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On the calculus of parameterized fractal curves

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Abstract: In this paper, we apply F^α-calculus on the fractal Koch and Cesàro curves with different dimensions. A generalized Newton's second law on the fractal Koch and Cesàro curves is given. Density of the moving particles absorbed on fractal Cesàro are derived. Illustrative examples are given to present the details of F^α-integrals and F^α-derivatives.

Key words: F^α-calculus, fractal Koch curve, staircase function; fractal Cesàro curve

1. Introduction

Fractional calculus includes the derivatives and integrals with arbitrary orders and it has been applied in science and engineering [1, 2]. The fractional derivatives were used to model non-conservative systems, processes with memory effect, and anomalous diffusion [3-5]. The fractional derivatives are nonlocal but most of the measurements in physics are local [6]. As a result, in view of the fractional local derivatives and the Chapman–Kolmogorov condition, a new Fokker–Planck equation was given [7]. The local fractional derivatives lead to a new measure on fractal sets [8]. Fractal geometry that generalizes Euclidean geometry has an important role in science, engineering, and medical science. Fractals are geometrical objects that have self-similar properties and fractional dimensions. Fractal geometry was used in biology and medicine for describing pathogenetic processes in medicine [9]. Fractal dust was utilized to model the distribution of stars and galaxies in the universe [10]. General topology and measure theory were used to generalize analysis on the fractal sets. Heat diffusion on a fractal medium and the vibration of a material with fractal structure were modeled by defining Laplacians on self-similar sets [11]. Using harmonic analysis and probability theory, differential equations on fractal sets were suggested and solved [12]. Many methods have been used to develop a formal analytical description of fractal sets and processes [13]. Fluid flow in a fractal porous medium was mapped into fractal continuum flow for describing stress and strain distributions in elastic fractal bars [14]. Frictional properties of Weierstrass–Mandelbrot surfaces were given using fractal Koch surfaces [15].

Recently, F^α-calculus (F^α.C.) was formulated in a seminal paper as a framework by Parvate and Gangal, which is the generalized standard calculus. F^α.C. is calculus on fractals with the algorithmic property [16–18]. Researchers have explored this area giving new insight into F^α.C. [19, 20]. The fractal Cantor sets were considered as grating in the diffraction phenomena [21]. Following this line of research we apply F^α.C. to the fractal Koch and Cesàro curves. The differential equation characterizing the motion of the particles on fractal curves is studied.

The paper is organized as follows: in Section 2, we summarize F^α.C. on the fractal Koch and Cesàro

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curves. In Section 3, we develop the new results including the equation of motion of the particles. Section 3 contains our conclusion.

2. Preliminaries

In this section, we summarize the F^{α}.C. on parameterized fractal curves and use it in the case of the fractal Koch and Cesàro curves (see [18] for a review).

Calculus on the fractal Koch and Cesàro curves: Let us consider the fractal Koch and Cesàro curves, which are denoted by $F \subset R^3$, and define the corresponding staircase function. The fractal Koch and Cesàro curves are called continuously parameterizable if there exists a function $\mathbf{w}(t) : [a_0, b_0] \to F,\ a_0, b_0 \in R$ that is continuous one to one and onto F [18].

Definition 1 *For fractal curves F and a subdivision $P_{[a,b]}, a < b, [a,b] \subset [a_0, b_0]$, the mass function is defined as [18]:*

$$\gamma^{\alpha}(F,a,b) = \lim_{\delta \to 0} \inf_{\{P_{[a,b]}:|P| \leq \delta\}} \sum_{i=0}^{n-1} \frac{|\boldsymbol{w}(t_{i+1}) - \boldsymbol{w}(t_i)|^{\alpha}}{\Gamma(\alpha+1)}, \tag{1}$$

where $|.|$ indicates the Euclidean norm on R^3 and $P_{[a,b]} = \{a = t_0, ..., t_n = b\}$.

Definition 2 *The staircase functions for the fractal Koch and Cesàro curves are defined as:*

$$S_F^{\alpha}(t) = \begin{cases} \gamma^{\alpha}(F, p_0, t) & t \geq p_0, \\ -\gamma^{\alpha}(F, t, p_0) & t < p_0, \end{cases} \tag{2}$$

where $p_0 \in [a_0, b_0]$ is an arbitrary point.

