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Dimension of quantum mechanical path, chain rule, and extension of Landau's energy straggling method using F^{α}-calculus

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Abstract: F^{α}-calculus was recently presented for fractals. We show that the F^{α}-derivative satisfies the chain rule and affirms that the dimension of a quantum mechanical path is two. F^{α}-calculus allows us to extend quantum mechanics to fractal curves. To show the applicability of F^{α}-calculus, we study the fractal model of a particle in a box. F^{α}-calculus is suitable for describing the motion of particles with a fractal route through matter. In addition, we extend Landau's energy straggling method of charged particles to fractals.

Key words: F^{α}-calculus, fractal quantum paths, staircase function, energy straggling

1. Introduction

For many years, authors have tried to establish a calculus to fractals [1–3] and its applications [4, 5]. The most powerful and applicable one, called F^{α}-calculus, was developed by Gangal and his coworkers for fractal curves [6], on fractal sets [7], local fractional operators [8], and its applications [9–12], with other applications by some other authors [13–16]. With F^{α}-calculus, they have successfully explained anomalous diffusion, one of the most famous problems of physics, using local operators on fractals; this is one of the successes of F^{α}-calculus [7–10]. The previous theories of anomalous diffusion are nonlocal theories [17–24].

In this manuscript, we present a brief introduction to F^{α}-calculus and attempt to apply this calculus to physical fractal systems. It is not our purpose to deal with the details of the proofs of definitions and theorems of this calculus; for more study on proofs we refer enthusiasts to the references [6–14]. F^{α}-calculus is an undeveloped theory. We prove that the definition of the F^{α}-derivative satisfies the chain rule, and then we develop F^{α}-equations and give their solutions. It is important to note that F^{α}-calculus has two important application areas: fractal sets and fractal curves [6, 7]. In this paper, both fractal sets and curves are considered. We first study quantum mechanical motions. The fractal structure of the quantum mechanical paths has been studied by some authors using the previous theories of fractals [25–28]. We show that F^{α}-calculus confirms that the path of a particle in quantum mechanics is a fractal of dimension two. In addition, we apply F^{α}-calculus to derive the fractal Schrödinger equation and give a fractal quantum model for a particle in a box. Finally, to give an equation for the macroscopic cross-section of fractal interactions through matter, we expand the formulation of the theory of energy straggling to fractal materials.In the next paragraph, we introduce the classical theory of the energy straggling phenomenon.

When a beam of fast charged particles passes through a layer of matter, the particles lose their energy

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stochastically from ionization through matter. Indeed, for a fixed given layer, the energy loss will fluctuate [29–34]. The first study and formulation of this phenomenon was done by Landau [35]. Since the theory that Landau proposed is based on the continuity and smoothness of classical mathematics, the matter and energy loss are considered continuous variables. Therefore, this formalization fails for fractal quantities. F^α-calculus has motivated us to present a fractal model for the phenomenon of the passing particles losing energy. This fractal model will be an alternative method to the continuous model of Landau. The theory of energy straggling is used to calculate $h(E,x)$, the fraction of heavy charged particles with energy between E-$E+dE$ at x, the path length of the penetrated particles. This function satisfies the following master equation:

$$\frac{\partial h(E,x)}{\partial x} = \int_0^\infty \big[t(E,\epsilon)h(E+\epsilon,x) - t(E,\epsilon)h(E,x)\big]d\epsilon, \tag{1}$$

where $t(E,\epsilon)$ is the probability that a heavy charged particle with energy E will lose energy in the boundary of ϵ-$\epsilon+d\epsilon$.

This paper is divided into five sections. The second section is devoted to a brief introduction to F^α-calculus on fractal sets and curves, and it proves that the chain rule can be applied to F^α-derivatives. Section 3 discusses the application of F^α-calculus on quantum mechanics. The fourth section looks at the treatment of energy straggling using F^α-calculus theory. The fifth section presents a brief summary.

2. A brief introduction to F^α-calculus

2.1. Fractal calculus of subset of R

This section contains our given introduction to F^α-calculus. We choose the important definitions and theorems of F^α-calculus, and for more study we refer the readers to the references of the introduction section. In this paper, we assume that F represents all fractal structures with dimension α. In addition, the notation of this calculus belongs to its establishers [6, 7].

