Turkish Journal of Physics

http://journals.tubitak.gov.tr/physics/

Research Article

Turk J Phys
(2018) 42: 386 – 401

doi:10.3906/fiz-1802-39

Noether symmetry analysis of anisotropic universe in $f(T,B)$ gravity

Yusuf KÜÇÜKAKÇA*
Department of Physics, Faculty of Science, Akdeniz University, Antalya, Turkey

Received: 27.02.2018 • **Accepted/Published Online:** 30.05.2018 • **Final Version:** 15.08.2018

Abstract: The purpose of the present work was to investigate the Noether symmetries of the locally rotationally symmetric Bianchi type I space time in $f(T,B)$ gravity theory, which depends on the torsion scalar T and the boundary term B. In this theory, we consider some particular models and investigate their Noether symmetry generators. Besides, we get exact cosmological solutions of the considered models including the matter dominant universe using the Noether symmetry technique. The obtained results coincide with the accelerated expansion behavior of the universe.

Key words: Noether symmetry, modified gravity, Bianchi space-time

1. Introduction

Astrophysical observations in recent years indicated that our universe is expanding at an accelerated phase. Various cosmological scenarios have been proposed to clarify this interesting behavior of the universe [1–4]. In this context, two categories have been considered in the literature. The first category is to introduce in the framework of the General Theory of Relativity (GR) an exotic liquid, called dark energy, that has a repulsive gravitational feature because it creates a negative pressure. It is believed that the late-time accelerating expansion of the universe may be due to the existence of dark energy. Although the underlying physics of the dark energy is still unclear, one of the remarkable nominees is the cosmological constant, which yields a negative pressure with the equation of state (EoS) parameter, $\omega = -1$. However, since the cosmological constant causes some problems such as extreme fine-tuning and coincidence problems, it has gradually lost its popularity [5]. In order to overcome these problems, many dark energy models that are some kinds of the scalar field such as quintessence [6], quintom [7], phantom energy [8], fermion [9–11], or tachyon field [12] were proposed. The second category is mainly based on the modifications of GR as a purely geometric effect. These modified theories may be considered the most popular candidates to reveal the mysterious nature of the dark energy. The most important one of these theories is $f(R)$ gravity that is constructed by inserting an arbitrary function of the curvature scalar to the Einstein–Hilbert action. It has recently been put forward in several forms of $f(R)$ gravity and discussed in many fields including the early- and late-time cosmic acceleration, solar system test, and black hole solution [13–16].

Teleparallel gravity (TG) is equivalent to GR in that its modified form is an alternative to explain the cosmic acceleration providing a gravitational alternative to the dark energy. This theory, called $f(T)$ gravity theory, is constructed by inserting an arbitrary function of the torsion scalar to the action of TG [17–19]. In this formalism of gravity, one could use the Weitzenböck connection that has torsion but not curvature instead of

*Correspondence: ykucukakca@akdeniz.edu.tr

utilizing curvature constructed by the Levi-Civita connection in GR. The dynamical variables in TG are tetrad fields. An important advantage of this theory is that it has the second-order field equations and, therefore, it is easy to deal with when compared to $f(R)$ gravity theory with the fourth-order field equations. On the other hand, Li et al. demonstrated that $f(T)$ gravity theory and its field equations are not invariant under local Lorentz transformations [20]. Recently, by this motivation, in the framework of teleparallel gravity, a new modified gravitational theory called $f(T,B)$ gravity that is reduced to both $f(T)$ and $f(R)$ gravity by the special selection of its form has been formulated [21]. Lagrangian density of $f(T,B)$ theory depends on both the torsion scalar T and the boundary term B. Bahamonde et al. [22] discussed different cosmological features for this theory such as the use of the reconstruction technique and examination of the validity of the laws of thermodynamics. Moreover, some cosmological solutions were examined by using the Noether symmetry approach in the spatially FRW metric [23].

The most suitable model for identifying the large-scale structure of the universe may be FRW space-time, which has a spatially homogeneous and isotropic nature. On the other hand, there are some indications in the CMB temperature anisotropy studies that may break the isotropic nature of the universe, leading to some interesting anomalies [24]. Therefore, it is important to explore the Bianchi space times giving information about the anisotropy in the early- and late-time universe on the current observations [25]. We also note that these models are the more generalized forms of the FRW universe. Akarsu and Kılınç [26] examined the anisotropic dark energy model in the LRS Bianchi type I cosmological analysis. Sharif and Shamir [27] discussed some anisotropic solutions in the context of $f(R)$ gravity. Bianchi cosmological models have been studied both in GR and in modified theories of gravity to understand the dynamics of the universe [28–32].

