

Symmetry reduction of the first heavenly equation and 2 + 1-dimensional bi-Hamiltonian system

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Abstract: The first heavenly equation of Plebanski in the two-component form is known to be a 3 + 1-dimensional tri-Hamiltonian system. We show that a particular choice of symmetry reduction applied to the first heavenly equation yields a 2 + 1-dimensional bi-Hamiltonian system. For this tri-dimensional system, we present Lagrangian, Hamiltonian, and recursion operators; point symmetries; and integrals of motions.

Key words: First heavenly equation, symmetry reduction, recursion operator, bi-Hamiltonian, 2 + 1-dimensional systems

1. Introduction

In reference [1] we showed that the 3 + 1-dimensional first heavenly equation (FHE) of Plebanski possesses a tri-Hamiltonian structure. FHE in the one component form,

$$u_{\tilde{t}\tilde{y}}u_{\tilde{x}\tilde{z}} - u_{\tilde{t}\tilde{z}}u_{\tilde{x}\tilde{y}} = 1, \quad (1)$$

can be presented in the two-component form

$$u_t = v, \quad u_x = u_{xx} + \frac{1}{u_{\tilde{x}\tilde{z}}} [(v_{\tilde{z}} + u_{x\tilde{z}})(v_{\tilde{x}} - u_{x\tilde{x}}) + 1] \quad (2)$$

where $t = \tilde{t} + \tilde{y}$, $x = \tilde{t} - \tilde{y}$ and subscripts t , x , \tilde{x} , \tilde{z} denote partial derivatives with respect to corresponding variables. We have shown that second heavenly and asymmetric heavenly equations are reduced to 2 + 1-dimensional bi-Hamiltonian system by using the method of symmetry reduction [2–4].

In [1] we found all point symmetries of the FHE equation. In general, symmetry reduction of Eq. (1) has no Hamiltonian structure. However, if we choose a particular combination of symmetries given in [1] we obtain a 2 + 1-dimensional reduced bi-Hamiltonian system. For the new 2 + 1-dimensional system we present Hamiltonian structures, recursion operator, Lie point symmetries, and integrals of motion.

2. Symmetry reduction of FHE and reduced 2 + 1-dimensional system

Point Lie symmetries of system (2) are determined by the following basic generators:

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$$\begin{aligned}
 X_1 &= \partial_{\tilde{z}}, \quad X_2 = \partial_{\tilde{x}}, \quad X_3 = \tilde{z}\partial_{\tilde{z}} - \tilde{x}\partial_{\tilde{x}}, \quad X_4 = t\partial_t + x\partial_x + u\partial_u \\
 X_5 &= 2\tilde{z}\partial_{\tilde{z}} + u\partial_u + v\partial_v, \quad Y_a = a(\tilde{x})(\partial_t + \partial_x), \quad Z_b = b(\tilde{z})(\partial_x - \partial_t) \\
 V_{f,g} &= \{f(t+x, \tilde{x}) + g(t-x, \tilde{z})\}\partial_u + \{f_t(t+x, \tilde{x}) + g_t(t-x, \tilde{z})\}\partial_v
 \end{aligned} \tag{3}$$

where a, b and f, g are arbitrary smooth functions of two variables, respectively, and subscripts denote partial derivatives [1].

We combine X_1 and X_2 as $X = X_2 - X_1$ and we obtain

$$X = \partial_{\tilde{x}} - \partial_{\tilde{z}} \tag{4}$$

The invariants of X are determined by the characteristic system as

$$\tilde{X} = \tilde{x} + \tilde{z}, \quad X = x, \quad T = t, \quad U = u, \quad V = v \tag{5}$$

The symmetry reduction implies the ansatz: $u = U(\tilde{X}XT)$ and $v = V(\tilde{X}XT)$. Considering Eq. (5) the total derivatives in terms of new variables change as

$$D_{\tilde{x}} = D_{\tilde{X}}, \quad D_{\tilde{z}} = D_{\tilde{X}}, \quad D_x = D_X, \quad D_t = D_T \tag{6}$$

Substituting this into the original system (2) and renaming $U \rightarrow u, V \rightarrow v, T \rightarrow t, U \rightarrow u, \tilde{X} \rightarrow y, X \rightarrow x$ we obtain the new 2 + 1-dimensional reduced system in two component form as

$$u_t = v, \quad u_t = u_{xx} + \frac{1}{u_{yy}} [v_y^2 - u_{xy}^2 + 1] \equiv Q \tag{7}$$

where Q is the right-hand side of the second equation.