Definition 2.1 *The basis of F^α-calculus is on the integral staircase function, $S_F^\alpha(x)$, of order α for a fractal set F, which is given as follows:*

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F,a,x), & \textit{if } x \geq a; \\ -\gamma^\alpha(F,x,a), & \textit{otherwise}, \end{cases} \tag{2}$$

where γ^α is the mass function of the fractal set F, all $\alpha, x, a \in \Re$ and $0 < \alpha \leq 1$ [7].

It is worth pointing out that the two important properties of S_F^α, the continuity and monotonic increasing properties, are the essence of the definitions of the F^α-derivative and F^α-integral on fractals.

Definition 2.2 *Let $F \subset R$, $f : R \to R$, and $x \in F$. A number l is said to be the limit of f through the points of F, or simply the F-limit of f, as $y \to x$, and for any ϵ there exists $\delta > 0$ such that [7]*

$$y \in F, |y-x| < \delta \Rightarrow |f(y) - l| < \epsilon, \tag{3}$$

and then it is denoted by

$$l = F - lim_{y\to x} f(y). \tag{4}$$

Now the definitions of the F^α-derivative and F^α-integral of fractals are as follows:

Definition 2.3 *If F is an α-perfect set [36], then the F^{α}-derivative of f at x is [7]*

$$D_F^{\alpha} f(x) = \begin{cases} F-\lim_{y\to x} \frac{f(y)-f(x)}{S_F^{\alpha}(y)-S_F^{\alpha}(x)}, & x \in F; \\ 0, & otherwise. \end{cases} \tag{5}$$

Theorem 2.1 *A function h is α-integrable over $[a,b]$ if and only if $g=\phi[h]$ (g is conjugate function of h) [36] is Riemann integrable over $K=[S^{\alpha}(a), S^{\alpha}(b)]$ [7]:*

$$\int_a^b h(x) d^{\alpha}x = \int_{S^{\alpha}(a)}^{S^{\alpha}(b)} g(u) du. \tag{6}$$

Theorem 2.2 *Let h be a function such that the image $g=\phi[h]$ of h is ordinary differential on K. Then*

$$D_F^{\alpha} h(x) = \frac{dg(t=S^{\alpha}(x))}{dt}, \tag{7}$$

for all $x \in F$ [7].

The useful relation that the stair function satisfies is

$$ax^{\alpha} \leq S_F^{\alpha}(x) \leq bx^{\alpha}, \tag{8}$$

where a and b are constants [6, 7].

2.2. Fractal calculus of fractal curves

In this subsection, we introduce F^{α}-calculus for fractal curves.

Definition 2.4 *A fractal curve $F \subset R^n$ is said to be continuously parameterizable if there exists a function $w:[a,b] \to F \subset R^n$ that is continuous one to one and on to F [6].*

Definition 2.5 *A subdivision $P_{[a,b]}$ of interval $[a,b]$, $a<b$ is the finite set of points $a=t_0, t_1, t_2, ..., t_n = b$, $t_i < t_{i+1}$, and any interval of the form $[t_i, t_{i+1}]$ is called a component of the form of subdivision P. Moreover, if Q is a subdivision such that $P \subset Q$, then Q is called a refinement of P [6].*

Definition 2.6 *For a set F and subdivision $P_{[a,b]}$, $a<b$, $[a,b] \subset [a_0, b_0]$,*

$$\sigma^{\alpha}[F,P] = \sum_{i=0}^{n-1} \frac{|w(t_{i+1}) - w(t_i)|^{\alpha}}{\Gamma(\alpha+1)}, \tag{9}$$

where $|.|$ denotes the Euclidean norm on R^n [6].

Definition 2.7 *Given δ and $a_0 \leq a \leq b \leq b_0$, the coarse-grained mass $\gamma_{\delta}^{\alpha}(F,a,b)$ is given by*

$$\gamma_{\delta}(F,a,b) = inf_{\{P_{[a,b]}:|P|<\delta\}} \sigma^{\alpha}[F,P], \tag{10}$$

where $|P| = max_{0\leq i \leq n-1}(t_{i+1}-t_i)$ for a subdivision P [6].

