Localization of a nonlocal interaction

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Abstract: A simple method for the localization of a separable nonlocal interaction is formulated in terms of Green's functions and the solutions with regular and irregular boundary conditions. The constructed energy–momentum-dependent local potential with regular boundary condition is real while that for the irregular boundary condition is complex in nature. The phase function method is exploited to compute elastic scattering phases for the nucleon–nucleon and alpha–nucleon systems. Reasonable agreements in scattering phase shifts with experimental data are obtained, particularly, in the low energy range for the systems under consideration. The phase shifts for the imaginary parts of the potentials derived from the irregular solutions for the alpha–nucleon systems, however, give indications of resonances at very low energies.

Key words: Localization of separable interaction, energy–momentum-dependent local potential, phase function method, nucleon–nucleon and alpha–nucleon systems, elastic scattering phase shifts

1. Introduction

In 1949, the first attempt to establish the relationship between scattering phase shifts and potential was made by Levinson [1]. He showed that two potentials that decrease rapidly enough give rise to no bound states and those that give the same phase shifts for all angular momenta are identical. Later that same year, Bargmann [2] conducted a study on phase equivalent potentials and introduced a technique for constructing them. He discovered a manifold of potentials V(r) with the same spectral density for positive energies as a given potential $V_0(r)$. Each member of V(r)s is phase equivalent to $V_0(r)$ but may differ in the number of bound states [3,4]. In 1951, Gel'fand and Levitan [5] presented an integral equation that relates the phase shifts as functions of energy to another function from which the scattering potential can be obtained. The phase shifts are to be described by what is called a spectral function and the potential is derived from the kernel that solves their integral equation. Later on, the algorithms of the inverse scattering problem, within the framework of supersymmetric quantum mechanics, have been exploited by a number of researchers [6–10] to generate phase equivalent potentials.

The use of separable nonlocal interactions to fit two-nucleon phase shifts in various angular momentum states is well established. Attempts were made by several groups [11–13] to construct equivalent local potentials to nonlocal interactions or localization of nonlocal potentials. An equivalent local potential analysis to a nonlocal one is quite common in optical model studies. These methods include a comparison between the characteristics of nonlocal potentials and the familiar phenomenology of the local potentials. In general, the phase shifts or the T-matrix elements of the nonlocal and its equivalent local potentials are compared. Coz et al. [14] have

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applied the method of energy-dependent equivalent potential to several nonlocal Hartree–Fock nucleon–nucleus potentials. The energy dependence of such local potentials varies slightly over the entire energy range for which the original nonlocal potentials are applicable. There exist experimental situations that involve scattering by additive interactions, some of which must for various physical reasons be treated exactly, whereas others may be treated as a relatively small perturbation [15,16]. A typical example of this kind is the scattering of particles under the combined influence of electromagnetic and nuclear forces like proton–proton (p-p), alpha–proton (α-p), and alpha–alpha (α-α) [17–19]. In view of the importance of experiments that involve charged hadrons, the interest in studying potentials consisting the sum of a short-range finite-rank separable potential and an electromagnetic potential is increased. The short range interaction is of nuclear origin while the electromagnetic potential takes care of the charges. The alpha particles are tightly bound and have no low lying excited states. In view of this, one can use the simple Schrödinger equation with electromagnetic plus separable nonlocal potentials to compute scattering phase shifts for alpha–nucleon systems. In atomic and plasma physics screened and cut-off Coulomb potentials are important. Many standard results in nonrelativistic scattering theory for the short-range potentials have to be modified for charged particle scattering as the particles interacting via the Coulomb potential never behave like free particles. Even the asymptotic condition for a well-behaved potential does not hold and as a consequence the concept of a phase shift is ill defined for Coulomb scattering. In reality, the Coulomb potential does not exist in nature and becomes somewhat screened at a certain distance. Moreover, the traditional approach to the phase function method (PFM) for the local potential does not hold good for the pure Coulomb interaction and it needs separate treatment. The Hulthén potential at small values of r behaves like a Coulomb potential, whereas for large values of r it decreases exponentially so that its capacity for bound states is smaller than that of the Coulomb potential (for attractive cases). Thus, to circumvent the difficulties for computing scattering phase shifts by use of the traditional phase function method the electromagnetic interaction here is defined by a screened Coulomb potential, the atomic Hulthén one.

In this text, we present a simple method of localization of nonlocal interaction by using the regular and irregular solutions and judge the merit of our approach through some model calculations. Our approach to the problem based on simple rearrangements of the Schrödinger equation with electromagnetic plus separable nonlocal interactions and the phase function method (PFM) permits a rigorous inclusion of the electromagnetic effect to the elastic scattering of charged hadron systems. In section 2 we describe the method of localization of local plus a separable nonlocal interaction. Section 3 is devoted to the results and discussions. Finally, the conclusions are given in section 4.

2. Localization of Hulthén-modified separable nonlocal interaction

Separable potentials have been frequently used in different areas of physics because of the simplicity involved in analytical calculation. A nonlocal potential is, in general, a function of two coordinate variables. In the separable model $V_\ell(r, r') = \sum_{N}^{i=1} \lambda_\ell^i \left| g_\ell^i(r) \right\rangle \left\langle g^i(r') \right|$ with λ_ℓ^i and $g_\ell^i(r)$ represent the state dependent strength parameter and form factor of the interaction. The attractive part of the nucleon–nucleon interaction involves a phenomenological intermediate region and a one pion exchange tail [20]. Therefore, for a correct description of the nucleon–nucleon interaction within the formalism of a separable model one needs at least two terms in the potential with the strength parameters having opposite signs. Since low energy scattering experiments sample out only the outer region of the potential, one term separable potential may be of importance for this energy range. For intermediate and high energy ranges one has to consider higher rank potential because of

the sensitivity of scattering data to the choice of inner core irrespective of whether the separable interaction is symmetric or nonsymmetric [21–24], and the associated Schrödinger equation can be solved in closed form.

