

Analytical solution of the local fractional Klein–Gordon equation for generalized Hulthen potential

Hasibe Hale KARAYER^{1,*}, Ahmet Doğan DEMİRHAN², Fevzi BÜYÜKKILIÇ³
¹Department of Physics, Faculty of Arts and Sciences, Kırklareli University, Kırklareli, Turkey
²Faculty of Sport Sciences, Ege University, İzmir, Turkey

³Department of Physics, Faculty of Science, Ege University, İzmir, Turkey

Received: 11.07.2017

Accepted/Published Online: 12.09.2017

Final Version: 18.12.2017

Abstract: The one-dimensional Klein–Gordon (KG) equation is investigated in the domain of conformable fractional calculus for one-dimensional scalar potential, namely generalized Hulthen potential. The conformable fractional calculus is based on conformable fractional derivative, which is the most natural definition in noninteger order calculus. Fractional order differential equations can be solved analytically by means of this derivative operator. We obtained exact eigenvalue and eigenfunction solutions of the local fractional KG equation and investigated the evolution of relativistic effects in correspondence with the fractional order.

Key words: Local fractional Klein–Gordon equation, conformable fractional calculus, conformable fractional Nikiforov–Uvarov method, generalized Hulthen potential.

1. Introduction

Relativistic wave equations, namely the Dirac equation and Klein–Gordon (KG) equation, have great importance in efforts to determine the dynamics of a relativistic particle in relativistic quantum mechanics. Solution of the KG equation explains the behavior of a spinless particle of rest mass m at high energies and velocities comparable to the speed of light. Bound state solutions of the KG equation have been studied by many authors in the literature; see [1–6] and references therein. Different methods can be used to obtain exact or approximate solutions of KG equations written for various potential functions. The Nikiforov–Uvarov (NU) method, supersymmetric quantum mechanics, factorization method, and asymptotic iteration method are the most frequently used methods [7].

Using the theory of fractional calculus, which is based on noninteger order differentiations and integrations, many physical phenomena can be described successfully [8–11]. Consequently differential equations that describe physical systems are handled in the fractional domain. Various definitions have been proposed for the fractional order differential and integral operators. Predominant definitions are Riemann–Liouville and Caputo definitions [12].

$${}^{RL}D_a^\mu f(x) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\mu-1} f(t) dt,$$

*Correspondence: hale.karayer@gmail.com

$${}^C D_a^\mu f(x) = \frac{1}{\Gamma(n - \mu)} \int_a^x (x - t)^{n-\mu-1} f^{(n)}(t) dt,$$

where $\mu \in R$, $n - 1 \leq \mu < n$, and the superscripts *RL* and *C* stand for Riemann–Liouville and Caputo, respectively. These are nonlocal operators and do not satisfy classical properties such as chain, product, and quotient rules, which allow us to achieve an analytical solution in the standard calculus. In 2014 a local form of fractional derivative operator was defined by Khalil et al. [13]. This is the most natural fractional order derivative operator that provides the above-mentioned rules. Thereafter conformable fractional calculus theory [14], applicability of this definition in quantum mechanics [15], and solution of the fractional Schrödinger equation [16] are studied in view of this local fractional derivative definition. Although the definition fails some properties that are pointed out by Ortigueira and Machado [17], it is more suitable for applications as compared with Riemann–Liouville or Caputo fractional derivative operators for real physical problems [18].

In a recent work, we derived the conformable fractional form of the NU method and solved the local fractional Schrödinger equation for harmonic oscillator potential, Hulthen potential, and Woods–Saxon potential in order to present the accuracy of the method [19].

The aim of the present work was to solve the fractional order one-dimensional time independent KG equation for the generalized Hulthen potential in the scalar coupling scheme using the conformable fractional NU method. The manuscript is organized as follows: in Sec. 2 the formalism of the KG equation for the generalized Hulthen potential is briefly outlined. In Sec. 3 the definition of the conformable fractional derivative operator and conformable fractional NU method are reviewed. In Sec. 4 we present the analytical solution of the fractional KG equation for the generalized Hulthen potential. Finally, conclusions are discussed in the last section.