Definition 2.8 *For $a_0 \leq a \leq b \leq b_0$, the mass function $\gamma^{\alpha}(F,a,b)$ is given by [6]*

$$\gamma^{\alpha}(F,a,b) = \lim_{\delta \to 0} \gamma^{\alpha}_{\delta}(F,a,b). \tag{11}$$

Definition 2.9 *Let F be a fractal curve. Then the F^{α}-derivative of function f at $\theta \in F$ is defined as [6]*

$$D^{\alpha}_{F} f(\theta) = F - \lim_{\theta' \to \theta} \frac{f(\theta') - f(\theta)}{J(\theta') - J(\theta)}. \tag{12}$$

Now the definition of the Taylor expansion of fractal functions: if $g = \phi[h]$ then the ordinary Taylor expansion of g is given by

$$g(u) = \sum_{n=1}^{\infty} \frac{(x-y)^n}{n!} \frac{d^n g(y)}{dy^n}, \tag{13}$$

for $u, y \in [S^{\alpha}(a), S^{\alpha}(b)]$. Then for $\theta, \theta' \in F$, fractal variables, the fractal expansion is

$$h(\theta) = \sum \frac{(J(\theta) - J(\theta'))^n}{n!} (D^{\alpha}_{F})^n h(\theta'). \tag{14}$$

2.3. What is new in F^{α}-calculus theory?

F^{α}-calculus is different from the other fractal theories. This theory gives dimension to countable sets that are dense. For example, we set up a set, C', from the end points of the intervals of the ith stage of construction of the Cantor set. C^i, which is countable, includes 2^{i+2} points and $C' = \cup_{i=1}^{\infty} C_i$, and then we have

$$\dim_{\gamma} C' = \frac{\ln 2}{\ln 3}, \tag{15}$$

whereas the Hausdorff dimension of the set is zero,

$$\dim_{H} C' = 0. \tag{16}$$

Other features of this calculus are its similarity to the Riemann–Stieltjes approach in defining the integral and derivative, and its simplicity and algorithmic point of view [6, 7].

2.4. Chain rule of F^{α}-derivative

Consider two functions f and g where g is F^{α}-differentiable at point x and f is F^{α}-differentiable at point $g(x) = y$. Now we are interested in the F^{α}-derivative of the function $f(g(x))$ at point x. To begin, let $\epsilon = S^{\alpha}_{F}(y) - S^{\alpha}_{F}(x)$ and $g(x)$ be α-differentiable, so

$$\frac{g(y) - g(x)}{\epsilon} - D^{\alpha}_{F} g(x) \to 0, \tag{17}$$

as $\epsilon \to 0$. We define a new variable, t,

$$t = \frac{g(y) - g(x)}{\epsilon} - D^{\alpha}_{F} g(x), \tag{18}$$

where $t \to 0$ as $\epsilon \to 0$. Similarly, since f is an α-differentiable function at $y = g(x)$ we have

$$s = \frac{f(w) - f(z)}{\eta} - D_F^\alpha f(z), \tag{19}$$

where $s \to 0$ as $\eta \to 0$. From Eq. (18) and Eq. (19), we have

$$g(y) = g(x) + \Big[D_F^\alpha g(x) + t\Big]\epsilon \tag{20}$$

and

$$f(w) = f(z) + \Big[D_F^\alpha f(z) + s\Big]\eta, \tag{21}$$

where $z = g(x)$ and $w = g(y)$. If we choose $\eta = [D_F^\alpha g(x) + t]\epsilon$ and use

$$f(g(y)) = f(g(x) + [D_F^\alpha g(x) + t]\epsilon) \tag{22}$$

and Eq.(21), we have

$$f(g(y)) = f(g(x)) + [D_F^\alpha f(g(x)) + s][D_F^\alpha g(x) + t]\epsilon. \tag{23}$$