At a centre of mass energy $E = k^2 + i\varepsilon$, the radial Schrödinger equation for the Hulthén plus rank N separable potential in all partial waves ℓ is written as

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} - V_H(r)\right]\psi_\ell(k,r) = \sum_{i=1}^{N}\lambda_{\ell i}g_{\ell i}(\beta_{\ell i},r)\int_{0}^{\infty}dr'g_{\ell i}(\beta_{\ell i},r')\psi_\ell(k,r'), \tag{1}$$

where the atomic Hulthén interaction

$$V_H(r) = V_0\frac{e^{-r/a}}{1 - e^{-r/a}} \tag{2}$$

with V_0, the strength and a, the screening radius of the atomic Hulthén potential. In the limit $a \to \infty$, the potential in Eq. (2) goes over to Coulomb potential if $V_0a^2 = e^2 = 1$ (in atomic unit, au = 5.291772×10^{-11} m). The quantities $g_{\ell i}(\beta_{\ell i}, r)$s are the form factors of the separable potential and $\lambda_{\ell i}$ and $\beta_{\ell i}$ stand for the strength and inverse range parameters. Eq. (1) may be rewritten as

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2}\right]\psi_\ell(k,r) = \left[V_H(r) + \frac{1}{\psi_\ell(k,r)}\sum_{i=1}^{N}\lambda_{\ell i}g_{\ell i}(\beta_{\ell i},r)\,d_{\ell i}(\beta_{\ell i},k)\right]\psi_\ell(k,r) \tag{3}$$

with

$$d_{\ell i}(\beta_{\ell i},k) = \int_{0}^{\infty}dr'g_{\ell i}(\beta_{\ell i},r')\psi_\ell(k,r'). \tag{4}$$

The wave function $\psi_\ell(k,r)$ involved in Eq. (3) is not the solution of a local interaction while it denotes the solution for the Hulthén plus rank N separable potential. Thus, on comparing Eq. (3) with the Schrödinger equation for a local potential the term within the braces on the right-hand side is considered as an approximate energy-dependent equivalent local interaction for the Hulthén plus rank N separable potential identified as

$$V_{EQ}(k,r) = V_H(r) + \frac{1}{\psi_\ell(k,r)}\sum_{i=1}^{N}\lambda_{\ell i}g_{\ell i}(\beta_{\ell i},r)\,d_{\ell i}(\beta_{\ell i},k). \tag{5}$$

If the electromagnetic interaction is turned off the above equation reduces to equivalent local potential for the pure separable potential. If the potential is less singular at $r = 0$ than r^{-2}, that is $\lim_{r\to 0} r^2V(r) = 0$, the point $r = 0$ is regular in the theory of ordinary second-order differential equation. The solution that vanishes at $r = 0$ is termed as regular and the one that does not is called irregular. We shall apply both the regular and irregular boundary conditions for the construction of approximate equivalent local potentials. As the Hulthén or Hulthén-like potentials are exactly solvable for the S-wave only we shall treat here the case for $\ell = 0$ and hereafter omit the subscript ℓ throughout the text.

2.1. Regular boundary condition

The integral equation corresponding to Eq. (1) for the regular solution $\phi^{HS}(k,r)$ for the Hulthén plus rank N separable potential is written as

$$\phi^{HS}(k,r) = \phi^H(k,r) + \sum_{i=1}^{N} \lambda_i d_i^{(R)}(\beta_i,k) \int_0^r dr' g_i(\beta_i,r')G^{(R)H}(r,r'), \tag{6}$$

where the Hulthén Green's function $G^{(R)H}(r,r')$ for the regular boundary condition is given by [25]

$$G^{(R)H}(r,r') = \left[\phi^H(k,r)f^H(k,r') - \phi^H(k,r')f^H(k,r)\right]/f^H(k) \tag{7}$$

and

$$d_i^{(R)}(\beta_i,k) = \int_0^\infty dr g_i(\beta_i,r)\phi^{HS}(k,r). \tag{8}$$

Here $\phi^H(k,r)$ and $f^H(k,r)$ stand for the regular and irregular solutions of pure Hulthén potential [26,27] and $f^H(k)$ the corresponding Jost function defined as

$$\phi^H(k,r) = a\left(1 - e^{-r/a}\right)e^{ikr} \, {}_2F_1\left(1+A, \, 1+B; \, 2; \, 1 - e^{-r/a}\right), \tag{9}$$

$$f^H(k,r) = e^{ikr} \, {}_2F_1\left(A, \, B; \, C; \, e^{-r/a}\right) \tag{10}$$

and

$$f^H(k) = \frac{\Gamma(C)}{\Gamma(1+A)\Gamma(1+B)} \tag{11}$$

with

$$A = -iak + ia(k^2 + V_0)^{1/2}, \tag{12a}$$

$$B = -iak - ia(k^2 + V_0)^{1/2} \tag{12b}$$

and

$$C = 1 - 2iak. \tag{12c}$$

Eq. (6) represents an inhomogeneous integral equation with a degenerate kernel and can easily be solved to have

$$d_i^{(R)}(\beta_i,k) = \frac{1}{\det {}_N W^{HS}(\beta,k)} \sum_{j=1}^{N} w_{ij}^{HS}(\beta,k)Z_j^{HS}(k) \tag{13}$$

with

$$\det {}_N W^{HS}(\beta,k) = \begin{vmatrix} W_{11}^{HS}(\beta,k) & W_{12}^{HS}(\beta,k) & W_{13}^{HS}(\beta,k) \dots\dots\dots W_{1N}^{HS}(\beta,k) \\ W_{21}^{HS}(\beta,k) & W_{22}^{HS}(\beta,k) & W_{23}^{HS}(\beta,k) \dots\dots\dots W_{2N}^{HS}(\beta,k) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ W_{N1}^{HS}(\beta,k) & W_{N2}^{HS}(\beta,k) & W_{N3}^{HS}(\beta,k) \dots\dots\dots W_{NN}^{HS}(\beta,k) \end{vmatrix}, \tag{14a}$$

where

$$W_{ij}^{HS}(\beta, k) = \delta_{ij} - \lambda_j \int\limits_0^\infty \int\limits_0^r dr\, dr' g_i(\beta_i, r) G^{(R)H}(r, r') g_j(\beta_j, r'), \tag{14b}$$

$$Z_j^{HS}(k) = \int\limits_0^\infty dr\, \phi^H(k, r) g_j(\beta_j, r) \tag{15}$$

and $w_{ij}^{HS}(\beta, k)$s represent the cofactors of $W_{ij}^{HS}(\beta, k)$s. Thus, Eq. (6) together with Eqs. (7)–(15) gives the desired expression for the regular solution for Hulthén plus rank N separable potential as

$$\phi^{HS}(k, r) = \phi^H(k, r) + \frac{1}{\det\ _N W^{HS}(\beta, k)} \sum_{i,j=1}^N \lambda_i w_{ij}^{HS}(\beta, k) Z_j^{HS}(k) S_i^{HS}(\beta_i, r) \tag{16}$$

with

$$S_i^{HS}(\beta_i, r) = \int\limits_0^r dr' g_i(\beta_i, r') G^{(R)H}(r, r'). \tag{17}$$

Combination of Eqs. (2), (5), and (16) produces the desired result of equivalent energy-dependent local interaction for Hulthén plus rank N separable potential.