In Figure 1 we have sketched the fractal Koch and Cesáro curves and $S_F^{\alpha}(t)$ setting $\alpha = 1.26$ and $\alpha = 1.78$.

Definition 3 *The γ-dimensions of the fractal Koch and Cesáro curves (F) are defined as:*

$$\begin{aligned} \dim_{\gamma}(F) &= \inf\{\alpha : \gamma^{\alpha}(F,a,b) = 0\} \\ &= \sup\{\alpha :\ \gamma^{\alpha}(F,a,b) = \infty\}. \end{aligned} \tag{3}$$

Definition 4 *The F^{α}-derivative of a function f at $\theta \in F$ is defined as:*

$$D_F^{\alpha} f(\theta) = F - \lim_{\theta' \to \theta} \frac{f(\theta') - f(\theta)}{J(\theta') - J(\theta)}, \tag{4}$$

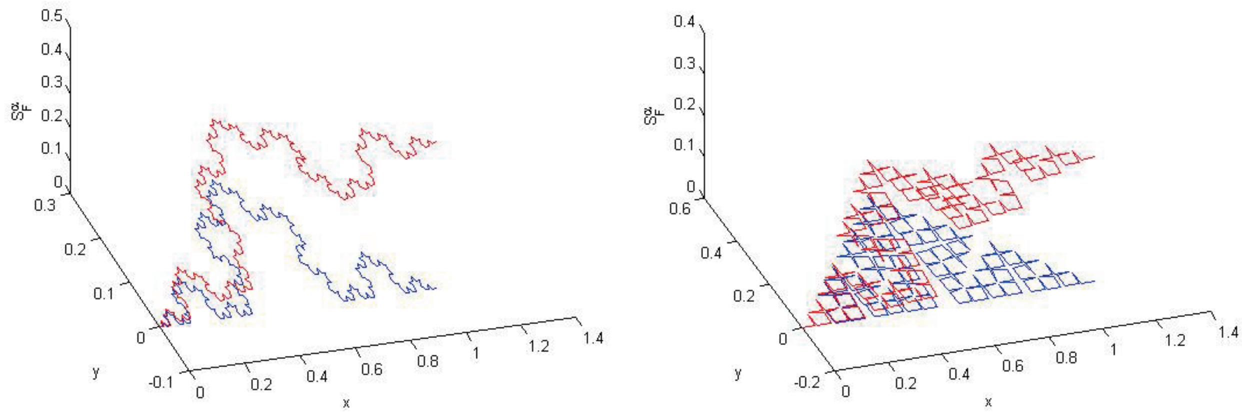
where $J(\theta) = S_F^{\alpha}(w^{-1}(\theta)), \theta \in F$ and if the limit exists [18].

Definition 5 *A number l is the F-limit of a function f if we have*

$$\theta' \in F \quad and \quad |\theta' - \theta| < \delta \Rightarrow |f(\theta') - l| < \epsilon. \tag{5}$$

If such a number exists [18], it is indicated by

$$l = F - \lim_{\theta' \to \theta} f(\theta'). \tag{6}$$



(a) The fractal Koch curve and $S_F^{1.26}(t)$ (b) The fractal Cesáro curve and $S_F^{1.7}(t)$

Figure 1. The graph of the fractal Koch curve and fractal Cesáro curve (in blue) and the corresponding staircase functions (in red).

A segment $C(t_1,t_2)$ of the fractal Koch and Cesáro curve is defined as:

$$C(t_1,t_2)=\{w(t'):t'\in[t_1,t_2]\}, \tag{7}$$

and M, m are defined as follows [18]:

$$M[f,C(t_1,t_2)]=\sup_{\theta\in C(t_1,t_2)} f(\theta), \tag{8}$$

$$m[f,C(t_1,t_2)]=\inf_{\theta\in C(t_1,t_2)} f(\theta). \tag{9}$$

Definition 6 *The upper and the lower F^α-sum for the function f over the subdivision P are defined as:*

$$U^\alpha[f,F,P]=\sum_{i=0}^{n-1} M[f,C(t_i,t_{i+1})][S_F^\alpha(t_{i+1})-S_F^\alpha(t_i)], \tag{10}$$

$$L^\alpha[f,F,P]=\sum_{i=0}^{n-1} m[f,C(t_i,t_{i+1})][S_F^\alpha(t_{i+1})-S_F^\alpha(t_i)]. \tag{11}$$