From Eq.(23), we have

$$\frac{f(g(y)) - f(g(x))}{\epsilon} = [D_F^\alpha f(g(x)) + s][D_F^\alpha g(x) + t], \tag{24}$$

and then we obtain

$$\lim_{\epsilon \to 0} \frac{f(g(y)) - f(g(x))}{\epsilon} = D_F^\alpha f(g(x)) D_F^\alpha g(x). \tag{25}$$

3. Dimension of quantum mechanical path using F^α-calculus

Abbott and Wise showed that the dimension of a quantum mechanical path is two [25]. First we review their method and then show that the same result can be obtained using F^α-calculus. Suppose that we measure the position of a particle at times t_0, $t_1 = t_0 + \Delta t$, $\dots$, $t + N\Delta t$, with $T = t_N - t_0 = N\Delta t$. Since in quantum mechanics particles move in statistical manner, we take the average value of physical quantum variables. Assume that $\langle \Delta l \rangle$ is the average distance that the particle moves in time Δt. Then the length of the path of the particle is

$$\langle l \rangle = N \langle \Delta l \rangle, \tag{26}$$

and from the uncertainty principle, we obtain

$$\langle \Delta l \rangle \propto \frac{\hbar \Delta t}{m \Delta x}, \tag{27}$$

and then

$$\langle \Delta l \rangle \propto \frac{\hbar T}{m \Delta x}. \tag{28}$$

Requiring that the length be independent of resolution, Hausdorff defined the length, L, as follows:

$$L = l(\Delta x)^{D-1}, \tag{29}$$

where l is the length measured with resolution Δx. Then, to define the dimension of the path using the Hausdorff formula, the average length is

$$\langle \Delta L \rangle = \langle \Delta l \rangle (\Delta x)^{D-1}. \tag{30}$$

Since $\langle \Delta L \rangle$ must be independent of Δx, using Eq. (28), $D = 2$ results. Now we treat the problem using F^α-calculus. We first consider the definition of mass function, and then using the average distance value, we have

$$\gamma = \sum_{i=1}^{N} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha+1)} = \frac{N\langle \Delta l \rangle^\alpha}{\Gamma(\alpha+1)}. \tag{31}$$

Assuming that the quantum mechanics path is self-similar, $\Delta l \propto \Delta x$, we have

$$\Delta t \propto \frac{m(\Delta x)^2}{\hbar}. \tag{32}$$

By replacing $N = \frac{T}{\Delta t}$ in Eq. (31), we have

$$\gamma = \frac{N\langle \Delta l \rangle^\alpha}{\Gamma(\alpha+1)} = \frac{\hbar T \langle \Delta l \rangle^\alpha}{m(\Delta x)^2 \Gamma(\alpha+1)}, \tag{33}$$

and requiring that γ be independent of the resolution Δx, we must have $\alpha = 2$, the same result of the Hausdorff definition of fractal length.

3.1. Quantum mechanics of a fractal curve

This section discusses the formulation of quantum mechanics of a fractal curve using F^α-calculus. Let θ represent a point on a fractal curve. Supposing that $\psi(\theta, t)$ is the wave function of a particle on a fractal curve, the time evolution of the wave function is as follows:

$$\psi(\theta, t+\epsilon) = \langle \theta | \psi(t+\epsilon) \rangle = \langle \theta | e^{-iH\epsilon/\hbar} | \psi(t) \rangle, \tag{34}$$

and applying the F^α-integral

$$= \int d_F^\alpha \theta_0 \langle \theta | e^{-iH\epsilon/\hbar} | \theta_0 \rangle \langle \theta_0 | \psi(t) \rangle \tag{35}$$

and using $\langle \theta | e^{-iH\epsilon/\hbar} | \theta_0 \rangle = U(\theta, t+\epsilon; \theta_0, t)$, we obtain

$$= \int d_F^\alpha \theta_0 U(\theta, t+\epsilon; \theta_0, t) \psi(\theta_0, t), \tag{36}$$

where H is the fractal Hamiltonian of the system. Now we consider the Hamiltonian of the system as follows:

$$\hat{H} = \frac{\hat{p}_\theta^2}{2m} + V(\hat{\theta}), \tag{37}$$

and using $\hat{\theta}|\theta\rangle = J(\theta)|\theta\rangle$, $\hat{p}_\theta = \frac{\hbar}{i}\frac{\partial}{\partial J(\theta)}$ and by some algebraic calculations, we have

$$U(J(\theta), t+\epsilon; J(\theta_0), t) = \left(\frac{m}{2\pi\hbar i \epsilon}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\left[\frac{m(J(\theta) - J(\theta_0))^2}{2\epsilon} - \epsilon V\left(\frac{J(\theta) + J(\theta_0)}{2}\right)\right]\right\}. \tag{38}$$