2.2. Irregular boundary condition

The irregular solution $f^{HS}(k, r)$ for the Hulthén plus rank N separable potential satisfies the integral equation

$$f^{HS}(k, r) = f^H(k, r) + \sum_{i=1}^N \lambda_i d_i^{(I)}(\beta_i, k) \int\limits_r^\infty dr' g_i(\beta_i, r') G^{(I)H}(r, r') \tag{18}$$

with the irregular Hulthén Green's function [15] $G^{(I)H}(r, r')$

$$G^{(I)H}(r, r') = - \left[\phi^H(k, r) f^H(k, r') - \phi^H(k, r') f^H(k, r) \right] / f^H(k) \tag{19}$$

and

$$d_i^{(I)}(\beta_i, k) = \int\limits_0^\infty dr g_i(\beta_i, r) f^{HS}(k, r). \tag{20}$$

Solving Eq. (18) one gets

$$d_i^{(I)}(\beta_i, k) = \frac{1}{\det\ _N W^{HS}(\beta, k)} \sum_{i,j=1}^N w_{ij}^{HS}(\beta, k) Y_j^{HS}(k), \tag{21}$$

where

$$Y_j^{HS}(k) = \int\limits_0^\infty dr\, f^H(k, r) g_j(\beta_i, r) \tag{22}$$

Eq. (18) together with Eqs. (21) and (22) yields

$$f^{HS}(k,r) = f^H(k,r) + \frac{1}{\det \,_N W^{HS}(\beta,k)} \sum_{i,j=1}^{N} \lambda_i w_{ij}^{HS}(\beta,k) Y_j^{HS}(k) T_i^{HS}(\beta_i,r) \tag{23}$$

with

$$T_i^{HS}(\beta_i,r) = \int_r^{\infty} dr' g_i(\beta_i,r') G^{(I)H}(r,r'). \tag{24}$$

Combining Eqs. (2), (5), and (23) one can construct the expression for $V_{EQ}(k,r)$ with the irregular boundary condition. When the electromagnetic interaction $V_H(r)$ is turned off one obtains the equivalent local interactions for the pure nonlocal potential for both the boundary conditions under consideration.

2.2.1. Case study: Hulthén plus Yamaguchi potential

Since its appearance the Yamaguchi potential [21] has become an immensely popular tool in dynamical calculation of two-nucleon or nucleon–nucleus systems. The rank one Yamaguchi potential with symmetric form factors is written as

$$V(r,r') = \lambda\, g(\beta,r)\, g(\beta,r') = \lambda\, e^{-\beta r}\, e^{-\beta r'}, \tag{25}$$

where λ is the strength and β the inverse range parameter. From Eqs. (16) and (23) the regular and irregular solutions [28] for the Hulthén plus Yamaguchi potential is obtained as

$$\phi^{HY}(k,r) = \phi^H(k,r) + \frac{\lambda}{W^{HY}(\beta,k)} Z^{HY}(k) S^{HY}(\beta,r) \tag{26}$$

and

$$f^{HY}(k,r) = f^H(k,r) + \frac{\lambda}{W^{HY}(\beta,k)} Y^{HY}(k) T^{HY}(\beta,r). \tag{27}$$

The quantity $W^{HY}(\beta,k)$ is the Fredholm determinant associated with the regular and irregular boundary conditions, $Z^{HY}(k)$ and $Y^{HY}(k)$ are the transforms of the regular and irregular solutions of the pure Hulthén interaction by the form factors of the Yamaguchi potential, and $S^{HY}(\beta,r)$ and $T^{HY}(\beta,r)$ denote the transforms of the regular and irregular Hulthén Green's functions by the Yamaguchi form factors. In the following we shall evaluate the above quantities by exploiting the standard integrals associated with the special functions of mathematical physics. The factors

$$Z^{HY}(k) = \lambda \int_0^{\infty} dr\, e^{-\beta r}\, \phi^H(k,r) = a^2 \frac{\Gamma\left((\beta - ik)a\right) \Gamma\left((\beta + ik)a\right)}{\Gamma\left(1 + (\beta - ik)a - A\right) \Gamma\left(1 + (\beta - ik)a - B\right)} \tag{28}$$

and

$$Y^{HY}(k) = \lambda \int_0^{\infty} dr\, e^{-\beta r}\, f^H(k,r) = \frac{1}{(\beta - ik)}\; _3F_2\left(A,\, B,\, (\beta - ik)a;\, C,\, 1 + (\beta - ik)a;\, 1\right). \tag{29}$$

In evaluating the above integrals we have used the following standard integral [29–31]:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1} \, {}_2F_1\left(\alpha, \beta; \gamma; x\right) \, dx = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)} \, {}_3F_2\left(\alpha, \beta, \rho; \gamma, \rho+\sigma; 1\right) ;$$

$$[Re\rho > 0, Re\sigma > 0, (\gamma + \sigma - \alpha - \beta) > 0]$$

(30)

The evaluation of two indefinite integrals $S^{HY}(\beta, r)$ and $T^{HY}(\beta, r)$ is rather tricky. From Eqs. (7), (9)–(11) the regular Hulthén Green's function $G^{(R)H}(r, r')$ is expressed as [28]

$$G^{(R)H}(r, r') = \frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(C)} a \, e^{ik(r+r')} \left[(1 - e^{-r/a}) \, {}_2F_1\left(1 + A, 1 + B; 2; 1 - e^{-r/a}\right)\right.$$

$$\left. \times \, {}_2F_1\left(A, B; C; e^{-r'/a}\right) - (1 - e^{-r'/a}) \, {}_2F_1\left(A, B; C; e^{-r/a}\right) \, {}_2F_1\left(1+A, 1+B; 2; 1 - e^{-r'/a}\right)\right]$$

(31)

Transforming the $_2F_1\left(A, B; C; *\right)$ in Eq. (31) by the recurrence relation [29–31]

$$_2F_1\left(a, b; c; z\right) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \, {}_2F_1\left(a, b; a + b - c + 1; 1 - z\right) + (1 - z)^{c-a-b}$$

$$\times \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \, {}_2F_1\left(c - a, c - b; c - a - b + 1; 1 - z\right)$$

(32)

we get

$$G^{(R)H}(r, r') = \lim_{\varepsilon \to 0} a e^{ik(r+r')} \left[\left(1 - e^{-r/a}\right) \, {}_2F_1\left(1 + A, 1 + B; 2; 1 - e^{-r/a}\right) \times \right.$$

$$\times \, {}_2F_1\left(A, B; \varepsilon; 1 - e^{-r'/a}\right) - \left(1 - e^{-r'/a}\right) \, {}_2F_1\left(1 + A, 1 + B; 2; 1 - e^{-r'/a}\right) \times$$

$$\left. \times \, {}_2F_1\left(A, B; \varepsilon; 1 - e^{-r/a}\right)\right]$$

(33)