Symmetries play an important role in finding some exact solutions to dynamical systems. In particular, Noether symmetry that can be related to differential equations having a Lagrangian is a useful approach that leads to the existence of conserved quantities. In addition, this method is very useful for determining the unknown functions that exist in the Lagrangian. To date, this approach has been extensively studied in cosmological models such as scalar-tensor theories [33–37], the teleparallel dark energy model [38,39], models of fermionic fields [9–11], and $f(R)$ and $f(T)$ theories [40–45]. Furthermore, the technique has also been performed in different Bianchi space times [46–48], Gauss-Bonnet gravity [49–51], and others [52–54]. On the other hand, Sharif and Nawazish [55] investigated the existence of the Noether symmetry for the anisotropic models in $f(R)$ theory. Aghamohammadi [56] found the exact solution of the anisotropic space time with the $f(T)$ power-law model using the Noether symmetry approach. Recently, Bahamonde and Capozziello [23] explored some cosmological solutions by fixing the forms of $f(T,B)$ gravity with the presence of the Noether symmetry for the FRW space-time. In the present study, following the calculations performed in [23], we searched the Noether symmetries for the LRS Bianchi type I model in the $f(T,B)$ theory. We also examined some important cosmological parameters to determine how the universe evolved over time.

The outline of this work is as follows: In Section 2, we give the basic formalism of the teleparallel formulation of general relativity and its modified theories. We derive the modified field equations of $f(T,B)$ gravity for the LRS Bianchi type I model in Section 3. In Section 4, we investigate the Noether symmetries of the model and analyze the cosmological solutions. Finally, in Section 5, we give the basic results of this work.

2. $\mathbf{f(T,B)}$ theories

We will now briefly review the teleparallel formulation of GR and its modifications. In the TG theories, the fundamental dynamical objects are the tetrad fields (vierbeins) e^a_μ and inverse tetrad fields E^μ_a. The tetrad and the inverse tetrad fields satisfy the following orthogonality conditions:

$$E^\mu_m e^n_\mu = \delta^n_m, \qquad E^\nu_m e^m_\mu = \delta^\nu_\mu \tag{1}$$

The metric tensor $g_{\mu\nu}$ can be generated from the tetrad fields as

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \tag{2}$$

where η_{ab} is the Minkowski metric with the signature -2. As it is well known, GR is based on the symmetric Levi-Civita connection used to construct the covariant derivative. In contrast to the GR, teleparallel gravity utilizes the antisymmetric Weitzenböck connection defined as [57]:

$$W_{\mu}{}^{a}{}_{\nu} = \partial_\mu e^a{}_\nu \tag{3}$$

which yields zero curvature but nonzero torsion. The torsion tensor is the antisymmetric part of this connection as follows:

$$T^a{}_{\mu\nu} = W_{\mu}{}^{a}{}_{\nu} - W_{\nu}{}^{a}{}_{\mu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu \tag{4}$$

The Weitzenböck connection of TG can be expressed in terms of the usual Levi-Civita connection, which we denote by ${}^0\Gamma$ of GR as:

$$W^{\mu}_{\lambda\ \rho} = {}^0\Gamma^\mu_{\lambda\rho} + K^{\mu}_{\lambda\ \rho} \tag{5}$$

where K is called the contorsion tensor, which is defined by the following torsion tensor:

$$2K^{\lambda}_{\mu\ \nu} = T^\lambda{}_{\mu\nu} - T_{\mu\nu}{}^{\lambda} + T_{\mu}{}^{\lambda}{}_{\nu} \tag{6}$$

One also defines the following tensor:

$$S_\sigma{}^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_\sigma - \delta^\mu_\sigma T^\nu + \delta^\nu_\sigma T^\mu\right) \tag{7}$$

where T_μ is called the torsion vector, which is obtained by the contraction of the torsion tensor. The combination of Eq. (7) with the torsion tensor of Eq. (4) leads to the following torsion scalar:

$$T = T^\alpha{}_{\mu\nu} S_\alpha{}^{\mu\nu}, \tag{8}$$

and then the action of TG reads as:

$$\mathbf{S} = \frac{1}{\kappa^2}\int d^4x e T + \mathbf{S_m}, \tag{9}$$

where e is the volume element of the metric tensor that is equal to $\sqrt{-g}$ and $\mathbf{S_m}$ is the action of the standard matter content. Using the definitions given above, one can easily achieve the relation between the Ricci scalar related to the Levi-Civita connection and the torsion scalar [21]:

$$R = -T + \frac{2}{e}\partial_\mu\left(eT^\mu\right) = -T + B, \tag{10}$$

where $B = \frac{2}{e}\partial_\mu\left(eT^\mu\right)$ is a boundary term. This relationship given by Eq. (10) tells us that the action of the TG of Eq. (9) is dynamically equivalent to the standard action of GR, since they only differ by a total derivative.