2. Formalism of the KG equation with the generalized Hulthen potential in scalar scheme

The one-dimensional time independent KG equation for a spinless particle of rest mass m in the presence of vector and scalar potentials is given by

$$\psi''(x) + \frac{1}{\hbar^2 c^2} \left[(E - V(x))^2 - (mc^2 + S(x))^2 \right] \psi(x) = 0, \tag{1}$$

where $V(x)$ and $S(x)$ are vector and scalar potentials, respectively. For the existence of bound state solutions it is required that $S(x) > V(x)$ [2]. When $V(x) = 0$, the one-dimensional KG equation for a given scalar potential $S(x)$ is reduced to the following form:

$$\psi''(x) + \frac{1}{\hbar^2 c^2} \left[E^2 - (mc^2 + S(x))^2 \right] \psi(x) = 0, \tag{2}$$

In this case Eq. (2) can be transformed to a second-order Schrödinger-like differential equation:

$$\psi''(x) + \frac{2m}{\hbar^2} \left[E_{eff} - U_{eff}(x) \right] \psi(x) = 0, \tag{3}$$

where E_{eff} and U_{eff} are effective energy and effective potential given by

$$E_{eff} = \frac{E^2 - m^2 c^4}{2mc^2}, \quad U_{eff}(x) = \frac{S^2(x)}{2mc^2} + S(x). \tag{4}$$

Therefore, Eq. (2) can be rewritten in the following form for $\hbar = c = 1$ [2]:

$$\psi''(x) + [-S^2(x) - 2mS(x) - (m^2 - E^2)]\psi(x) = 0. \tag{5}$$

In order to specify the dynamics of a relativistic particle in a scalar potential, the potential function $S(x)$ is inserted in this equation. Here the potential function is chosen as generalized Hulthen function, which is given by

$$S(x) = -S_0 \frac{e^{-\alpha x}}{1 - qe^{-\alpha x}}, \tag{6}$$

where q is a deformation parameter. This potential transforms to exponential potential, standard Hulthen potential, and Woods–Saxon potential for $q = 0$, $q = 1$ and $q = -1$, respectively. Substituting the potential function given by Eq. (6) in Eq. (5) and using a transformation $z = S_0 e^{-\alpha x}$ the following hypergeometric type differential equation is obtained [2]:

$$\begin{aligned} \psi_q''(z) + \frac{S_0 - qz}{z(S_0 - qz)}\psi_q'(z) + \frac{1}{[z(S_0 - qz)]^2}[-(\gamma^2 + \\ q\beta^2 + q^2\epsilon^2)z^2 + S_0(\beta^2 + 2q\epsilon^2)z - S_0^2\epsilon^2]\psi_q(z) = 0 \end{aligned} \tag{7}$$

for which

$$\gamma^2 = \frac{S_0^2}{\alpha^2}, \quad \beta^2 = \frac{2mS_0}{\alpha^2}, \quad \epsilon^2 = \frac{1}{\alpha^2}(m^2 - E^2). \tag{8}$$

3. A brief review of the conformable fractional derivative operator and conformable fractional NU method

The local fractional derivative operator, which is a natural extension of the standard derivative definition, was introduced by Khalil et al. for the first time:

$$D^\mu[f(t)] = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\mu}) - f(t)}{\epsilon}, \quad t > 0 \tag{9}$$

$$f^{(\mu)}(0) = \lim_{t \rightarrow 0^+} f^{(\mu)}(t) \tag{10}$$

where $0 < \mu \leq 1$ and D^μ is the local fractional derivative operator [13]. This operator provides the basic rules such as product, quotient, and chain rules, which are valid in standard calculus:

$$D^\mu[af + bg] = aD^\mu[f] + bD^\mu[g] \quad \textit{linearity}$$

$$D^\mu[f g] = f D^\mu[g] + g D^\mu[f] \quad \textit{product rule}$$

$$D^\mu[f(g)] = \frac{df}{dg} D^\mu[g] \quad \textit{chain rule}$$

$$D^\mu[f] = t^{1-\mu} f' \quad \textit{where } f' = \frac{df}{dt}.$$

The last property is called the key property of the definition. If f is differentiable, then the μ th order derivative of f is equal to the product of its first-order derivative with $t^{1-\mu}$. Since fractional order differential equations

have great importance in describing physical systems with a realistic approach, some appropriate methods are derived to solve these equations. The NU method is a well known method that gives exact solutions of second-order linear differential equations. In quantum mechanics the method has been used to solve Schrödinger-like differential equations for various potentials. The method is based on reducing the handled equation to a hypergeometric type second-order differential equation:

$$\psi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z) = 0, \tag{11}$$

where $\tilde{\tau}(z)$ is a polynomial of at most first-degree, $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials of at most second-degree and $\psi(z)$ is a function of hypergeometric-type [20]. Then the reduced equation, which is called the basic equation of the method, can be solved systematically by means of special orthogonal functions and eigenstate solutions can be achieved completely [20–23].

The conformable fractional form of this method is introduced in order to solve the conformable fractional order Schrödinger equation and was presented in our recent work [19]. In the case of the conformable fractional NU method fractional orders are inserted in the basic equation. Then using the key property of the conformable fractional derivative operator one can obtain the following second-order differential equation:

$$\psi''(z) + \frac{\tilde{\tau}_f(z)}{\sigma_f(z)}\psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma_f^2(z)}\psi(z) = 0, \tag{12}$$

where $\tilde{\tau}_f(z) = (1-\mu)z^{-\mu}\sigma(z) + \tilde{\tau}(z)$ and $\sigma_f(z) = z^{1-\mu}\sigma(z)$ and the subscript f stands for fractional. Boundary conditions of the conformable fractional NU method are determined by the degrees of the coefficients in the basic equation of the method given by Eq. (12). Here $\tilde{\tau}_f(z)$ is a function of at most μ th degree (which means that this function can also be equal to a constant), $\sigma_f(z)$ is a function of at most $(\mu + 1)$ th (i.e. the degree of this function can also be equal to 1) and $\tilde{\sigma}(z)$ is a function of at most 2μ th degree (i.e. the degree of this function can also be equal to 0 or μ). If any fractional order differential equation is reduced to the basic equation using the key property of the local fractional derivative operator, then it can be solved analytically by the conformable fractional NU method.

After determining the following newly defined functions related to the initial functions in the basic equation, the eigenvalue and eigenfunction solution of Eq. (12) can be obtained:

$$\pi_f(z) = \frac{\sigma'_f(z) - \tilde{\tau}_f(z)}{2} \pm \sqrt{\left(\frac{\sigma'_f(z) - \tilde{\tau}_f(z)}{2}\right)^2 - \tilde{\sigma}(z) + k(z)\sigma_f(z)}. \tag{13}$$

Recall that $\pi_f(z)$ is a function of at most μ th degree. Providing this condition the expression under the square root sign must be the square of a μ th-order function. Thus the function $k(z)$ under the square root sign must be chosen properly.

$$\tau_f(z) = \tilde{\tau}_f(z) + 2\pi_f(z). \tag{14}$$

$$\lambda(z) = k(z) + \pi'_f(z). \tag{15}$$

$$\lambda_n(z) = -n\tau'_f(z) - \frac{n(n-1)}{2}\sigma''_f(z) \quad (n = 0, 1, 2, \dots). \tag{16}$$

In order to obtain the eigenvalue solution, the function $\lambda(z)$ in Eq. (15) is taken equal to $\lambda_n(z)$ in Eq. (16). For the eigenfunction solution, functions $\phi(z)$ and $y_n(z)$ given by

$$\frac{\phi'(z)}{\phi(z)} = \frac{\pi_f(z)}{\sigma_f(z)}, \tag{17}$$

$$(\sigma_f(z)\rho(z))' = \tau_f(z)\rho(z). \tag{18}$$

$$y_n(z) = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma_f^n(z)\rho(z)], \tag{19}$$

are inserted in $\psi(z) = \phi(z)y(z)$.