Substitution of Eq. (33) in Eq. (17), transformation of the independent variable $z' = \left(1 - e^{-r'/a}\right)$ along with the series expansion of z' as

$$(1 - z')^{(\beta+ik)a - 1} = \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - (\beta + ik)a)}{\Gamma(1 - (\beta + ik)a)} \frac{z'^n}{n!}$$

(34)

leads to

$$S^{HY}(\beta, r) = a^2 \, e^{ikr} \left(1 - e^{-r/a}\right) \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - (\beta + ik)a)}{\Gamma(1 - (\beta + ik)a) \, n!} \, f_{n+1}\left(A + 1, B + 1; 2; 1 - e^{r/a}\right).$$

(35)

In deriving the above expression we have made use of the following standard integral [32]:

$$f_\sigma\left(a, b; c; z\right) = \frac{1}{(c-1)} \left[{}_2F_1\left(a, b; c; z\right) \int_0^z dz' \, z'^{\sigma-1}(1 - z')^{a+b-c} \, {}_2F_1\left(a - c + 1, b - c + 1; 2 - c; z'\right)\right.$$

$$\left. - z^{1-c} \, {}_2F_1\left(a - c + 1, b - c + 1; 2 - c; z\right) \int_0^z dz' \, z'^{\sigma+c-2}(1 - z')^{a+b-c} \, {}_2F_1\left(a, b; c; z'\right)\right]$$

(36)

Combining Eqs. (19) and (24) the indefinite integral $T^{HY}(\beta, r)$ may be rewritten as

$$
\begin{aligned}
T^{HY}(\beta, r) &= \int\limits_{r}^{\infty} dr' g(\beta, r') G^{(I)H}(r, r') = \int\limits_{0}^{\infty} dr' g(\beta, r') G^{(I)H}(r, r') + \int\limits_{0}^{r} dr' g(\beta, r') G^{(R)H}(r, r') \\
&= \int\limits_{0}^{\infty} dr' g(\beta, r') G^{(I)H}(r, r') + S^{HY}(\beta, r)
\end{aligned}
\tag{37}
$$

Substitution of Eq. (19) together with Eqs. (9)–(12c) in Eq. (37), evaluation of the definite integrals in the light of Eq. (30), and some algebraic manipulation lead to

$$
T^{HY}(\beta, r) = S^{HY}(\beta, r) - \frac{Y^{HY}(k)}{f^H(k)} \phi^H(k, r) + \frac{Z^{HY}(k)}{f^H(k)} f^H(k, r).
\tag{38}
$$

The double transform of the regular Hulthén Green's function $G^{(R)H}(r, r')$ by the form factors of the Yamaguchi potential can easily be obtained by straightforward integration of Eq. (35). From Eq. (14) we write

$$
W^{HS}(\beta, k) = 1 - \lambda \, G^{(R)H}(\beta, k),
\tag{39}
$$

where

$$
G^{(R)H}(\beta, k) = \int\limits_{0}^{\infty} \int\limits_{0}^{r} dr \, dr' \, e^{-\beta r} G^{(R)H}(r, r') \, e^{-\beta r'} = \int\limits_{0}^{\infty} dr \, e^{-\beta r} \, S^{HY}(\beta, r).
\tag{40}
$$

Substitution of Eq. (35) in (40) and change of independent variable by $z = \left(1 - e^{-r/a}\right)$ yields

$$
G^{(R)H}(\beta, k) = a^3 \sum_{n=0}^{\infty} \frac{\Gamma(n+1-(\beta+ik)a)\Gamma(n+2)\Gamma((\beta-ik)a)}{n! \, \Gamma(1-(\beta+ik)a)\Gamma(n+2+(\beta-ik)a)} f_{n+1}\left(1+A, \, 1+B; \, 2+(\beta-ik)a; \, 1\right).
\tag{41}
$$

In deriving Eq. (40) the following standard integral has been used [32]:

$$
\begin{aligned}
&\int\limits_{0}^{1} dz \, z^{c-1} \, (1-z)^{\nu-1} f_\sigma\left(a, \, b; \, c; \, pz\right) = \tfrac{\Gamma(\sigma+c-1)\Gamma(\nu)}{\Gamma(\sigma+c+\nu-1)} f_\sigma\left(a, \, b; \, c+\nu; \, p\right), \\
&[Re\,\nu > 0, \;\; Re\,\sigma > 0, \;\; Re\,(\sigma+c) > 1, \;\; |p| < 1]
\end{aligned}
\tag{42}
$$

By exploiting the relation [32]

$$
f_\sigma\left(a, \, b; \, c; \, z\right) = \frac{z^\sigma}{\sigma\,(\sigma+c-1)} \; {}_3F_2\left(1, \, \sigma+a, \, \sigma+b; \, \sigma+1, \, \sigma+c; \, z\right)
\tag{43}
$$

we arrive at

$$
\begin{aligned}
G^{(R)H}(\beta, k) &= a^3 \sum_{n=0}^{\infty} \tfrac{\Gamma(n+1-(\beta+ik)a)\Gamma((\beta-ik)a)}{\Gamma(1-(\beta+ik)a)\Gamma(n+3+(\beta-ik)a)} \times \\
&\quad {}_3F_2\left(1, \, n+A+2, \, n+B+2; \, n+2, \, n+3+(\beta-ik)a; \, 1\right)
\end{aligned}
\tag{44}
$$

Eq. (44) involves an unpleasing infinite sum over ${}_3F_2(*)$ function and may not be suitable for numerical computation. To that end we shall express it in its maximal reduced form. Using the following transformation

formulas [33] for the $_3F_2(*)$ function

$$
\begin{aligned}
{}_3F_2\left(a,\,b,\,c;\,e,\,f,\,1\right) \;=\; & \tfrac{\Gamma(e)\Gamma(e-a-b)}{\Gamma(e-a)\Gamma(e-b)}\;{}_3F_2\left(a,\,b,\,f-c;\,a+b-c+1,\,f,\,1\right)+\tfrac{\Gamma(e)\Gamma(f)\Gamma(a+b-e)\Gamma(e+f-a-b-c)}{\Gamma(a)\Gamma(b)\Gamma(f-c)\Gamma(e+f-a-b)} \\
& \times\;{}_3F_2\left(e-a,\,e-b,\,e+f-a-b-c;\,e-a-b+1,\,e+f-a-b,\,1\right)
\end{aligned}
, \tag{45}
$$