One of the most popular generalizations of the teleparallel gravity is $f(T)$ gravity. The action integral for this theory is given by [17]:

$$\mathbf{S} = \frac{1}{\kappa^2}\int d^4x e f(T) + \mathbf{S_m}\,, \tag{11}$$

where $f(T)$ is a function of T. It is clear that $f(T)$ is a linear function of T and then the action of Eq. (10) is recovered. The gravitational field equations are derived by taking the variation according to the tetrad field of the action of Eq. (11). The resulting field equations are second order because the torsion scalar consists of the first derivatives of the tetrad fields. A new and interesting modified teleparallel theory of gravity has recently been proposed by Bahamonde et al. to combine these two theories. The new action has the following form [21]:

$$\mathbf{S} = \frac{1}{\kappa^2}\int d^4x e f(T,B) + \mathbf{S_m}\,, \tag{12}$$

where f depends on T and B. From the action of Eq. (12), one can show that the $f(T)$ and $f(R)$ gravity can be obtained by selecting $f\left(T,B\right) = f(T)$ and $f\left(T,B\right) = f\left(-T+B\right) = f(R)$, respectively. The gravitational field equations for the theory given by Eq. (12) are as follows [21]:

$$\begin{aligned} &2e\delta_\nu^\lambda \Box f_B - 2e\nabla^\lambda\nabla_\nu f_B + eBf_B\delta_\nu^\lambda + 4e\left[\partial_\mu f_B + \partial_\mu f_T\right]\, S_\nu{}^{\mu\lambda} \\ &+4e_\nu^a\partial_\mu\left(eS_a{}^{\mu\lambda}\right)f_T - 4ef_T T^\sigma{}_{\mu\nu}S_\sigma{}^{\mu\lambda} - ef\delta_\nu^\lambda = 16\pi e\Theta_\nu^\lambda \end{aligned} \tag{13}$$

where $\Theta_\nu^\lambda = \mathrm{e}_\nu^a\Theta_a^\lambda$ is the standard energy momentum tensor, ∇_ν stands for the covariant derivative with respect to the Levi-Civita connection, and $f_T = \partial f/\partial T$, $f_B = \partial f/\partial B$. In the next section, we will focus on the anisotropic Bianchi type I cosmological model for the abovementioned $f(T,B)$ theories.

3. Anisotropic $\mathbf{f(T,B)}$ cosmology

In the present work, we explore the cosmological consequences of $f(T,B)$ theory. In particular, we deal with $f(T,B)$ anisotropic cosmology in spatially homogenous Bianchi type I space-time such that the LRS line element is given by:

$$ds^2 = dt^2 - X\left(t\right)^2dx^2 - Y\left(t\right)^2\left(dy^2+dz^2\right) \tag{14}$$

where directional scale factors X and Y are functions of time t. The field equations of $f(T,B)$ cosmology are obtained either by the help of Eq. (13) or by using a point-like Lagrangian associated with the action of Eq. (12). Using Eqs. (2) and (14), we find the diagonal tetrad components as follows:

$$e_\mu^a = diag\left(1,X,Y,Y\right)\,. \tag{15}$$

For this tetrad component, the torsion scalar and boundary term can be calculated in their respective forms as follows:

$$T = -2\left(2\frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{Y}^2}{Y^2}\right)\,,\quad B = -2\left(\frac{\ddot{X}}{X} + 2\frac{\ddot{Y}}{Y} + 4\frac{\dot{X}\dot{Y}}{XY} + 2\frac{\dot{Y}^2}{Y^2}\right) \tag{16}$$

where the dot represents derivatives with respect to t. One can obtain the point-like Lagrangian related to the action of Eq. (12) if one uses the Lagrange multiplier approach to set T and B as a constraint of the dynamics. Therefore, inserting Eqs. (15) and (16) into the action of Eq. (12), we write Eq. (12) again in physical units as follows:

$$\mathbf{S} = \int dt XY^2 \left[f - \lambda_1 \left[T + 2\left(2\frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{Y}^2}{Y^2} \right) \right] - \lambda_2 \left[B + 2\left(\frac{\ddot{X}}{X} + \frac{2\ddot{Y}}{Y} + 4\frac{\dot{X}\dot{Y}}{XY} + 2\frac{\dot{Y}^2}{Y^2} \right) \right] + L_m \right], \quad (17)$$

where λ_1 and λ_2 are the Lagrange multipliers and L_m is the standard matter Lagrangian. We note that since there is no single definition of the matter Lagrangian, we can choose it as $L_m = -\rho_m = -\rho_{m0}(XY^2)^{-1}$, which corresponds to a matter-dominated universe [43]. The variation of the action of Eq. (17) with respect to T and B leads to $\lambda_1 = XY^2 f_T$ and $\lambda_2 = XY^2 f_B$. After some calculations, we obtain the point-like Lagrangian as follows:

$$L = XY^2 (f - Tf_T - Bf_B) - 2f_T \left(2Y\dot{X} + X\dot{Y} \right) \dot{Y}$$

$$+2Y \left(Y\dot{X} + 2X\dot{Y} \right) (f_{BB}\dot{B} + f_{T,B}\dot{T}) - \rho_{m0}\,. \quad (18)$$