3. First Hamiltonian structure of the reduced system

In order to conclude that the reduction is conducted correctly we should perform the procedure from the beginning. This means that we should start with the Lagrangian of the FHE and continue in this order. Therefore, we apply (6) to Lagrangian L given in [1] and we get reduced L^{red} for system (7) as follows:

$$L^{red} = \left(vu_t - \frac{v^2}{2} \right) u_{yy} + u_t \left(\frac{1}{3} u_x u_{yy} + \frac{2}{3} u_y u_{xy} \right) - \frac{1}{2} u_x^2 u_{yy} + u \tag{8}$$

Following the result of [1], symplectic operator K becomes

$$K^{red} = \begin{pmatrix} D_y v_y + v_y D_y & -u_{yy} \\ u_{yy} & 0 \end{pmatrix} \tag{9}$$

and the reduced first Hamiltonian operator $J_0 = K^{-1}$ is reduced as

$$J_0^{red} = \begin{pmatrix} 0 & \frac{1}{u_{yy}} \\ -\frac{1}{u_{yy}} & \frac{1}{u_{yy}} (D_y v_y + v_y D_y) \frac{1}{u_{yy}} \end{pmatrix} \tag{10}$$

We note that K^{red} and J_0^{red} change only with y coordinate and superscript *red* denotes the variables in the *reduced* system. The corresponding Hamiltonian for J_0^{red} can be obtained from L^{red} or by the direct reduction of H_1 from [1]. After the reduction we get

$$H_1^{red} = \frac{1}{2} (v^2 + u_x^2) u_{yy} - u \quad (11)$$

Having first Hamiltonian operator J_0^{red} and Hamiltonian function H_1^{red} , reduced system (7) can be written in the Hamiltonian form as

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0^{red} \begin{pmatrix} \delta_u H_1^{red} \\ \delta_v H_1^{red} \end{pmatrix} = \begin{pmatrix} v \\ Q \end{pmatrix} \quad (12)$$

where $\delta_u = \frac{\delta}{\delta u}$ and $\delta_v = \frac{\delta}{\delta v}$ denote Euler–Lagrange operators related to variational derivatives of Hamiltonian functional [5].

4. Recursion operator and second Hamiltonian structure of reduced system

According to reference [1] the recursion operator obtained by Lax pair of FHE reads as

$$R_\varepsilon = \begin{pmatrix} R_\varepsilon^{11} & (D_{\bar{x}}^{-1} - \varepsilon D_{\bar{z}}^{-1}) u_{\bar{x}\bar{z}} \\ R_\varepsilon^{21} & - (D_{\bar{x}}^{-1} + \varepsilon D_{\bar{z}}^{-1}) D_x u_{\bar{x}\bar{z}} + v_{\bar{z}} + u_{x\bar{z}} - \varepsilon (v_{\bar{x}} - u_{x\bar{x}}) \end{pmatrix} \quad (13)$$

Here

$$R_\varepsilon^{11} = - (D_{\bar{x}}^{-1} + \varepsilon D_{\bar{z}}^{-1}) u_{\bar{x}\bar{z}} D_x - D_{\bar{x}}^{-1} (v_{\bar{x}} - u_{x\bar{x}}) D_{\bar{z}} + \varepsilon D_{\bar{z}}^{-1} (v_{\bar{z}} + u_{x\bar{z}}) D_{\bar{x}} \quad (14)$$