$$
{}_3F_2\left(a,\,b,\,c;\,e,\,f,\,1\right)=\frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\;{}_3F_2\left(s,\,e-a,\,f-a;\,s+b,\,s+c,\,1\right) \tag{46}
$$

and

$$
{}_3F_2\left(a,\,b,\,c;\,e,\,f,\,1\right)=\frac{\Gamma(e+f-a-b-c)\Gamma(f)}{\Gamma(f-c)\Gamma(e+f-a-b)}\;{}_3F_2\left(e-a,\,e-b,\,c;\,e,\,e+f-a-b;\,1\right) \tag{47}
$$

in Eq. (44) we obtain

$$
\begin{aligned}
G^{(R)H}(\beta,k) \;=\; & -a^3\frac{\Gamma((\beta-ik)a)}{(1+A)(1+B)\Gamma(1-(\beta+ik)a)}\sum_{n=0}^{\infty}\frac{(n+1)\Gamma(n+1-(\beta+ik)a)}{\Gamma(n+2+(\beta-ik)a)}\times{}_3F_2\left(-n,\,1,\,1-(\beta+ik)a;\,A+2,\,B+2;\,1\right) \\
& +a^2\frac{\Gamma(1+A)\Gamma(1+B)\Gamma((\beta-ik)a)\Gamma((\beta+ik)a)}{(\beta-ik)\Gamma(C)\Gamma(1+(\beta-ik)a-A)\Gamma(1+(\beta-ik)a-B)}\times\;{}_3F_2\left(A,\,B,\,(\beta-ik)a;\,1+(\beta-ik)a,\,C;\,1\right)
\end{aligned}
. \tag{48}
$$

Eqs. (39) and (48) express the Fredholm determinant for the regular and irregular boundary conditions for motion in Hulthén plus Yamaguchi potential in the maximal reduced form. Thus, combination of Eqs. (2), (5), (9), (12), (13), (25), (26), (28), (35), (39), and (48) produces the desired expression for the energy-dependent local interaction for the Hulthén plus Yamaguchi potential $V_{EQ}^{HY(R)}(k,r)$ with the regular boundary condition written as

$$
V_{EQ}^{HY(R)}(k,r)=V_H(r)+\lambda\frac{Z^{HY}(k)}{\phi^{HY}(k,r)W^{HY}(\beta,k)}e^{-\beta\,r}. \tag{49}
$$

While from Eqs. (2), (5), (10), (12), (21), (25), (27), (29), (38), (39), and (48) the same for the irregular boundary condition is obtained as

$$
V_{EQ}^{HY(I)}(k,r)=V_H(r)+\lambda\frac{Y^{HY}(k)}{f^{HY}(k,r)W^{HY}(\beta,k)}e^{-\beta\,r}. \tag{50}
$$

For future use we designate the above two interactions, constructed from the exact regular and irregular solutions of the Hulthén plus Yamaguchi potential, as exact equivalent potentials $V_{EX}^{(R)}(k,r)=V_{EQ}^{HY(R)}(k,r)$ and $V_{EX}^{(I)}(k,r)=V_{EQ}^{HY(I)}(k,r)$.

2.2.2. Case study: pure Yamaguchi potential

The regular and irregular solutions of the Yamaguchi potential are given by [34]

$$
\phi^{(Y)}(k,r)=k^{-1}\sin kr+\frac{\lambda\,d^{(R)Y}(\beta,k)}{(\beta^2+k^2)}\left(e^{-\beta\,r}+\beta\,k^{-1}\sin kr-\cos kr\right) \tag{51}
$$

$$
f^{(Y)}(k,r)=e^{ikr}+\frac{\lambda\,d^{(I)Y}(\beta,k)}{(\alpha^2+k^2)}e^{-\beta\,r} \tag{52}
$$

with

$$d^{(R)}(\beta, k) = \frac{1}{W^{(Y)}(\beta, k)(\beta^2 + k^2)}, \tag{53}$$

$$d^{(I)}(\beta, k) = \frac{(\beta + ik)}{W^{(Y)}(\beta, k)(\beta^2 + k^2)} \tag{54}$$

and

$$W^{(Y)}(\beta, k) = 1 - \frac{\lambda}{2\beta(\beta^2 + k^2)}. \tag{55}$$

Therefore, Eq. (5) with $V_H(r) = 0$ in conjunction with Eqs. (49)–(53) produces the equivalent potentials for the nonlocal Yamaguchi one. For charged hadron scattering one deals with the problem by adding the electromagnetic interaction to the nuclear part. Thus, the total energy-dependent potentials, hereby designated as approximate energy-dependent local interactions, for (p-p) and (α-p) systems for the regular and irregular boundary conditions are written as

$$V_{AP}^{(R)}(k, r) = V_H(r) + \frac{\lambda}{(\beta^2 + k^2)\phi^Y(k, r)W^Y(\beta, k)} e^{-\beta r} \tag{56}$$

and

$$V_{AP}^{(I)}(k, r) = V_H(r) + \frac{\lambda(\beta + ik)}{(\beta^2 + k^2)f^Y(k, r)W^Y(\beta, k)} e^{-\beta r}. \tag{57}$$

Thus far we have constructed energy-dependent local interactions for the Hulthén plus nonlocal Yamaguchi potential with regular and irregular boundary conditions and these will be applied to compute scattering phase shifts for the nucleon–nucleon and alpha–nucleon systems.

3. Results and discussions

The phase function method is an efficient approach for computing the scattering phase shifts for quantum mechanical problems involving local [35] and nonlocal interactions [36,37]. We shall compute the phase shifts for the systems under consideration by applying the phase equation

$$\delta_\ell'(k, r) = -k^{-1} V(r) [\hat{j}_\ell(kr) \cos \delta_\ell(k, r) - \hat{\eta}_\ell(kr) \sin \delta_\ell(k, r)]^2, \tag{58}$$

where $\hat{j}_\ell(kr)$ and $\hat{\eta}_\ell(kr)$ are the Riccati Bessel functions with $\hat{h}_\ell^{(1)}(x) = -\hat{\eta}_\ell(x) + i\hat{j}_\ell(x)$. The scattering phase shift $\delta_\ell(k)$ is obtained by solving Eq. (54) from origin to asymptotic region with the initial condition $\delta_\ell(k, 0) = 0$.

Table. Strength $(\lambda\, fm^{-3})$ and range $(\beta\, fm^{-1})$ parameters for nucleon–nucleon and alpha–nucleon systems.