It is well known that the basic properties of a dynamical system can be determined by the Euler–Lagrange equation, given by:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0\,, \quad (19)$$

where q_i and $\dot{q}_i$ are generalized coordinates and velocities of the configuration space. The configuration space of the Lagrangian of Eq. (18) is $Q = (X, Y, T, B)$, and its tangent space is given by $TQ = (X, Y, T, B\dot{X}, \dot{Y}, \dot{T}, \dot{B})$. Inserting the Lagrangian of Eq. (18) into the Euler–Lagrange equation for the variables X and Y we obtain the following:

$$f_T \left(\frac{2\ddot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} \right) + 2\frac{\dot{Y}}{Y}\dot{f}_T - \ddot{f}_B + \frac{1}{2}(f - Tf_T - Bf_B) = 0\,, \quad (20)$$

$$f_T \left(\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} \right) + \left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} \right) \dot{f}_T - \ddot{f}_B + \frac{1}{2}(f - Tf_T - Bf_B) = 0. \quad (21)$$

The modified Friedmann equation for $f(T, B)$ cosmology is obtained by imposing that the Hamiltonian related to the Lagrangian of Eq. (18) vanishes, i.e.

$$\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = 0.$$

$$\Rightarrow\ f_T \left(\frac{2\dot{X}\dot{Y}}{XY} + \frac{\dot{Y}^2}{Y^2} \right) - \left(\frac{\dot{X}}{X} + \frac{2\dot{Y}}{Y} \right) \dot{f}_B + \frac{1}{2}(f - Tf_T - Bf_B - \rho_m) = 0 \quad (22)$$

We now consider the relation between the scale factors as $X = Y^m$; $m \neq 0, 1$ where m measures the deviation from the isotropy. When $m = 1$, the universe is isotropic; otherwise, it is anisotropic. This physical condition comes from the assumption that the ratio of the shear scalar to the expansion scalar is constant. The physical importance of this condition by considering the perfect fluid having a barotropic equation of state is discussed by Collins [58]. Several researchers have also used this relation to obtain the cosmological solutions to the field equations [59–61]. Thus, we can rewrite the Lagrangian of Eq. (18) as follows:

$$L = Y^{m+2}\left(f - Tf_T - Bf_B\right) - 2\left(2m+1\right) f_T Y^m \dot{Y}^2 + 2(m+2)Y^{m+1}\dot{Y}(f_{BB}\dot{B} + f_{T,B}\dot{T}) - \rho_{m0} \tag{23}$$

which depends on Y, T, and B. For this Lagrangian, the field equations reduce to the following equations:

$$(2m+1)\left[f_T\left(\frac{2\ddot{Y}}{Y} + \frac{m\dot{Y}^2}{Y^2}\right) + 2\frac{\dot{Y}}{Y}\dot{f}_T\right] - (m+2)\,\ddot{f}_B + \frac{m+2}{2}\left(f - Tf_T - Bf_B\right) = 0\,, \tag{24}$$

$$(2m+1)\,f_T\frac{\dot{Y}^2}{Y^2} - (m+2)\,\frac{\dot{Y}}{Y}\dot{f}_B + \frac{1}{2}\left(f - Tf_T - Bf_B - \rho_m\right) = 0. \tag{25}$$

Since these equations are nonlinear differential equations, their solutions are very difficult. In order to find cosmological solutions to these equations, we also need to choose the form of the unknown function $f(T,B)$. In the next section, we utilize the Noether symmetry approach to determine the form of $f(T,B)$.

4. Noether symmetry approach and cosmological solutions

This section deals with the Noether symmetry technique for the Lagrangian given by Eq. (23). This technique is very useful for obtaining conserved quantities relevant to the dynamical system as well as for choosing the form of the unknown functions in the theory. Following the work by Capozziello and de Ritis [34], we define a vector field for Eq. (23) as follows:

$$\mathbf{X} = \alpha\frac{\partial}{\partial Y} + \beta\frac{\partial}{\partial T} + \gamma\frac{\partial}{\partial B} + \dot{\alpha}\frac{\partial}{\partial \dot{Y}} + \dot{\beta}\frac{\partial}{\partial \dot{T}} + \dot{\gamma}\frac{\partial}{\partial \dot{B}}\,, \tag{26}$$

where α, β, and γ depend on the generalized coordinates Y, T, and B. The Noether theorem tells us that the Lie derivative of any Lagrangian along a vector field is zero, i.e.