$$\begin{aligned} R_\varepsilon^{21} &= (D_{\bar{x}}^{-1} - \varepsilon D_{\bar{z}}^{-1}) D_x u_{\bar{x}\bar{z}} D_x - \{v_{\bar{z}} + u_{x\bar{z}} + \varepsilon (v_{\bar{x}} - u_{x\bar{x}})\} D_x \\ &+ D_x \{D_{\bar{x}}^{-1} (v_{\bar{x}} - u_{x\bar{x}}) D_{\bar{z}} + \varepsilon D_{\bar{z}}^{-1} (v_{\bar{z}} - u_{x\bar{z}}) D_{\bar{x}}\} - (Q - u_{xx}) (D_{\bar{z}} - \varepsilon D_{\bar{x}}). \end{aligned} \quad (15)$$

and the second Hamiltonian operator given in [1] is

$$J_\varepsilon = \begin{pmatrix} - (D_{\bar{x}}^{-1} - \varepsilon D_{\bar{z}}^{-1}) & J_\varepsilon^{12} \\ J_\varepsilon^{21} & J_\varepsilon^{22} \end{pmatrix} \quad (16)$$

$$\begin{aligned} J_\varepsilon^{12} &= -J_\varepsilon^{21} = - (D_{\bar{x}}^{-1} + \varepsilon D_{\bar{z}}^{-1}) D_x + \frac{\{v_{\bar{z}} + u_{x\bar{z}} + \varepsilon (v_{\bar{x}} - u_{x\bar{x}})\}}{u_{\bar{x}\bar{z}}} \\ J_\varepsilon^{22} &= (D_{\bar{x}}^{-1} - \varepsilon D_{\bar{z}}^{-1}) D_x^2 - \{v_{\bar{z}} + u_{x\bar{z}} + \varepsilon (v_{\bar{x}} - u_{x\bar{x}})\} D_x \frac{1}{u_{\bar{x}\bar{z}}} \\ &- \frac{1}{u_{\bar{x}\bar{z}}} D_x \{v_{\bar{z}} + u_{x\bar{z}} + \varepsilon (v_{\bar{x}} - u_{x\bar{x}})\} - \frac{1}{u_{\bar{x}\bar{z}}} (D_{\bar{z}} - \varepsilon D_{\bar{x}}) \frac{1}{u_{\bar{x}\bar{z}}} \\ &+ \frac{(v_{\bar{z}} + u_{x\bar{z}})}{u_{\bar{x}\bar{z}}} D_{\bar{x}} \frac{(v_{\bar{z}} + u_{x\bar{z}})}{u_{\bar{x}\bar{z}}} - \varepsilon \frac{(v_{\bar{x}} + u_{x\bar{x}})}{u_{\bar{x}\bar{z}}} D_{\bar{z}} \frac{(v_{\bar{x}} + u_{x\bar{x}})}{u_{\bar{x}\bar{z}}} \end{aligned}$$

where $\varepsilon = \pm 1$ both for R_ε and J_ε . We know that the FHE in 3 + 1-dimension admits a tri-Hamiltonian structure; hence we also expect to find a tri-Hamiltonian structure in the reduced 2 + 1-dimensional system.

Unfortunately in the reduced form we have only one bi-Hamiltonian structure instead of a tri-Hamiltonian one. Since we started reduction by using the symmetry generator $X = \partial_{\tilde{x}} - \partial_{\tilde{z}}$ a bi-Hamiltonian structure for the reduced system can only be obtained for $\varepsilon = -1$. In order to obtain a bi-Hamiltonian structure for $\varepsilon = +1$ our starting point should be $X = \partial_{\tilde{x}} + \partial_{\tilde{z}}$, but in both cases we have the same second Hamiltonian structure, which means we have only a bi-Hamiltonian structure instead of a tri-Hamiltonian structure in 2+1-dimension. Under this consideration we take $\varepsilon = -1$ and if we perform the reduction using Eq. (6) to R_{-1} in (13) and J_{-1} in (16) respectively we get

$$R_{-1}^{red} = \begin{pmatrix} -2D_y^{-1}v_y D_y & 2D_y^{-1}u_{yy} \\ 2D_y^{-1}D_x u_{yy} D_x - 2D_x D_y^{-1}u_{yy} D_y & \\ -2u_{xy} D_x - 2(Q + u_{xx}) D_y & 2v_y \end{pmatrix} \quad (17)$$

and

$$J_{-1}^{red} = \begin{pmatrix} -2D_y^{-1} & \frac{2v_y}{u_{yy}} \\ -\frac{2v_y}{u_{yy}} & J_{22}^{red} \end{pmatrix} \quad (18)$$