4. Solution of the conformable fractional KG equation for the generalized Hulthen potential

The conformable fractional form of the one-dimensional KG equation for the generalized Hulthen potential given by Eq. (7) is written by replacing integer orders with fractional orders:

$$D^\mu D^\mu \psi_q(z) + \frac{S_0 - qz^\mu}{z^\mu(S_0 - qz^\mu)} D^\mu \psi_q(z) + \frac{1}{[z^\mu(S_0 - qz^\mu)]^2} [-\gamma^2 + q\beta^2 + q^2\epsilon^2]z^{2\mu} + S_0(\beta^2 + 2q\epsilon^2)z^\mu - S_0^2\epsilon^2] \psi_q(z) = 0 \tag{20}$$

Using the key property of the conformable fractional derivative definition, Eq. (20) can be transformed to a second-order differential equation:

$$\psi_q''(z) + \frac{(S_0 - qz^\mu)(2 - \mu)}{z(S_0 - qz^\mu)} \psi_q'(z) + \frac{1}{[z(S_0 - qz^\mu)]^2} [-\gamma^2 + q\beta^2 + q^2\epsilon^2]z^{2\mu} + S_0(\beta^2 + 2q\epsilon^2)z^\mu - S_0^2\epsilon^2] \psi_q(z) = 0. \tag{21}$$

Comparing this equation with the basic equation of the method, the parameters in Eq. (12) are determined as

$$\begin{aligned} \tilde{\tau}_f(z) &= (S_0 - qz^\mu)(2 - \mu) \\ \sigma_f(z) &= z(S_0 - qz^\mu) \\ \tilde{\sigma}(z) &= -(\gamma^2 + q\beta^2 + q^2\epsilon^2)z^{2\mu} + S_0(\beta^2 + 2q\epsilon^2)z^\mu - S_0^2\epsilon^2. \end{aligned} \tag{22}$$

Since $\tilde{\tau}_f(z)$, $\sigma_f(z)$, and $\tilde{\sigma}(z)$ are μ th-, $(\mu + 1)$ th-, and 2μ th-order, the conformable fractional NU method can be used in order to obtain the bound state solutions of the local fractional KG equation for the generalized Hulthen potential. After substituting the parameters given by Eq. (22) into Eq. (13), the function $\pi_f(z)$ can be obtained as

$$\begin{aligned} \pi_f(z) &= \frac{1}{2} \{ (\mu - 1)S_0 - qz^\mu(2\mu - 1) \pm \\ &[[q^2(2\mu - 1)^2 + 4(\gamma^2 + q\beta^2 + q^2\epsilon^2) - 4k_\mu q]z^{2\mu} + [-2(\mu - 1)(2\mu - 1)qS_0 - \\ &4S_0(\beta^2 + 2q\epsilon^2) + 4k_\mu S_0]z^\mu + 4S_0^2\epsilon^2 + (\mu - 1)^2 S_0^2]^\frac{1}{2} \}. \end{aligned} \tag{23}$$

For the requirement of $\pi_f(z)$ to be a μ th-degree function, parameter k_μ , which is given by $k = k_\mu z^{\mu-1}$, must be chosen properly:

$$k_{\mu 1,2} = \frac{1}{2}[2\beta^2 + \mu(\mu - 1)q \pm \sqrt{(\mu^2 q^2 + 4\gamma^2)((\mu - 1)^2 + 4\epsilon^2)}] \quad (24)$$