$$\pounds_{\mathbf{X}} L = 0\,. \tag{27}$$

If this condition satisfies, then $\mathbf{X}$ is a symmetry and the following constant of motion (conserved quantity, first integral) will be generated:

$$I_0 = \alpha\frac{\partial L}{\partial \dot{Y}} + \beta\frac{\partial L}{\partial \dot{T}} + \gamma\frac{\partial L}{\partial \dot{B}}\,, \tag{28}$$

Hence, implementing the Noether symmetry condition of Eq. (27) for the Lagrangian of Eq. (23), we find the system of partial differential equations as follows:

$$(2m+1)\,f_T\left(m\alpha + 2Y\frac{\partial\alpha}{\partial Y}\right) + f_{TB}Y\left((2m+1)\,\gamma - (m+2)Y\frac{\partial\beta}{\partial Y}\right) + (2m+1)\,Y f_{TT}\beta - (m+2)\,Y^2\frac{\partial\gamma}{\partial Y} = 0, \tag{29}$$

$$f_{TB}\frac{\partial\alpha}{\partial T}=0,\;\; f_{TB}\frac{\partial\alpha}{\partial T}=0, \tag{30}$$

$$(m+2)\,f_{TB}\left((m+1)\,\alpha+Y\frac{\partial\alpha}{\partial Y}+Y\frac{\partial\beta}{\partial T}\right)-2\,(2m+1)\,f_T\frac{\partial\alpha}{\partial T}+(m+2)\,Y f_{BB}\frac{\partial\beta}{\partial B}+(m+2)Y\left(f_{TTB}\beta+f_{TBB}\gamma\right)=0\,, \tag{31}$$

$$(m+2)\,f_{BB}\left((m+1)\,\alpha+Y\frac{\partial\alpha}{\partial Y}+Y\frac{\partial\gamma}{\partial B}\right)-2\,(2m+1)\,f_T\frac{\partial\alpha}{\partial B}+(m+2)\,Y f_{TB}\frac{\partial\beta}{\partial B}+(m+2)Y\left(f_{TBB}\beta+f_{BBB}\gamma\right)=0\,, \tag{32}$$

$$(m+2)\,(f-Tf_T-Bf_B)\,\alpha-Y\,(Tf_{TT}+Bf_{TB})\,\beta-Y\,(Tf_{TB}+Bf_{BB})\,\gamma=0\,. \tag{33}$$

There are two different ways to solve the Noether symmetry equations given by Eqs. (29)–(33): the first is to choose a particular shape of $f(T,B)$, and then to find the components of the vector field accordingly. The second method is to solve the equations directly and find the unknown functions. From a physical perspective, the first method is more preferable because it permits studying credible models. Thus, we prefer the second method to study the anisotropic $f(T,B)$ models.

4.1. Case 1: $\mathrm{f}(\mathbf{T},\mathbf{B})=\mathbf{f}(\mathbf{T})$

The first important model is the $f(T)$ gravity. In this case, the Lagrangian of Eq. (23) does not include the boundary term B. From the Noether symmetry of Eqs. (29)–(33), we can easily find the following solution for α, β, and $f(T)$:

$$\alpha=\alpha_0 Y^{1-\frac{m+2}{2n}},\;\;\beta=-\frac{\alpha_0(m+2)}{n}Y^{\frac{m+2}{2n}}T\,, \tag{34}$$

$$f\,(T)=T_0T^n\,, \tag{35}$$

where n, α_0, and T_0 are integration constants. In Eq. (28), the first integral associated with the Noether symmetry corresponding to this vector field has the following form:

$$Y^{\frac{m+2-2n}{2n}}\dot{Y}=k_0\,, \tag{36}$$

where we define

$$k_0=\left(\frac{I_0}{-4\,(-2)^{n-1}\,\alpha_0T_0(2m+1)^n}\right)^{\frac{1}{2n-1}}$$

The general solution of Eq. (36) is

$$Y\,(t)=\left[\frac{k_0(m+2)}{2n}t+c_1\right]^{\frac{2n}{m+2}}\,, \tag{37}$$

where c_1 is an integration constant and $m\neq-2$. From the condition $X=Y^m$, we obtain the scale factor along the x-direction as follows:

$$X\,(t)=\left[\frac{k_0(m+2)}{2n}t+c_1\right]^{\frac{2mn}{m+2}}\,. \tag{38}$$

Consequently, we have a power-law form for the scale factors. Such models suitable for Noether symmetry have been studied extensively in the literature for both isotropic [44] and anisotropic [56] space time. For $m=-2$ in Eq. (36) and using the definition of the average factor, we obtain $a\left(t\right)=a_0e^{k_0t}$, which is a de Sitter solution.