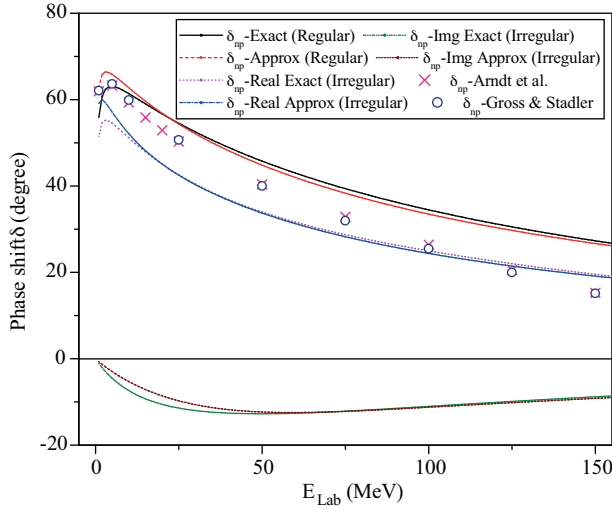| Nucleon–nucleon				Alpha–nucleon			
$^1S_0$ (n-p/p-p)		$^3S_1$ (n-p)		$1/2^{(+)}$ (α-n)		$1/2^{(+)}$ (α-p)	
λ	β	λ	β	λ	β	λ	β
−5.237	1.1045	−7.533	1.4054	−9.995	1.2	−13.56	1.3

Figure 1. $^1S_0$ phase shifts for the (n-p) system. Experimental data from Refs. [38] and [39].
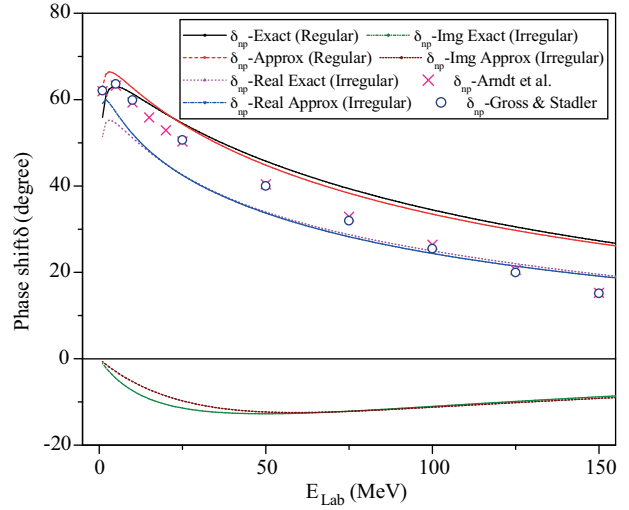

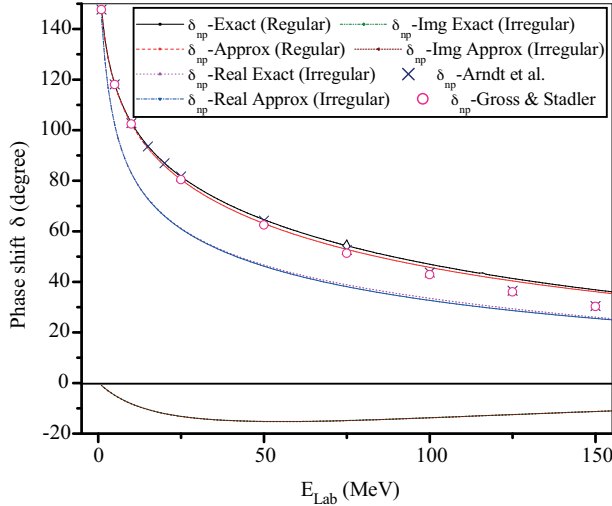**Figure 2.** $^1S_0$ phase shifts for the (p-p) system. Experimental data from Ref. [38].

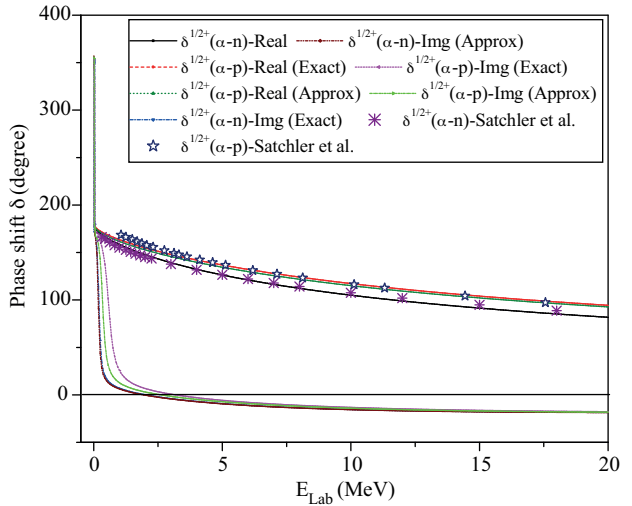**Figure 3.** $^3S_1$ phase shifts for the (n-p) system. Experimental data from Refs. [38] and [39].

Figure 4. S-wave phase shifts for the (α-n) and (α-p) system. Experimental data from Ref. [40].

With the parameters given in the Table we have computed the phase shifts for the systems under consideration by applying Eq. (58) and they are portrayed in Figures 1–5 with experimental data [38–40]. Moreover, the associated potentials as expressed in Eqs. (49), (50), (56), and (57) for those systems are depicted in Figures 6–10. We have chosen to work with $\frac{\hbar^2}{2m} = 41.47\,MeV\,fm^2$, $V_0a = 0.03472\,fm^{-1}$ and $\frac{\hbar^2}{2m} = 25.92\,MeV\,fm^2$, $V_0a = 0.05516\,fm^{-1}$ for nucleon–nucleon and alpha–nucleon systems [41,42] respectively. The scattering phase shifts for the (n-p) and (α-n) systems are obtained by putting $V_0 = 0$ in our numerical routine for the (p-p) and (α-p) systems, respectively, with proper parameters. Looking closely at Figures 1 and 2 it is observed that our computed $^1S_0$ phase shifts δ_{np}, δ_{pp}-Exact (Regular) and δ_{np}, δ_{pp}-Approx. (Regular) for the nucleon–nucleon systems are comparable with the experimental results [38,39] up to 50 MeV. Beyond 50 MeV our phase shift values started diverging from standard data [38,39]. This is not quite unexpected as the
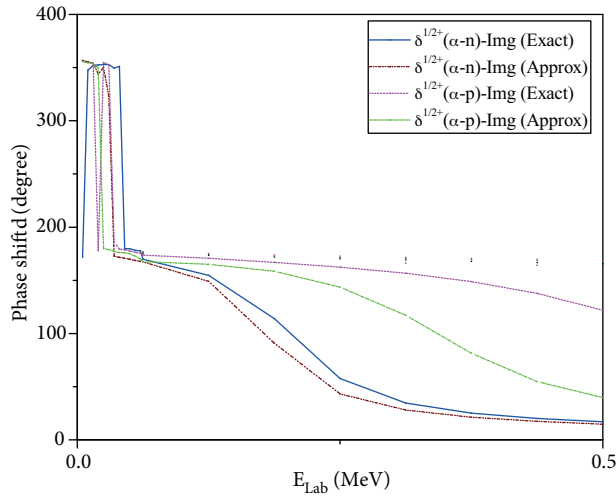
457