Taking into account the \pm signs in Eq. (24), four different forms of $\pi_f(z)$ are obtained. The function $\pi_f(z)$, which is chosen as the function $\tau_f(z)$ given by Eq. (14), has a negative derivative for physical validity [20]. This condition is provided by

$$k_\mu = \frac{1}{2}[2\beta^2 + \mu(\mu - 1)q - \sqrt{(\mu^2 q^2 + 4\gamma^2)((\mu - 1)^2 + 4\epsilon^2)}]. \quad (25)$$

Using the chosen k_μ in Eq. (25) the function $\pi_f(z)$ is obtained as

$$\pi_f(z) = \frac{1}{2} \left[S_0(\mu - 1 + \sqrt{(\mu - 1)^2 + 4\epsilon^2}) - (q(2\mu - 1 + \sqrt{(\mu - 1)^2 + 4\epsilon^2}) + \sqrt{\mu^2 q^2 + 4\gamma^2})z^\mu \right]. \quad (26)$$

After determining $\pi_f(z)$, one can obtain the functions $\tau_f(z)$, $\lambda(z)$, and $\lambda_n(z)$ from Eq. (14), Eq. (15), and Eq. (16), respectively:

$$\tau_f(z) = S_0(1 + \sqrt{(\mu - 1)^2 + 4\epsilon^2}) - (q(\mu + 1 + \sqrt{(\mu - 1)^2 + 4\epsilon^2}) + \sqrt{\mu^2 q^2 + 4\gamma^2})z^\mu, \quad (27)$$

$$\lambda(z) = \frac{1}{2} [2\beta^2 - \mu^2 q - \sqrt{(\mu^2 q^2 + 4\gamma^2)((\mu - 1)^2 + 4\epsilon^2)} - \mu \sqrt{\mu^2 q^2 + 4\gamma^2} - \mu q \sqrt{(\mu - 1)^2 + 4\epsilon^2}] z^{\mu-1}, \quad (28)$$

$$\lambda_n(z) = n\mu [q(\mu + 1 + \sqrt{(\mu - 1)^2 + 4\epsilon^2}) + \sqrt{\mu^2 q^2 + 4\gamma^2} + \frac{(n - 1)(\mu + 1)q}{2}] z^{\mu-1}. \quad (29)$$

For $\lambda(z) = \lambda_n(z)$, the eigenvalue spectra of the problem are established by recalling the equalities given by Eq. (8):

$$E^2 - m^2 = \frac{\alpha^2}{4} \left\{ (\mu - 1)^2 - \left[\frac{4mS_0 - \mu^2 \alpha^2 q - \mu \alpha \sqrt{\mu^2 \alpha^2 q^2 + 4S_0^2(1 + 2n)} - \mu(\mu + 1)n(n + 1)q\alpha^2}{\mu \alpha^2 q(2n + 1) + \alpha \sqrt{\mu^2 \alpha^2 q^2 + 4S_0^2}} \right]^2 \right\} \quad (30)$$

In order to obtain the eigenfunction solution, the function $\phi(z)$ is determined by using Eq. (17):

$$\phi(z) = z^{\frac{1}{2}(\mu-1+\sqrt{(\mu-1)^2+4\epsilon^2})} (S_0 - qz^\mu)^{\frac{1}{2\mu q}(\mu q + \sqrt{\mu^2 q^2 + 4\gamma^2})}. \quad (31)$$