4.2. Case 2: $\mathbf{f\left(T,B\right)=b_0B^k+t_0T^n}$

Second, we assume that $f\left(T,B\right)=b_0B^k+t_0T^n$, where b_0, t_0, k, and n are the arbitrary constants. Substituting this form of $f\left(T,B\right)$ in the Noether symmetry of Eqs. (29)–(33), a trivial solution is obtained by $\alpha=\beta=\gamma=0$ for $k=0$, which means that there is no Noether symmetry. For $k=1$, we have $f\left(T,B\right)=b_0B+t_0T^n$, which is the same as in the previous case. At this point, we can note that if the function $f(T,B)$ is linear with respect to B, then there is no change in the field equations.

4.3. Case 3: $\mathbf{f\left(T,B\right)=b_0B^kT^n}$

In this case, we choose the form of $f(T,B)$ as a product of power law forms of T and B as $f\left(T,B\right)=b_0B^kT^n$, where b_0, k, and n are redefined nonzero constants. Using this form of $f(T,B)$ in Eqs. (29)–(33), we find the following solution:

$$\alpha=-\frac{\beta_0}{(m+2)}Y^{-(m+1)},\ \beta=2\beta_0Y^{-(m+2)}T,\ \gamma=\beta_0Y^{-(m+2)}B \tag{39}$$

where β_0 is an integration constant and we have a constraint as $n=\frac{1-k}{2}$ ($k\neq 1$ and $n\neq 0$, which yields a trivial case). Let us try to find some analytical solutions for this case. To do this, we consider three arbitrary functions z, u, and w which depend on the variables of configuration space as $z=z(Y,T,B)$, $u=u(Y,T,B)$, and $w=w(Y,T,B)$, respectively. Such a transformation allows us to find a cyclic variable so that the new Lagrangian can be rewritten in a form such that $L=L(u,w,\dot{z},\dot{u},\dot{w})$. This transformation is always possible if there is Noether symmetry. Following this process described in detail by Capozziello and de Ritis [34], one can find the corresponding variables transformation as:

$$z=-\frac{Y^{(m+2)}}{(m+2)\,\beta_0},\ u=Y^{2(m+2)}T,\ w=Y^{(m+2)}B \tag{40}$$

where we chose z as a variable cyclic. The original variables are obtained from Eq. (40) by converting to the new variables as follows:

$$Y=\left[-\left(m+2\right)\beta_0z\right]^{\frac{1}{m+2}},\ T=\frac{u}{\left[-\left(m+2\right)\beta_0z\right]^2},\ B=\frac{w}{\left[-\left(m+2\right)\beta_0z\right]} \tag{41}$$

When the point-like Lagrangian is rewritten with respect to these new variables, one can obtain it in the following form:

$$L=u^{-\frac{k+1}{2}}w^{k-2}\left[\beta_0k\left(m+2\right)\left(w\dot{u}-2u\dot{w}\right)\dot{z}+\beta_0^2\left(2m+1\right)w^2\dot{z}^2-\frac{uw^2}{2}\right]-\rho_{m0}. \tag{42}$$

We can easily see that the variable z is cyclic in the Lagrangian of Eq. (42). This Lagrangian yields the following Euler–Lagrange equations:

$$k\left(m+2\right)\left(w\dot{u}-2u\dot{w}\right)+2\beta_0\left(2m+1\right)w^2\dot{z}=\frac{I_0}{\beta_0}u^{\frac{k+1}{2}}w^{2-k} \tag{43}$$

$$(k-1)\,uw - 4\beta_0 k\,(m+2)\,u\ddot{z} - 2\beta_0^2(k+1)(2m+1)w\dot{z}^2 = 0, \tag{44}$$

$$uw - 4\beta_0\,(m+2)\,u\ddot{z} - 2\beta_0^2(k+1)(2m+1)w\dot{z}^2 = 0, \tag{45}$$

$$uw^2 + 2\beta_0 k\,(m+2)\,(w\dot{u} - 2u\dot{w})\,\dot{z} + 2\beta_0^2\,(2m+1)\,w^2\dot{z}^2 + 2\rho_{m0}u^{\frac{k+1}{2}}w^{2-k} = 0, \tag{46}$$

where I_0 is a constant of the motion associated with the coordinate z. Now we can rewrite the variables u and w in terms of the variable z using Eq. (16) with the condition $X = Y^m$ and Eq. (41). Then, inserting the results obtained for the u and w into Eqs. (44) and (45), these equations are identically satisfied. The other equations can be written as follows:

$$2^{\frac{k+1}{2}}\left(-\left(2m+1\right)\right)^{\frac{1-k}{2}}\left(m+2\right)^k\beta_0\dot{z}^{-k}\ddot{z}^{k-2}\left[\left(k-1\right)\ddot{z}^2 - k\dot{z}z\right] = I_0\,, \tag{47}$$

$$\dot{z}^2\ddot{z}^2\left[\rho_{m0}\beta_0(-(2m+1)\dot{z}^2)^{\frac{k-1}{2}} + 2^{\frac{k+1}{2}}k(m+2)^k\ddot{z}^{k-2}\left(\ddot{z}^2 - \dot{z}z\right)\right] = 0\,. \tag{48}$$