Here

$$J_{22}^{red} = 2D_y^{-1}D_x^2 - 2u_{xy}D_x \frac{1}{u_{yy}} - \frac{2}{u_{yy}}D_x v_y - \frac{2}{u_{yy}}D_x \frac{1}{u_{yy}} + \frac{2v_y}{u_{yy}}D_y \frac{v_y}{u_{yy}} + \frac{2u_{xy}}{u_{yy}}D_y \frac{u_{xy}}{u_{yy}}$$

J_{-1}^{red} can also be obtained by $J_{-1}^{red} = R_{-1}^{red} J_0^{red}$. Since it will not change the results we can skip over all (-2) and we rewrite Eq. (18) as

$$J_{-1}^{red} = \begin{pmatrix} D_y^{-1} & -\frac{v_y}{u_{yy}} \\ \frac{v_y}{u_{yy}} & J_{22}^{red} \end{pmatrix} \quad (19)$$

$$J_{22}^{red} = -D_y^{-1}D_x^2 + u_{xy}D_x \frac{1}{u_{yy}} + \frac{1}{u_{yy}}D_x v_y + \frac{1}{u_{yy}}D_x \frac{1}{u_{yy}} - \frac{v_y}{u_{yy}}D_y \frac{v_y}{u_{yy}} - \frac{u_{xy}}{u_{yy}}D_y \frac{u_{xy}}{u_{yy}}.$$

Note that J_{-1}^{red} is a skew symmetric differential operator and satisfies the Jacobi identity. The Jacobi identity of J_0 and J_ε for the FHE is checked in detail in [1]. Therefore, using these results it is straightforward to check that J_0^{red} , J_{-1}^{red} , and their linear combination also satisfy the Jacobi identity. The second Hamiltonian function given in [1] is

$$H_{0\varepsilon} = \frac{1}{2}(\varepsilon\tilde{x} - \tilde{z})vu_{\tilde{x}\tilde{z}} - \frac{1}{4}u_x(u_{\tilde{x}} + \varepsilon u_{\tilde{z}}) \quad (20)$$

From the same reason given above we chose $\varepsilon = -1$ and reduction of $H_{0(-1)}$ becomes

$$H_{0(-1)}^{red} = -yvu_{yy}$$

or

$$H_{0(-1)}^{red} = yvu_{yy} \quad (21)$$

Finally the second Hamiltonian structure

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_{-1}^{red} \begin{pmatrix} \delta_u H_{0(-1)}^{red} \\ \delta_v H_{0(-1)}^{red} \end{pmatrix} = \begin{pmatrix} v \\ Q \end{pmatrix} \quad (22)$$

gives a 2 + 1-dimensional system in (7). Together with the original Hamiltonian representation (12) of the system (7) we have a bi-Hamiltonian representation of this new 2 + 1-dimensional system [5].

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0^{red} \begin{pmatrix} \delta_u H_1^{red} \\ \delta_v H_1^{red} \end{pmatrix} = J_0^{-1} \begin{pmatrix} \delta_u H_{0(-1)}^{red} \\ \delta_v H_{0(-1)}^{red} \end{pmatrix} = \begin{pmatrix} v \\ Q \end{pmatrix}. \tag{23}$$

Hence we have proved that reduced system is a *bi-Hamiltonian system*. By repeated applications of the recursion operator to the first Hamiltonian operator J_0^{red} , according to Magri’s theorem we could generate an infinite sequence of Hamiltonian operators

$$J_n^{red} = R_{red}^n J_0^{red} \quad n = 0, 1, 2, \dots, \tag{24}$$

which shows that the reduced 2 + 1-dimensional equation considered in a two-component form is a multi-Hamiltonian system in the above sense [6,7].