Then the functions $\rho(z)$ and $y_n(z)$ are obtained from Eq. (18) and Eq. (19):

$$\rho(z) = z^{\sqrt{(\mu-1)^2+4\epsilon^2}} (S_0 - qz^\mu)^{\frac{1}{\mu q}(\sqrt{\mu^2 q^2 + 4\gamma^2})}. \quad (32)$$

$$y_n(z) = B_n z^{-\sqrt{(\mu-1)^2+4\epsilon^2}} (S_0 - qz^\mu)^{-\frac{1}{\mu q}(\sqrt{\mu^2 q^2+4\gamma^2})} \frac{d^n}{dz^n} \left[z^{n+\sqrt{(\mu-1)^2+4\epsilon^2}} (S_0 - qz^\mu)^{n+\frac{1}{\mu q}(\sqrt{\mu^2 q^2+4\gamma^2})} \right]. \tag{33}$$

The right-hand sides of Eq. (31) and Eq. (33) are inserted in the transformation $\psi(z) = \phi(z)y_n(z)$:

$$\psi(z) = B_n z^{\frac{1}{2}(\mu-1+\sqrt{(\mu-1)^2+4\epsilon^2})} (S_0 - qz^\mu)^{\frac{1}{2\mu q}(\mu q+\sqrt{\mu^2 q^2+4\gamma^2})} z^{-\sqrt{(\mu-1)^2+4\epsilon^2}} (S_0 - qz^\mu)^{-\frac{1}{\mu q}(\sqrt{\mu^2 q^2+4\gamma^2})} \frac{d^n}{dz^n} \left[z^{n+\sqrt{(\mu-1)^2+4\epsilon^2}} (S_0 - qz^\mu)^{n+\frac{1}{\mu q}(\sqrt{\mu^2 q^2+4\gamma^2})} \right]. \tag{34}$$

Consequently, the eigenvalue and the eigenfunction spectra of a spinless particle in the generalized Hulthen potential, which are identical to the results in Ref. [2] and Ref. [24] for $\mu = 1$, have been obtained completely in view of conformable fractional calculus.

5. Results and discussion

Fractionalization of the relativistic wave equations has been widely studied by using Riemann–Liouville or Caputo fractional derivative operators in general. Since all fractional derivative operators have a nonlocal character and they do not satisfy the Leibniz rule, the wave equations including these operators are so complicated in order to obtain an analytical solution related to the fractional dimension of the space. Herein, a local fractional derivative operator is needed to arrive at an exact solution. The local fractional form of the KG equation is proposed in order to describe the dynamics of a relativistic particle moving in the generalized Hulthen potential by means of a conformable fractional derivative operator. Therefore, variation in the energy and the wavefunction spectra with respect to the fractional order can be obtained in a more realistic manner. In the presented figures, evolution of the ground state energy of a spinless particle in deformed Hulthen potential is represented as a function of the fractional order μ for three different values of the deformation parameter q and for three different values of the range parameter α , namely 0.5, 1, and 2. It can be seen that the curves increase more rapidly with increasing q to a particular value of μ . Then they decrease to the well-known values of ground state energy at $\mu = 1$ when the green line in Figure (1) is excluded. In Figure (2) and Figure (3) the initial values of the curves start at $\mu \neq 0$ for all values of q . Moreover, maximum values of the curves are in evidence when μ reaches the value 1. On the whole, all curves intersect two by two at different points corresponding to the different values of μ and the curves reach maximum values more rapidly with increasing α . Furthermore, ground state energy for the deformed Hulthen potential is given numerically for fixed $S_0 = 0.25$ and given α in Table (1), Table (2) and Table (3).

Table 1. Ground-state energy of the local fractional KG equation for $S_0 = 0.25$ and $\alpha = 0.5$. Here α is expressed in units of Compton wavelength, $\alpha = 1/\lambda_C = mc/\hbar$.

	$\mu = 0.25$	$\mu = 0.5$	$\mu = 0.75$	$\mu = 1$
q	E_0	E_0	E_0	E_0
0.5	0.376295	0.441808	0.478148	0.496505
1	0.42148	0.48417	0.5	0.498157
1.5	0.45227	0.5	0.503884	0.490179

Table 2. Ground-state energy of the local fractional KG equation for $S_0 = 0.25$ and $\alpha = 1$. Here α is expressed in units of Compton wavelength, $\alpha = 1/\lambda_C = mc/\hbar$.