By the conjugate theorem of F^α-calculus, $y = J(\theta)$, $y_0 = J(\theta_0)$, $d_F^\alpha \theta_0 = dy_0$, and making the changing $\eta = y - y_0$, the wave function becomes

$$\psi(\theta, t+\epsilon) = \left(\frac{m}{2\pi\hbar i\epsilon}\right)^{1/2} \int d\eta \exp\left[im\eta^2/2\hbar\epsilon\right] \times \exp\left[-\frac{i}{\hbar}\epsilon V\left(y+\frac{\eta}{2}\right)\right]\psi(y+\eta, t). \tag{39}$$

Now by Taylor expansion of the wave function

$$\psi(y+\eta, t) = \psi(y,t) + \eta\frac{d\psi}{dy} + \eta^2\frac{d^2\psi}{dy^2} + \ldots, \tag{40}$$

and by some simple calculations, we obtain

$$i\hbar\left[\frac{\psi(\theta, t+\epsilon) - \psi(\theta, t)}{\epsilon}\right] = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + V(\theta)\right]\psi(y,t), \tag{41}$$

and then using the inversion of the transformation, we obtain the fractal equation

$$i\hbar\frac{\partial}{\partial t}\psi(\theta, t) = \left[-\frac{\hbar^2}{2m}(D_\theta^\alpha)^2 + V(\theta)\right], \tag{42}$$

which is called the fractal Schrödinger equation.

3.2. A particle in a fractal box

Consider a particle that is confined in a fractal well potential. A fractal well is fractal space in length L with walls such that the potential is zero inside length L but suddenly goes to infinity at the boundaries. As discussed before, the Hamiltonian operator for a fractal well is

$$\hat{H} = -\frac{\hbar^2}{2m}(D_x^\alpha)^2 + \chi_F V(x), \tag{43}$$

where F is the fractal space confined in length L, and

$$V(x) = \begin{cases} 0, & \text{for} \quad 0 < x < L; \\ \infty, & \text{for} \quad x < 0 \text{ and } x > L. \end{cases} \tag{44}$$

Particles cannot penetrate the infinite walls and then the only region where the particles can be found has the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}(D_x^\alpha)^2, \tag{45}$$

and then the stationary states are the solution of the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}(D_x^\alpha)^2\psi(x) = E\psi(x). \tag{46}$$

Now to solve Eq. (46) using F^α-calculus, suppose that ϕ is the conjugate of ψ and then $\phi(u) = \psi(x)$ for $u = S_F^\alpha(x)$. Using the theorem of the conjugate differential equation $d\phi/du = D_x^\alpha\psi$, we have

$$-\frac{\hbar^2}{2m}\frac{d^2}{du^2}\phi = E\phi, \tag{47}$$

which has the following general solution:

$$\phi(u) = C\cos(ku) + D\sin(ku) \qquad E = \hbar^2 k^2/2m. \tag{48}$$

By applying the boundary conditions, $\psi(L) = \psi(0) = 0$, and taking $L' = S_F^\alpha(L)$, we have

$$k = \frac{n\pi}{L'} = \frac{n\pi}{S_F^\alpha(L)} \approx \frac{n\pi}{L^\alpha}, \tag{49}$$

where we use Eq. (8), and then the wave function is

$$\psi = \sin(n\pi x/S_F^\alpha(L)) \approx \sin(n\pi x/L^\alpha), \tag{50}$$

and the discrete energy values are

$$E_n = n^2(h^2/8m(S_F^\alpha(L))^2) \approx h^2/(8mL^{2\alpha}), \tag{51}$$

where for $\alpha = 1$ we reobtain the classical values.