Figure 5. Low energy behavior of the phase shifts obtained from imaginary parts of the potentials for the (α-n) and (α-p) systems.
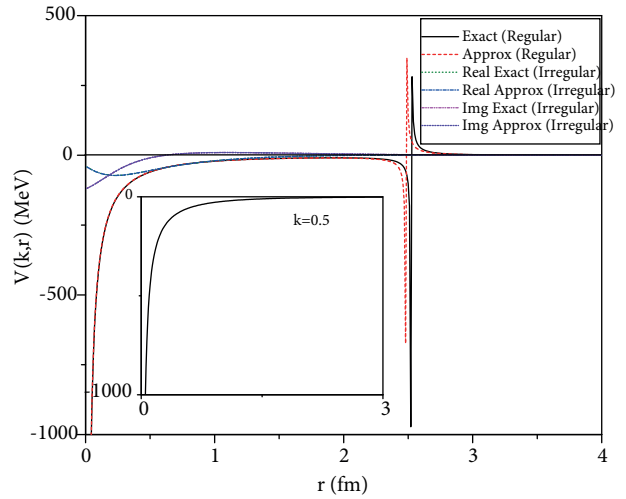

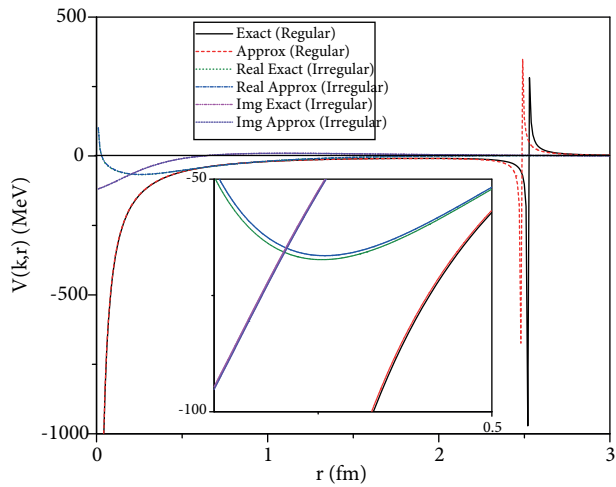**Figure 6.** $^1S_0$ potentials for the (n-p) system for k = 1.0 fm^{-1}.

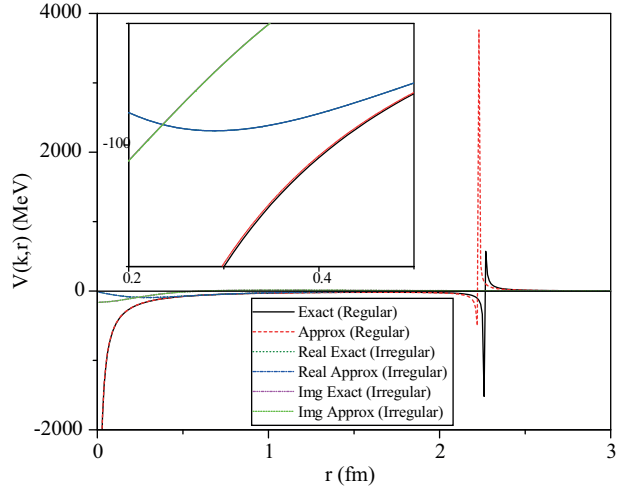**Figure 7.** $^1S_0$ potentials for the (p-p) system for k = 1.0 fm^{-1}.

Figure 8. $^3S_1$ potentials for the (n-p) system for k = 1.0 fm^{-1}.

pure nonlocal Yamaguchi [21] potential fits the nucleon–nucleon phase shifts in the low energy range. Those computed with the real potentials, derived from the irregular solutions, fit experimental results [38,39] beyond 75 MeV. It is noted that out of the two types of interactions, the $^1S_0$ potentials constructed with the regular solutions produce better results in the low energy range than the interactions developed from their irregular solutions. In Figure 3 it is observed that the $^3S_1$ potentials from regular solutions fit the experimental data accurately and are superior to their irregular counterparts. The potentials constructed via the regular solutions are real quantities while those from irregular solutions are complex. This is attributed to the fact that the regular solution for any potential is always a real quantity and the irregular one, in general, is complex in nature. We have also computed phase shifts for the imaginary part of the potential developed from irregular solutions and plotted them in Figures 1–3 for the nucleon–nucleon systems. All these phase shifts are negative within the entire range of energy. In Figure 4 the phase shifts for the alpha–nucleon systems are plotted with both kinds
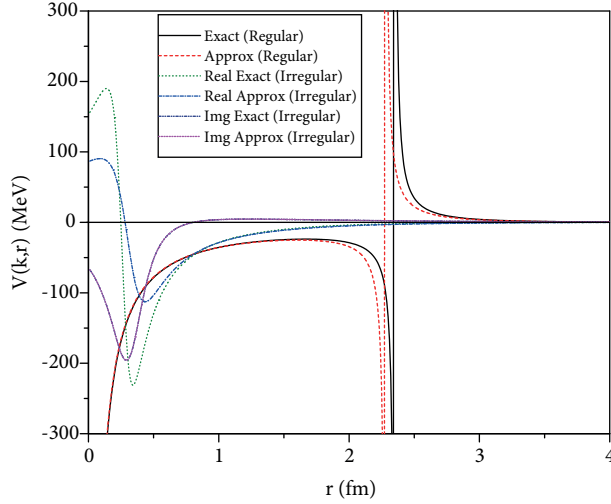
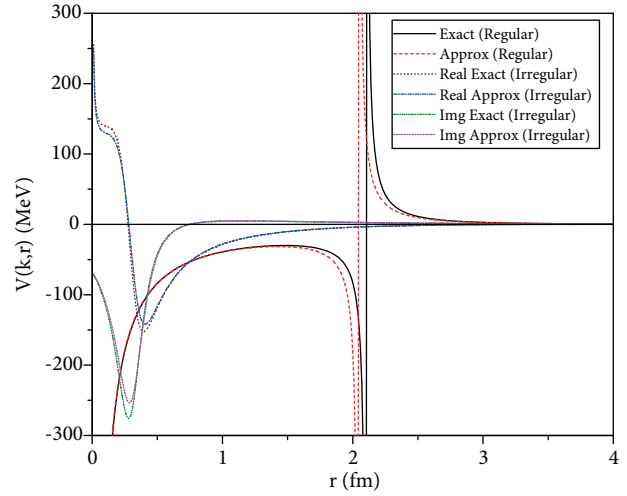**Figure 9.** S-wave potentials for the (α-n) system for k = 0.4 fm^{-1}.

