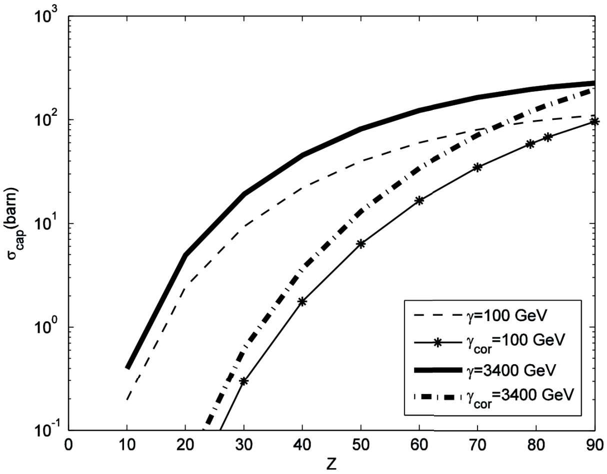


**Figure 1.** BFPP cross sections for two different systems as functions of the nuclear charge $Z$. BFPP cross sections (in barn) for the symmetric collision of bare ions with nuclear charge $Z$ at RHIC and LHC energies with and without the correction term.

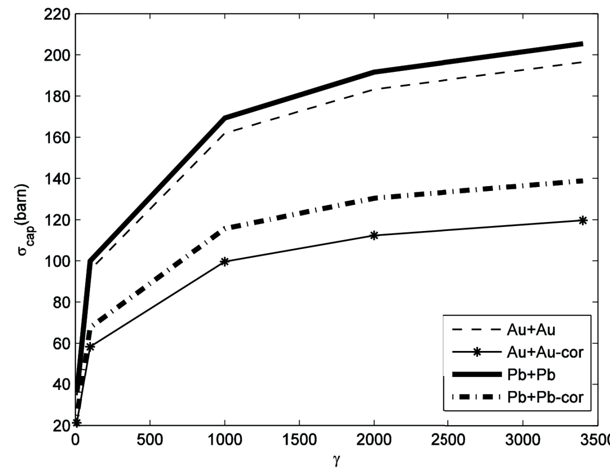


**Figure 2.** BFPP cross sections with and without the correction term for two different systems $Au+Au$ and $Pb+Pb$ as functions of the $\gamma$. The magnitude of $\gamma$ is going from 10 to 3400.

maximum value of the cross section is obtained when nuclear charge is about 65 at RHIC and LHC energies. The reason for this is while the ions are getting heavier, the contribution of the normalization constant in the antihydrogen wave function is inversely proportional to the charge of heavy ions [20]. This conflict is solved when we add the correction term to our cross section calculations and therefore the antihydrogen cross section value increases with the nuclear charge of the heavy ions.

Figure 4 shows the relativistic antihydrogen cross sections with and without the correction term for two different ions as functions of the Lorentz contraction factor. This figure displays that the probability of antihydrogen production increases with the $\gamma$ values. Without the correction term, the antihydrogen cross

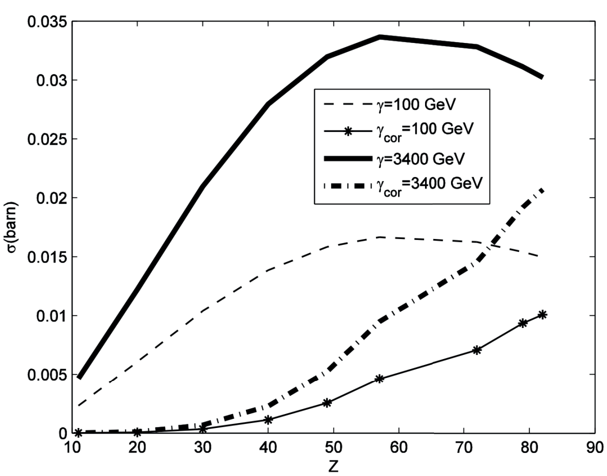


**Figure 3.** Relativistic antihydrogen cross sections with and without the correction term for two energy systems at 100 GeV/nucleon and 3400 GeV/nucleon as functions of the nuclear charge $Z$.

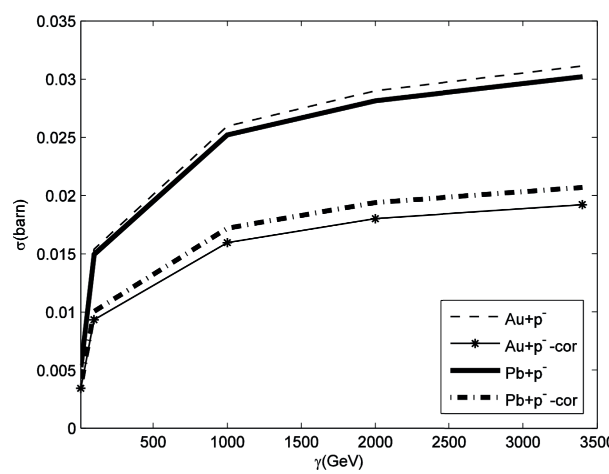


**Figure 4.** Relativistic antihydrogen cross sections with and without correction term for two different ions ($Au$ and $Pb$) as functions of $\gamma$ that the magnitude of it is going from 10 to 3400 ($p^-$ represents antiproton).

section results of $Au + \bar{p}$ collision are higher than the results of $Pb + \bar{p}$. While $Z$ is getting higher, the inverse effect of the square of the normalization constant makes the antihydrogen cross section values lower [20]. When we add correction term to the relativistic antihydrogen cross sections, this conflict disappears and, as expected, the cross section results of $Pb + \bar{p}$ get higher than the antihydrogen cross section results of $Au + \bar{p}$.

In this work we calculated bound-free electron–positron pair production cross section and antihydrogen production including the correction term. We added the correction term $\Psi^{'}$ to the electron wave function known as the Sommerfeld-Maue wave function and for the bound positron we used the Darwin wave function. Similarly, for the antihydrogen production mechanism, we added the correction term $\bar{\Psi}^{'}$ to the positron wave function and for the bound electron we used the Darwin wave function.

We presented our results in the above tables and figures. The BFPP cross section results with the correction term are smaller than the results without it, since the wave functions used are correct only to first order in $\alpha$. It is expected that the addition of higher-order terms may change our results. In future research, the inclusion of distortion corrections to higher order in $Z\alpha$ will be necessary to obtain more precise results.

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