4. Fractal Landau's method of the straggling of energy

Fractal materials are the realities of physics. If a beam of particles passes through a layer of fractal matter, its energy through matter will not be lost continuously. Even in continuous matter the energy loss is in quantum values. Then the master equation, Eq. (1), should be generalized such that it can explain more general phenomena, so we extend it as follows:

$$D_{x,F}^\alpha h(x,E) = \int_F t(\epsilon)\big[h(x,E-\epsilon) - h(x,E)\big]d_F^\alpha \epsilon, \tag{52}$$

where we suppose that $t(E,\epsilon)$ is independent of E. To solve this equation we follow Landau's method [35]. First, we calculate Laplace transformation $h(x,E)$,

$$\tilde{h}(x,q) = \int_F e^{-Eq} h(x,E) d_F^\alpha E. \tag{53}$$

By multiplying both sides of Eq. (52) by e^{-Eq} and integrating, we have

$$D_{x,q}^\alpha \tilde{h}(x,q) = -\tilde{h}(x,q)\int_F t(\epsilon)(1 - e^{-q\epsilon}) d_F^\alpha \epsilon. \tag{54}$$

By solving Eq. (54), we have

$$\tilde{h}(x,q) = \exp\Big[-S_F^\alpha(x)\int_F t(\epsilon)(1 - e^{-q\epsilon}) d_F^\alpha \epsilon\Big], \tag{55}$$

and then reversing the Laplace transformation, we have

$$h(x,E) = \frac{1}{2\pi i}\int_{+i\infty+\sigma}^{-i\infty+\sigma} \exp\big[Eq - S_F^\alpha(x)\int_F t(\epsilon)(1 - e^{-q\epsilon}) d_F^\alpha \epsilon\big] d_F^\alpha q. \tag{56}$$

Classical physics predicts $t(\epsilon)$ to be in the form of C/ϵ^2 [29]. Using this prediction, the most simple fractal model will be in the form of

$$t(\epsilon) = C\frac{1}{S_F^\alpha(\epsilon)^2}. \tag{57}$$

Applying Eq. (8) to Eq. (57), this equation can be written as

$$t(\epsilon) \approx C\frac{1}{\epsilon^{2\alpha}}, \tag{58}$$

where for $\alpha = 1$, we reobtain the classical one, $t(\epsilon) = C/\epsilon^2$.

4.1. Fractal Fokker–Planck equation

If energy loss in any collision is small compared with the other physical quantities of the system, the collision integral in Eq. (1) can be expanded as

$$\int_0^\infty \big[t(E+\epsilon,\epsilon)h(E+\epsilon,x) - t(E,\epsilon)h(E,x)\big]d\epsilon$$
$$= \sum_{n=1}^{\infty}(D_{E,F}^\alpha)^n\left[\int_F \frac{t(E,\epsilon)}{n!}\epsilon^n h(E,x)\right]d_F^\beta\epsilon, \tag{59}$$

and then we have

$$D_{x,F'}^\alpha h(E,x) = \sum_{n=1}^{\infty}(D_{E,F}^\alpha)^n\Big[M_n(E)h(E,x)\Big], \tag{60}$$

where $M_n(E) = 1/n!\int_F \epsilon^n t(\epsilon)d_F^\alpha\epsilon$ and $t(E+\epsilon,\epsilon)F(E+\epsilon,\epsilon)$ is considered to be a function of a fractal structure. If we assume that M_1, M_2 are nonzero, the resulting equation will be the fractal generalization of the Fokker–Planck equation:

$$D_{x,F'}^\alpha h(E,x) = (D_{E,F}^\alpha)\Big[M_1(E)h(E,x)\Big] + (D_{E,F}^\alpha)^2\Big[M_2(E)h(E,x)\Big]. \tag{61}$$

To solve this equation, we suppose that the second moment is small and ignorable, and then Eq.(61) becomes

$$D_{x,F'}^\alpha h(E,x) = (D_{E,F}^\alpha)\Big[M_1(E)h(E,x)\Big], \tag{62}$$

where it can be solved by the characteristic method, so we rewrite it as follows:

$$D_x^\alpha h(E,x) - M_1(E)D_E^\alpha h(E,x) = D_E^\alpha M_1 h(E,x). \tag{63}$$