Definition 7 *The F^α-integral of the function f is defined as*

$$\int_{C(a,b)} f(\theta)d_F^\alpha\theta=\underline{\int_{C(a,b)}} f(\theta)d_F^\alpha\theta=\sup_{P_{[a,b]}} L^\alpha[f,F,P]$$
$$=\overline{\int_{C(a,b)}} f(\theta)d_F^\alpha\theta=\inf_{P_{[a,b]}} U^\alpha[f,F,P]. \tag{12}$$

Fundamental theorems of F^{α}-calculus:

First part: If $f : F \rightarrow R$ is an F^{α}-differentiable function and $h : F \rightarrow R$ is F-continuous such that $h(\theta) = D_F^{\alpha} f(\theta)$, then we have

$$\int_{C(a,b)} h(\theta) d_F^{\alpha}\theta = f(w(b)) - f(w(a)). \tag{13}$$

Second part: If f is bounded and F-continuous on $C(a,b)$ and $g : F \rightarrow R$ then

$$g(w(t)) = \int_{C(a,t)} f(\theta) d_F^{\alpha}\theta, \qquad t \in [a,b], \tag{14}$$

where we suppose

$$D_F^{\alpha} g(\theta) = f(\theta). \tag{15}$$

For the proofs we refer the readers to [18].

Some properties:

1) If $f(\theta) = k \in R$, then $D_F^{\alpha} f = 0$.

2) If f is F-continuous and $D_F^{\alpha} f = 0$, then $f = k$.

3) The generalized Taylor series on the fractal Koch curves is written as

$$h(\theta) = \sum_{n=0}^{\infty} \frac{(J(\theta) - J(\theta'))^n}{n!} (D_F^{\alpha})^n h(\theta'), \qquad \theta \in F. \tag{16}$$

4) If $f(\theta) = 1$ is a constant function then

$$\begin{aligned} \int_{C(a,b)} f(\theta) d_F^{\alpha}\theta &= \int_{C(a,b)} 1 d_F^{\alpha}\theta \\ &= S_F^{\alpha}(b) - S_F^{\alpha}(a) = J((w(b)) - J((w(a)). \end{aligned} \tag{17}$$

Note: The F^{α}-integral and the F^{α}-derivative on the fractal Koch curves are linear operators.

Consider $f(t) : F \rightarrow R$ on the fractal Koch curves as

$$f(t) = (S_F^{\alpha}(t))^2. \tag{18}$$

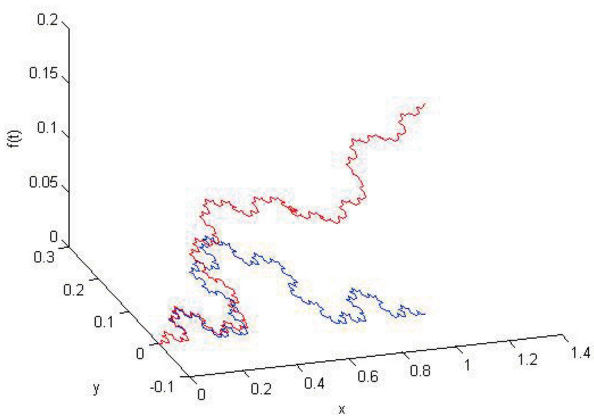
The F^{α}-derivative and the F^{α}-integral of f are