For $I_0 = 0$, the nontrivial solution can be easily found from Eq. (47) as follows:

$$z\left(t\right) = \frac{z_2\left(t - kz_1\right)^{k+1}}{k+1} + z_3 \tag{49}$$

where z_i are integration constants and $k \neq -1$. Inserting this solution into Eq. (48) we can find a constraint: $(2m+1)\left[k(m+2)\right]^k z_2\beta_0 = \rho_{m0}[-(2m+1)/2]^{(k+1)/2}$. Substituting the solution from Eq. (49) into Eq. (41), we obtain the solution for the scale factor on the y and z axes as follows:

$$Y\left(t\right) = \left[-\left(m+2\right)\beta_0\left(\frac{z_2\left(t-kz_1\right)^{k+1}}{k+1} + z_3\right)\right]^{\frac{1}{m+2}} \tag{50}$$

On the other hand, the scale factor in the direction of x can be found from the relation $X = Y^m$.

$$X\left(t\right) = \left[-\left(m+2\right)\beta_0\left(\frac{z_2\left(t-kz_1\right)^{k+1}}{k+1} + z_3\right)\right]^{\frac{m}{m+2}} \tag{51}$$

By means of the directional scale factors X and Y, the average scale factor for the universe is defined as $a\left(t\right) = (XY^2)^{\frac{1}{3}} = Y^{\frac{m+2}{3}}$ so that we get

$$a\left(t\right) = \left[-\left(m+2\right)\beta_0\left(\frac{z_2\left(t-kz_1\right)^{k+1}}{k+1} + z_3\right)\right]^{\frac{1}{3}} \tag{52}$$

To analyze the behavior of the obtained solution, we now examine some cosmological parameters such as the directional Hubble, average Hubble parameter, deceleration parameter, and equation of state parameter. The directional Hubble parameters $H_x = \frac{\dot{X}}{X}$, $H_y = H_z = \frac{\dot{Y}}{Y}$ and average Hubble parameter $H = \frac{\dot{a}}{a}$ are given by the following:

$$H_x = \frac{z_2(k+1)\left(t - kz_1\right)^k}{z_2\left(t - kz_1\right)^{k+1} + (k+1)z_3}, \; H_y = H_z = \frac{H_x}{m+2}, \tag{53}$$

$$H = \frac{z_2(k+1)\left(t - kz_1\right)^k}{3[z_2\left(t - kz_1\right)^{k+1} + (k+1)z_3]}. \tag{54}$$

The deceleration parameter defined by $q = -\frac{\ddot{a}a}{\dot{a}^2}$ plays a significant role in describing the nature of the expansion of the universe. The positive value of the deceleration parameter indicates a decelerating universe while the negative value shows an accelerating universe. In our model, it takes the following form:

$$q = -1 + \frac{3}{k+1} - \frac{3kz_3}{z_2\left(t - kz_1\right)^{k+1}} \tag{55}$$

The corresponding effective EoS parameter for this model reads as follows:

$$\omega = -1 + \frac{2}{k+1} - \frac{2kz_3}{z_2\left(t - kz_1\right)^{k+1}} \tag{56}$$

We demonstrate the characteristic behavior of the present model with respect to cosmic time t via the scale factors along the $x-$ and y-directions and the average scale factor in Figures 1–3 by giving some suitable values to the parameters with the initial condition $a(0) = 0$. In these figures, we observe that all scale factors increase monotonically when cosmic time increases and approach infinity as $t \to \infty$. In Figure 4, which represents the mean Hubble parameter, one can see that it decreases as t increases approaches zero as $t \to \infty$. The deceleration parameter q given by Eq. (55), plotted in Figure 5, tells us that in the early periods of the universe, there is a decelerating phase. However, over time it takes values from positive to negative depending on the values of k, which shows that our universe has a transition phase in previous time. On the other hand, the universe enters asymptotically the de Sitter universe for the large values of k. We also depict the EoS parameter ω as a function of the cosmic time for different values of k in Figure 6. If this parameter is less than $-1/3$, the

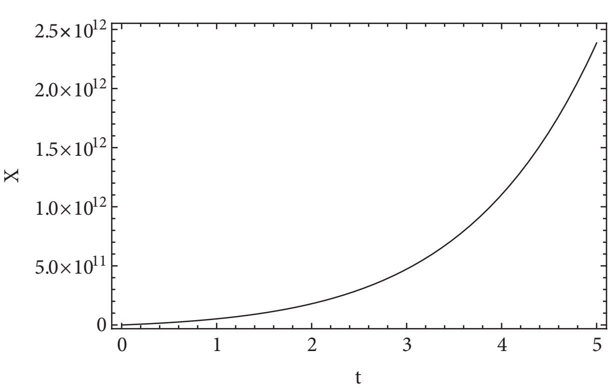

Figure 1. The behavior of the scale factor in the x-direction versus t for the numeric value of parameters $k = 10$, $m = 1.0672$, $z_1 = \beta_0 = -1$, $z_2 = 1$.