	$\mu = 0.25$	$\mu = 0.5$	$\mu = 0.75$	$\mu = 1$
q	E_0	E_0	E_0	E_0
0.5	0.84296	0.962835	1	0.996314
1	0.947387	1.01761	1.00519	0.964541
1.5	1	1.02942	0.994548	0.941246

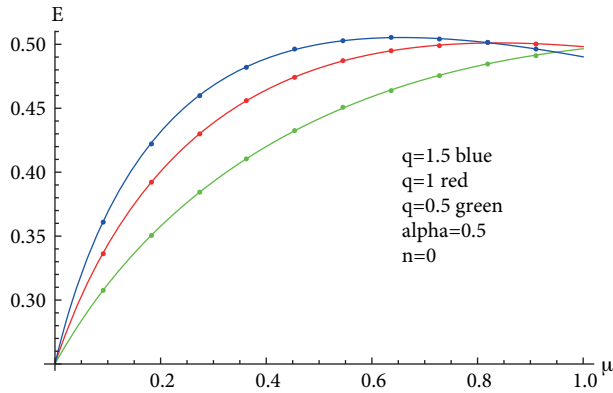


Figure 1. The variation in the ground state energy of a relativistic particle moving in the generalized Hulthen potential as a function of the fractional order μ for three different values of potential deformation parameter q , where $\alpha = 0.5$.

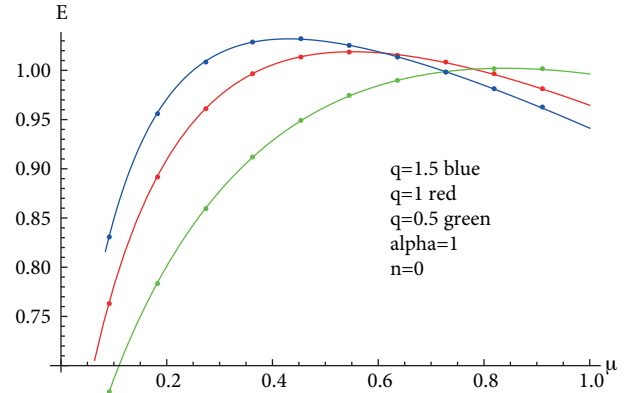


Figure 2. The variation in the ground state energy of a relativistic particle moving in the generalized Hulthen potential as a function of the fractional order μ for three different values of potential deformation parameter q , where $\alpha = 1$.

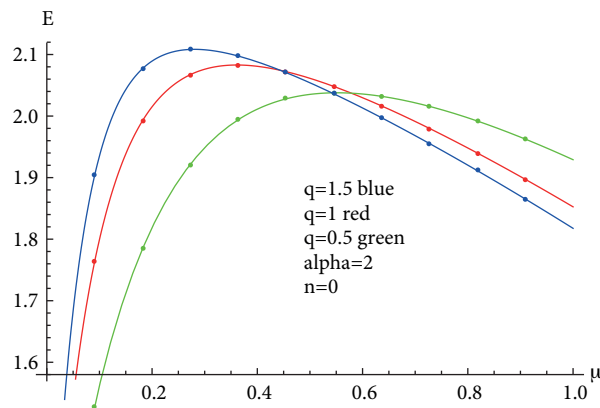


Figure 3. The variation in the ground state energy of a relativistic particle moving in the generalized Hulthen potential as a function of the fractional order μ for three different values of potential deformation parameter q , where $\alpha = 2$.

Table 3. Ground-state energy of the local fractional KG equation for $S_0 = 0.25$ and $\alpha = 2$. Here α is expressed in units of Compton wavelength, $\alpha = 1/\lambda_C = mc/\hbar$.