5. Symmetries and integrals of motion

Point Lie symmetries of the system (7) are determined by using the software packages LIEPDE and CRACK by Wolf [8], run under REDUCE 3.8, and we have calculated all point symmetries of the reduced system (7). The basis generators of one-parameter subgroups of the complete Lie group of point symmetries for reduced system (7) have the form

$$\begin{aligned} X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \partial_y, \quad X_4 = x\partial_y, \quad X_5 = t\partial_t + x\partial_x + u\partial_u \\ X_6 = y\partial_u, \quad X_7 = y\partial_y + u\partial_u + v\partial_v, \quad X_a = a(xt)\partial_u + a_t(xt)\partial_v \end{aligned} \tag{25}$$

where a is an arbitrary smooth function of two variables and satisfies $a_{xx} - a_{tt} = 0$. We note that the obvious translational symmetries are generated by X_1 , X_2 , and X_3 .

The Lie algebra of point symmetries is determined by the Table of commutators of the basis generators where the commutators $[X_i, X_j]$ stand at the intersection of the i th row and the j th column. Here we used the following short hand notation:

Table. Commutators of point symmetries of the reduced system.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_a
X_1	0	0	0	0	X_1	0	0	X_{a_t}
X_2	0	0	0	X_3	X_2	0	0	$X_{a'}$
X_3	0	0	0	0	0	$X_{a=1}$	X_3	0
X_4	0	$-X_3$	0	0	$-X_4$	$X_{a=x}$	X_4	0
X_5	$-X_1$	$-X_2$	0	X_4	0	$-X_6$	0	$X_{\hat{a}}$
X_6	0	0	$-X_{a=1}$	$-X_{a=x}$	X_6	0	0	0
X_7	0	0	$-X_3$	$-X_4$	0	0	0	$-X_a$
X_a	$-X_{a_t}$	$-X_{a'}$	0	0	$-X_{\hat{a}}$	0	X_a	0

$$a' = \frac{da}{dx} \text{ and } \hat{a} = ta_t + xa' - a.$$

We need symmetry characteristics determining symmetries in evolutionary form [5] with independent variables not being transformed under symmetry transformations. For the symmetry generator of the form $X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}$ where the summation over repeated indices is used the symmetry characteristic are defined as $\varphi^\alpha = \eta^\alpha - u_i^\alpha \xi^i$ with the subscripts i denoting derivative with respect to x^i . In our problem, $i = 1, 2, 3$, $\alpha = 1, 2$, $u^1 = u$, $u^2 = v$, $\eta^1 = \eta^u$, $\eta^2 = \eta^v$, $x^1 = t$, $x^2 = x$, $x^3 = y$, and $\varphi^1 = \varphi$ while $\varphi^2 = \psi$, where φ and ψ determine the transformation of u and v , respectively. We also use $u_t = v$ and $v_t = Q$, where Q is the right-hand side of the second equation (7). Symmetry characteristics have the form

$$\varphi = \eta^u - v\xi^t - u_x\xi^x - u_y\xi^y, \quad \psi = \eta^v - Q\xi^t - v_x\xi^x - v_y\xi^y \quad (26)$$

Applying the formula (26) to the generators (25), we obtain the characteristics of these symmetries

$$\begin{aligned} \varphi_1 &= -v, & \psi_1 &= -Q, & \varphi_2 &= -u_x, & \psi_2 &= -v_x \\ \varphi_3 &= -u_y, & \psi_3 &= -v_y, & \varphi_4 &= -xu_y, & \psi_4 &= -xv_y \\ \varphi_5 &= u - tv - xu_x, & \psi_5 &= -tQ - xv_x, & \varphi_6 &= y, & \psi_6 &= 0 \\ \varphi_7 &= u - yu_y, & \psi_7 &= v - yv_y, & \varphi_a &= a(tx), & \psi_a &= a_t(xt) \end{aligned} \quad (27)$$

The first Hamiltonian structure provides a link between symmetries in evolutionary form and integrals of motion conserved by the Hamiltonian flow (7) replacing time t by the group parameter τ in (12) and using $u_\tau = \varphi$, $v_\tau = \psi$ for symmetries in the evolutionary form we obtain the Hamiltonian form of the Noether theorem for any conserved density H_{red} of an integral of motion.