Figure 10. S-wave potentials for the (α-p) system for k = 0.4 fm^{-1}.

of energy-dependent interactions as a function of laboratory energy and are found to be in close agreement with experimental data [40] within the entire energy range under consideration. In contrast to nucleon–nucleon cases, the real part of the potentials constructed with irregular solutions for the alpha–nucleon systems produce better agreements with experimental data [40] compared to its counterparts expressed in terms of regular solutions. The imaginary parts of the potentials constructed from the irregular solutions for the Hulthén plus Yamaguchi potential produce abrupt changes in phase shifts by π modulo in certain low energy intervals, which give some indications of resonances. In Figure 5 it is noted that the phase shifts for both the exact and approximate potentials for the (α-p) system change by about 180° in the energy intervals $E_{cm} = 10 - 40\, KeV\,\&\,25 - 30\ KeV$, respectively, while those for the (α-n) system occur at $E_{cm} = 10 - 20\,KeV\,\&\,5 - 10\,KeV$, respectively. The resonance phenomenon is caused by a capture of the incident missiles in the scattering region and delay in their emergence. The low energy resonances are associated with the introduction of bound states. Generally, the low energy resonances for S-waves do not occur due to absence of the centrifugal barriers but if the potential contains its own barrier, it may happen in the S-wave also. The imaginary parts of our potentials for the (α-n) and (α-p) systems (Figures 9 and 10) have barriers and are consistent with the above statement. These S-wave resonances are described as nonphysical in nature as no bound states for ^{5}Li and ^{5}He exist at these energies for $1/2^+$ states. The ground states of ^{5}Li and ^{5}He correspond to 3/2-state. However, the real parts of the said potentials do not support any such resonances.

As observed, the energy-dependent equivalent potentials, constructed via the regular solutions for the various systems under consideration, as shown in Figures 6–10, exhibit finite discontinuities at certain points within their ranges. For instance, the nucleon–nucleon potentials change sharply at about $r = 2.5 fm$ for $^1S_0$ state with $k = 1.0 fm^{-1}$ (Figures 6 and 7) and at about $r = 2.25 fm$ for $^3S_1$ state with $k = 1.0 fm^{-1}$ (Figure 8). Figures 9 and 10 show that for alpha–nucleon systems sharp changes in the potentials occur at about $r = 2.3\,\&\,2.1 fm$ for $k = 0.4\,fm^{-1}$. We have also verified that these sharp peaks in the potentials occur at smaller values of r as k increases. These sharp peaks vary from –700 to 4000 MeV for the (α-n) system and from –1500 to 2000 MeV for the (α-p) system, which are not observed in the scale of Figures 9 and 10. However, for $k \leq 0.8\,fm^{-1}$ $(E_{Lab} \leq 53\,MeV)$, we have verified (plotted in Figure 6 [k = 0.5-Exact] only) that

no abrupt changes in the $^1S_0$, $^3S_1$ (n-p), or $^1S_0$ (p-p) potentials occur and they vary smoothly within the range of interactions. These observations are fully consistent with our phase shift studies as they are in good agreement with those of refs. 38 and 39. Thus, one may conclude by noting that our localization procedure is valid in the low and intermediate energy range.

The observed finite discontinuities in the potentials associated with the regular solutions are originated due to the behavior of the regular solutions with distance. As r becomes large the regular wave function $\phi^{HS}(k, r)$ behaves as $Sin\ (kr + \delta)$ while the irregular solution $f^{HS}(k, r)$ goes as e^{ikr} [25]. Although the energy-dependent interactions with regular solutions show some unexpected behavior, this is not reflected in the phase shift calculations. The phase shift values are smooth and very much consistent with standard data. This is attributed to the fact that the resultant contributions to scattering phase shifts from either side of the point of finite discontinuities in the related potentials are of definite values. On the other hand, the real and imaginary parts of the potentials computed via the irregular solutions exhibit no unforeseen behavior and change smoothly with distance.

4. Conclusion

In this paper we have localized the separable nonlocal interactions by the use of Green's functions with regular and irregular boundary conditions to fit nucleon–nucleon and alpha–nucleon phase shifts. Our energy-dependent equivalent local potentials derived via the regular and irregular solutions produce more or less the same phase shifts for the systems under consideration and are in reasonable agreement with the experimental data, particularly at low and intermediate energies. This is quite expected since the result of inversion from nonlocal to local potentials should not depend on the boundary conditions imposed on the input information. The alpha–nucleon problem has been one of the few meaningful two-body problems in nuclear physics. In this direction a large number of phenomenological potentials (both local and nonlocal) have been constructed for possible applications to alpha–nucleus scattering in the spirit of the folding model. It is noted that the Woods–Saxon potentials extracted from the phenomenological studies of Satchler et al. [40] fitted the tail of (n-α) potential thus generated in a larger part at higher energy. The first analysis of the (n-α) data with nonlocal separable potential was performed by Mitra et al. [43]. For the S-wave scattering they observed that the spin-orbit potential was much smaller than the central potential and the interaction admitted a bound state. It is worthwhile to mention that the nonlocal separable or energy-dependent local interactions of various shapes are generally used in the folding models for alpha–nucleus scattering. The present text also dealt with the energy-dependent interaction without spin–orbit potential. Thus, the alpha–nucleon scattering, apart from being sufficiently interesting by itself, is expected to provide a deeper understanding of the alpha–nucleus interaction. It would be desirable to extend our results to higher partial waves. Such an effort requires analytical solutions for the Hulthén potential for $\ell > 0$. With the formalism of supersymmetry inspired factorization method [27,42,44] one can construct higher partial wave solutions for the Hulthén potential only. However, the constructions of higher partial wave solutions for the Hulthén plus separable potentials involve inordinate complications. However, it is under our active consideration and will be communicated in a future correspondence. The present approach to the problem is much simpler and more straightforward than the earlier approaches by McTavish [12] and Talukdar et al. [13]. Our formalism can also be extended for nucleus–nucleus elastic scattering and restriction to symmetric form factors is not compelling. As it is of importance to have in the literature alternative approaches to the problem for calculation of physical observables of a particular system, it is our belief that the present treatment deserves some attention.

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