Let $d^\alpha x$ and $d^\alpha E$ be two fractal forms that are divided by their coefficients:

$$\frac{d^\alpha x}{1} = -\frac{d^\alpha E}{M_1(E)}, \tag{64}$$

and, for $d^\alpha f$ and $d^\alpha E$,

$$\frac{d^\alpha h}{hD_E^\alpha M} = -\frac{d^\alpha E}{M_1(E)}. \tag{65}$$

Now, if we change the fractal variable to a continuous variable by the stair function S_F^α and solve Eq. (64) and Eq. (65), we obtain

$$S^\alpha(x) - C_0 = \int -\frac{d^\alpha E}{M_1(E)} \tag{66}$$

and

$$\frac{D_E^\alpha M_1(E)}{M_1(E)} = -\frac{d^\alpha h}{h}. \tag{67}$$

If h' is the conjugate function of h, $E' = S_F^\alpha(E)$ and $x' = S_F^\alpha(x)$, the solution will be

$$\ln h' = \ln M_1'(S^\alpha(E)) + c_1 \tag{68}$$

and

$$h'(E', x') = C_1 M_1(E'). \tag{69}$$

From Eq. (66), we can obtain a relation between C_0 and C_1:

$$F(C_0) = C_1. \tag{70}$$

Finally, from Eq. (70) and Eq. (69), we have

$$h'(E', x') = F\left(S_F^\alpha(x) + \int_0^{S_F^\alpha(E)} \frac{-d^\alpha E''}{M_1(E'')} \right) M_1'(E'). \tag{71}$$

4.2. Energy straggling of a fractal curve

F^α-calculus can also be applied in continuous matter for the phenomenon of energy straggling. Since heavy particles take small effects from collisions through the matter, their total path is approximated by a straight line. This approximation will be useful for calculating physical quantities as stop power, $\frac{dE}{dx}$, by general mathematics. However, the mentioned approximation fails for light particles as electrons because of the multiple scattering by collisions. Indeed, the real paths of the particles are fractal curves. F^α-calculus enables us to offer a straggling energy theory for the particles along such fractal curves. To start, let F be a fractal curve that a particle moves on in matter, and let $h(E,\theta)$ be the probability that the particle will have energy E at point θ; then its derivative along the fractal curve will be

$$D_\theta^\alpha h(E,\theta) = F - \lim_{\theta' \to \theta} \frac{h(\theta') - h(\theta)}{J(\theta') - J(\theta)}, \tag{72}$$

where $J(\theta) = S_F^\alpha(x)$ and $J(\theta') = S_F^\alpha(x')$, and the continuous parameter of the curve is a straight line at position x. Then the master equation that it satisfies is

$$D_\theta^\alpha h(\theta, E) = \int_F t(\epsilon)\big[h(\theta, E-\epsilon) - h(\theta, E)\big] d_F^\alpha \epsilon, \tag{73}$$

and to determine $t(\epsilon)$ a theory should be given [29]. Now we want to calculate the range of the particles through the matter. Classical straggling theory defines the total distance, R, that a particle with initial value energy E would come to rest, by the following equation:

$$R(E) = \int_0^E \frac{dE'}{M_1(E')}, \tag{74}$$

where the path of the particles is considered to be a straight line. Using Eq. (66), we generalize Eq. (74) to

$$S_F^\alpha(R) - S_F^\alpha(0) = \int_0^E \frac{d^\alpha E}{M_1(E)}. \tag{75}$$

From Eq. (8) we have

$$R^\alpha \approx \int_0^E \frac{d^\alpha E}{M_1(E)}, \tag{76}$$

where for $\alpha = 1$ we are led to the classical Eq. (74) .

5. Conclusion

We showed that the F^α-derivative of F^α-calculus satisfies the chain rule. Using F^α-calculus, we affirmed that the dimension of a quantum mechanics path is two. The fractal Schrödinger equation was proposed, and the energy and wave function of a particle in a fractal box were calculated. Finally, we extended Landau's method of energy straggling to fractals.

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