$$\int_{C(0,t)} f(t) d_F^{\alpha} t = \frac{(S_F^{\alpha}(t))^3}{3} + k,$$

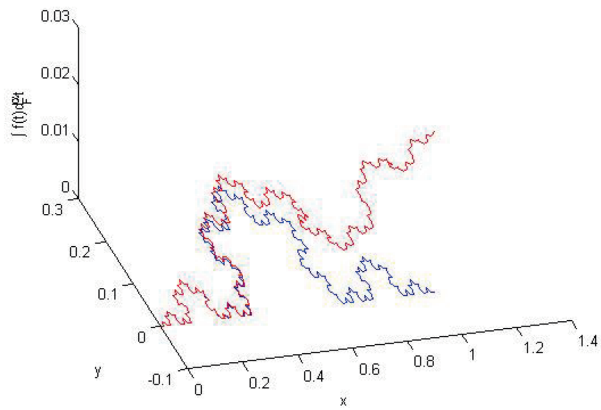
and

$$D_F^{\alpha} f(t) = 2\ S_F^{\alpha}(t),$$

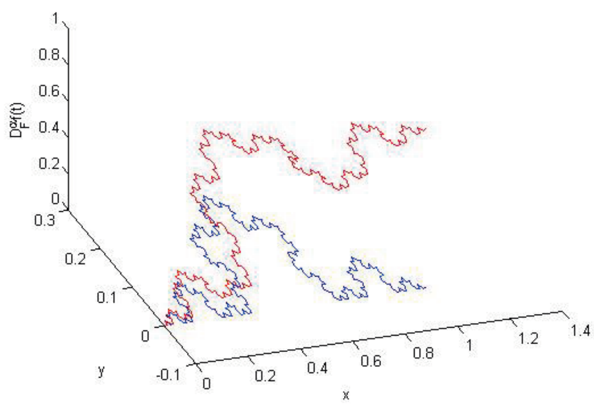
where k is constant. Figure 2 shows the graphs of f, the F^{α}-integral, and the F^{α}-derivative of f.



(a) The f on fractal Koch curve



(b) The $\int f(t) d_F^{\alpha} t$ on fractal Koch curve



(c) The $D_F^{\alpha} f(t)$ on fractal Koch curve

Figure 2. Graph of the F^{α}-integral and the F^{α}-derivative of f on the fractal Koch curve.

3. Equation of motion on fractal curves

The generalized Newton's second law on the fractal Koch and Cesáro curves is suggested as

$$m(D_F^{\alpha})^2\mathbf{r}_F^{\alpha}(t) = \mathbf{f}_F^{\alpha}, \tag{19}$$

where $\mathbf{r}_F^{\alpha} : F \to R$, $\mathbf{v}_F^{\alpha}(t) = D_F^{\alpha}\ \mathbf{r}_F^{\alpha}$, and $\mathbf{a}_F^{\alpha}(t) = (D_F^{\alpha})^2\ \mathbf{r}_F^{\alpha}$ are called the generalized position, generalized velocity, and generalized acceleration on the fractal Koch and Cesáro curves, respectively.

Example 1 *Consider a force* $\boldsymbol{f}_F^{\alpha} = k(\hat{i} + \hat{j})\ [\ ML^{\alpha}T^2\ ]$ *such that* $\boldsymbol{f}_F^{\alpha} : F \to R$ *is applied on a particle with mass* m *on the fractal Koch curves. One sees immediately that generalized acceleration, velocity, and position* *are*

$$\begin{aligned}\boldsymbol{a}_F^{\alpha}(t) &= \frac{k}{m},\\ \boldsymbol{v}_F^{\alpha}(t) &= \frac{k}{m}S_F^{\alpha}(t)+\boldsymbol{v}_F^{\alpha}(0),\\ \boldsymbol{r}_F^{\alpha}(t) &= \frac{k}{2m}S_F^{\alpha}(t)^2+\boldsymbol{v}_F^{\alpha}(0)S_F^{\alpha}(t)+\boldsymbol{r}_F^{\alpha}(0).\end{aligned} \tag{20}$$

We present the graph of Eq. (20) in Figure 3.

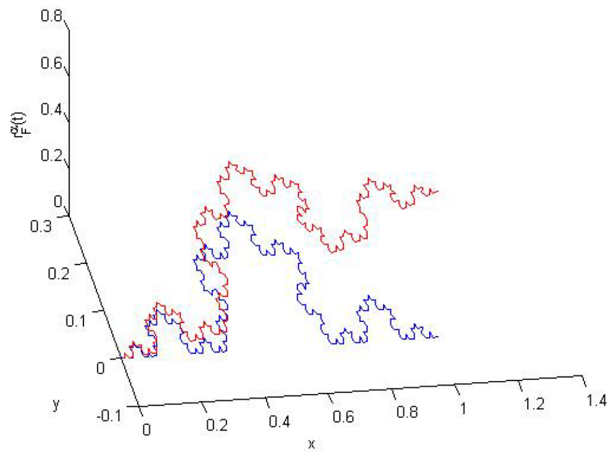


Figure 3. The graph of $\mathbf{r}_F^{\alpha}(t)$ setting $\mathbf{v}_F^{\alpha}(0)=\frac{k}{2m}=1,\ \mathbf{r}_F^{\alpha}(0)=0$.