	$\mu = 0.25$	$\mu = 0.5$	$\mu = 0.75$	$\mu = 1$
q	E_0	E_0	E_0	E_0
0.5	1.89477	2.03522	2.01038	1.92908
1	2.05619	2.06136	1.97015	1.85251
1.5	2.1062	2.05407	1.94451	1.81759

References

- [1] Yasuk, F.; Durmus A.; Boztosun, I. *J. Math. Phys.* **2006** *47*, 082302-082302-8.
- [2] Erifes, H.; Sever, R. *Int. J. Theor. Phys.* **2007**, *46*, 935-950.
- [3] Diao, Y. F.; Yi, L. Z.; Jia, C. S. *Phys. Lett. A* **2004**, *332*, 157-167.

- [4] Akbarieh, A. R.; Motavalli, H. *Mod. Phys. Lett. A* **2008**, *23*, 3005-3013.
- [5] Motavalli, H.; Akbarieh, A. R. *Int. J. Theor. Phys.* **2010**, *49*, 979-987.
- [6] Antia, A. D.; Ikot, A. N.; Akpan, I. O.; Awoga, O. A. *Indian J. Phys.* **2013**, *87*, 155-160.
- [7] Ikot, A. N.; Awoga, O. A.; Ita, B. I. *Few-Body Syst.* **2012**, *53*, 539-548.
- [8] Ganji, Z. Z.; Ganji, D. D.; Rostamiyan, Y. *Appl. Math. Model.* **2009**, *33*, 3107-3113.
- [9] Machado, J. A. T.; Silva, M. F.; Barbosa, R. S.; Jesus, I. S.; Reis, C. M.; Marcos, M. G.; Galhano, A. F. *Math. Prob. Eng.* **2010**, *2010*, 639801, 1-34.
- [10] Sierociuk, D.; Dzieliński, A.; Sarwas, G.; Petras, I.; Podlubny, I.; Skovranek, T. *Phil. Trans. R. Soc. A* **2012**, *371*, 20120146, 1-10.
- [11] Podlubny, I. *Fractional Differential Equations*. Academic Press: New York, NY, USA, 1999.
- [12] Miller, K. S. *An Introduction to Fractional Calculus and Fractional Differential Equations*. Wiley: New York, NY, USA, 1993.
- [13] Khalil, R.; Horani, M. A.; Yousef, A.; Sababheh, M. *J. Comput. Appl. Math.* **2014**, *264*, 65-70.
- [14] Abdaljawad, T. *J. Comput. Appl. Math.* **2014**, *279*, 57-66.
- [15] Anderson, D. R.; Ulness, D. J. *J. Math. Phy.* **2015**, *56*, 063502, 1-18.
- [16] Eslami, M. *Appl. Math. Comput.* **2016**, *285*, 141-148.
- [17] Ortigueira, M. D.; Machado, J. A. T. *J. Comput. Phys.* **2015**, *293*, 4-13.
- [18] Iyiola, O. S.; Tasbozan, O.; Kurt, A.; Cenesiz, Y. *Chaos Solitons Fractals* **2017**, *94*, 1-7.
- [19] Karayer, H.; Demirhan, D.; Buyukkilic, F. *Commun. Theor. Phys.* **2016**, *66*, 12-18.
- [20] Nikiforov, A. V.; Uvarov, V. B. *Special Functions of Mathematical Physics*. Birkhauser: Boston, MA, USA, 1988.
- [21] Egrifes, H.; Demirhan, D.; Buyukkilic, F. *Phys. Scripta* **1999a**, *60*, 195-198.
- [22] Egrifes, H.; Demirhan, D.; Buyukkilic, F. *Phys. Scripta* **1999b**, *59*, 90-94.
- [23] Ikhdaire, S. M.; Sever, R. *Int. J. Theor. Phys.* **2007**, *46*, 2384-2395.
- [24] Ikhdaire, S. M. *Int. J. Mod. Phys. C* **2009**, *20*, 25-45.