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = J_0^{red} \begin{pmatrix} \delta_u H_{red} \\ \delta_v H_{red} \end{pmatrix} \quad (28)$$

To determine the integrals H_{red} that correspond to a known symmetry with characteristics (φ, ψ) we use the inverse Noether theorem

$$\begin{pmatrix} \delta_u H_{red} \\ \delta_v H_{red} \end{pmatrix} = K^{red} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \quad (29)$$

where operator $K^{red} = (J_0^{red})^{-1}$ is defined in (9). (29) is obtained by applying K to both sides of (28).

We now apply the formula (29) to determine integrals H_{red}^i corresponding to all variational symmetries with characteristics (φ_i, ψ_i) from (27). Using the expression (9) in explicit form

$$\begin{pmatrix} \delta_u H_{red}^i \\ \delta_v H_{red}^i \end{pmatrix} = \begin{pmatrix} D_y v_y + v_y D_y & -u_{yy} \\ u_{yy} & 0 \end{pmatrix} \begin{pmatrix} \varphi_i \\ \psi_i \end{pmatrix} \quad (30)$$

gives the formulas for determining integrals H_{red}^i for the known symmetries (φ_i, ψ_i) ,

$$\delta_u H_{red}^i = (D_y v_y + v_y D_y) \varphi_i - u_{yy} \psi_i, \quad \delta_v H_{red}^i = u_{yy} \varphi_i$$

We have used (31) reconstructing conserved densities corresponding to all variational point symmetries X_1, X_2, X_3, X_4, X_6 , and X_a generated by the following integrals:

$$H_{red}^1 = u - \frac{1}{2} (v^2 u_{yy} + u_x^2 u_{yy})$$

$$\begin{aligned}
 H_{red}^2 &= -vu_xu_{yy} \\
 H_{red}^3 &= -vu_yu_{yy} \\
 H_{red}^4 &= -xvu_yu_{yy} \\
 H_{red}^6 &= yvu_{yy} \\
 H_{red}^\alpha &= \left(va - \frac{1}{2}a_tu \right) u_{yy}
 \end{aligned} \tag{31}$$

The symmetries X_5 and X_7 are not variational symmetries, because their generating integrals in (31) do not exist. Note that the first and second Hamiltonian functions are contained in the set as $H_{red}^1 = -H_1^{red}$ and $H_{0(-1)}^{red} = H_{red}^2$.

6. Discussion

We have proved that a certain symmetry reduction of the 3+1-dimensional FHE taken in a two-component form yields a two-component 2 + 1-dimensional bi-Hamiltonian system. Indeed the FHE has two different second Hamiltonian structures depending on $\varepsilon = \pm 1$ and therefore it possesses a tri-Hamiltonian structure. We also expected to have a tri-Hamiltonian structure for the reduced system. However, different choices of symmetry reduction give us the same second Hamiltonian structure both for $\varepsilon = \pm 1$. Therefore, we conclude that the reduced 2 + 1-dimensional system admits a bi-Hamiltonian structure. For these systems, we have presented explicitly two Hamiltonian operators, a recursion operators for symmetries, a complete set of point symmetries, and corresponding integrals of motion.

All the main objects for the reduced system: L^{red} , J_0^{red} , H_0^{red} , R_{-1}^{red} , J_{-1}^{red} , and $H_{0(-1)}^{red}$ are obtained directly by the symmetry reduction of L , K , J_0 , H_0 , R_ε , J_ε , and $H_{0\varepsilon}$ given in reference [1]. It is also checked that we can obtain a bi-Hamiltonian structure for the 2 + 1-dimensional system (7) applying the method of Dirac constraints theory as given in [1]. It seems that symmetry reduction is a simple method for discovering new integrable 2 + 1-dimensional systems. However, even a slight change in symmetry chosen for reduction ruins all these properties and creates difficulty in discovering a new integrable multi-Hamiltonian structure in three dimensions. If we choose more general symmetries for the reduction, for example from the optimal system of one-dimensional subalgebra from [2], then we shall not able to discover even a single Hamiltonian structure of the reduced system.

We think that symmetry reduction of multi-Hamiltonian structures will be an important and interesting topic in the future. Moreover, we know that there are very few examples about 2 + 1-dimensional integrable systems in the literature. This method gives us the opportunity to discover new integrable systems in tri-dimension.

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