Example 2 *Consider particles moving along the fractal Cesàro curve, which absorbs the particles. The mathematical model for this phenomenon is given by*

$$D_F^{\beta}\zeta(t)=-k\zeta(t),\quad \beta=1.78, \tag{21}$$

where ζ is the density of particles on the fractal Cesàro curve. Using the F^{α}-integral, it is easy to obtain the solution:

$$\zeta(t)=\zeta(0)e^{-kS_F^{\beta}(t)}. \tag{22}$$

Figure 4 shows the density of particles $\zeta(t)$ for the flux of particles on the fractal Cesàro curve with absorbtion.

Example 3 *Consider the following differential equation on the fractal Cesàro curve:*

$$(D_F^{1.78})^2\mu(t)+9\mu(t)=\cos(t),\quad t\in F, \tag{23}$$

with the boundary conditions

$$D_F^{1.78}\mu(t)\big|_{t=0} = 5, \tag{24}$$

$$\mu(\pi/2) = -5/3. \tag{25}$$

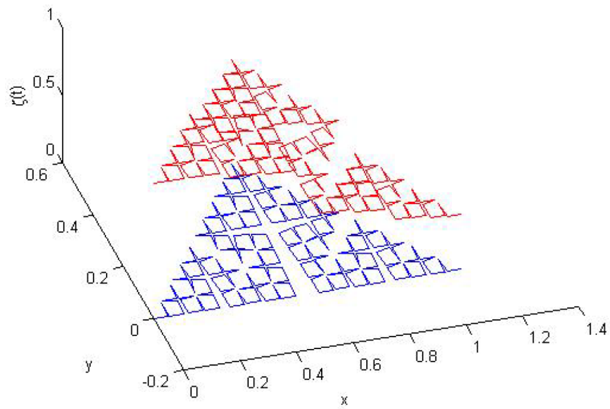


Figure 4. The graph of $\zeta(t)$ on the fractal Cesàro curve.

Using the conjugacy of F^{α}*.C. and ordinary calculus, the solution is*

$$\mu(t) = \lambda \cos(3\ S_F^{1.78}(t)) + \frac{5}{3}\sin(3\ S_F^{1.78}(t)) + \frac{1}{8}\cos(S_F^{1.78}(t)), \tag{26}$$

where λ *is constant.*

Example 4 *Consider the Langevin equation on the fractal Cesàro curve as follows:*

$$m\ D_F^{1.78}\xi(t) = -\kappa\ \xi(t) + \tau\ \eta(t) \tag{27}$$

where m, κ, *and* τ *are constants and* $\eta(t)$ *is a stationery Gaussian white noise. With the conjugacy of* F^{α}*.C. with ordinary calculus and using inverse transformation between them, we have*

$$\xi(t) = D_F^{1.78}\xi(t)\big|_{t=0}\ e^{-\kappa S_F^{1.78}(t)} + \frac{\tau}{m}\int_0^t e^{-\tau(S_F^{1.78}(t)-S_F^{1.78}(t'))}\eta(t')\ d_F^{1.78}t'. \tag{28}$$

Using $<\eta(t)>=0$, *we obtain*

$$<D_F^{1.78}\xi(t)>= D_F^{1.78}\xi(t)\big|_{t=0}\ e^{-\kappa S_F^{1.78}(t)}, \tag{29}$$

where $<.>$ *is denoted the mean function of a random process.*

4. Conclusion

In this work, we addressed the F^{α}.C. that generalizes standard calculus on fractals with fractional dimension and self-similar properties. In the sense of the standard calculus the fractal Koch and Cesàro curves are not differentiable and integrable. F^{α}.C. is used to define the F^{α}-integral and the F^{α}-derivative on the fractal Koch and Cesàro curves. Some illustrative examples are given and the main aspects discussed. Finally, generalized differential equations corresponding to the motions on the fractal Koch and Cesàro curves are suggested and solved.

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