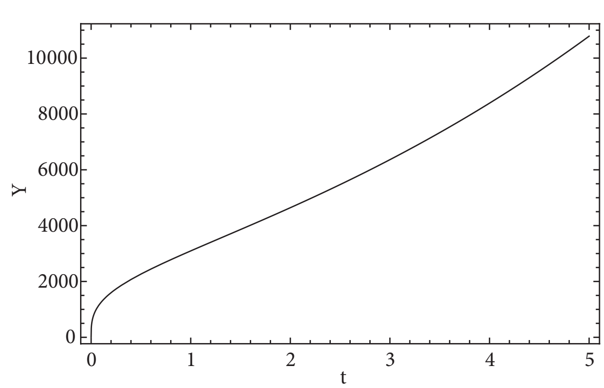

Figure 2. The behavior of the scale factors in the $y-$ and z-direction versus t for the numeric value of parameters $k = 10$, $m = 1.0672$, $z_1 = \beta_0 = -1$, $z_2 = 1$.

accelerating expansion of the universe can be generated. Furthermore, the observational constraints show that ω is around -1. When ω equals -1, the current universe is defined by the ΛCDM model where our universe is evolving towards an asymptotically de Sitter future. If $-1 < \omega < -1/3$, the dark energy models are known as quintessence, but the phantom dark energy models have an EoS parameter with $\omega < -1$. As can be seen in Figure 6, the effective EoS parameter shows quintessence behavior of the universe with time and in the late-time limit, it gets close to the ΛCDM model as the value of k increases.

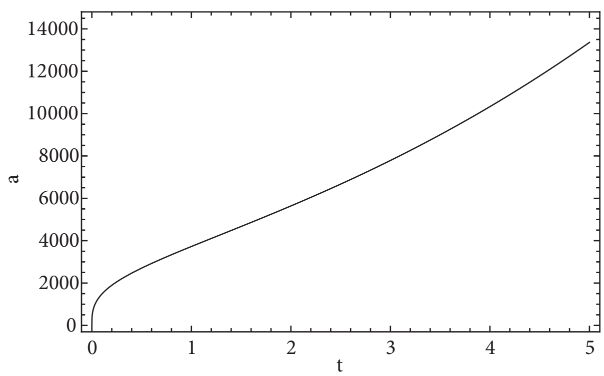

Figure 3. The behavior of the average scale factor versus t by taking $k = 10$, $m = 1.0672$, $z_1 = \beta_0 = -1$, $z_2 = 1$.

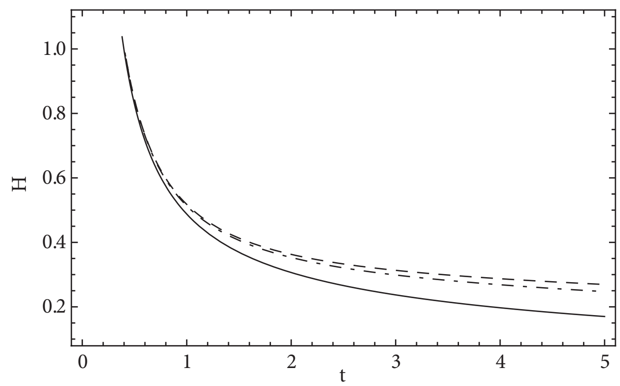

Figure 4. Evolutions of the average Hubble parameter versus t for different values of k. We set $k = 3$ (solid line), $k = 10$ (dashed line), $k = 15$ (dot dashed line) and $m = 1.0672$, $z_1 = \beta_0 = -1$, $z_2 = 1$.

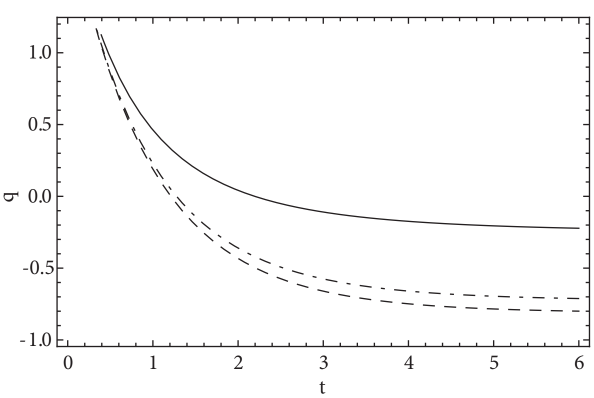

Figure 5. Plots of the deceleration parameter against t for different values of k. We set $k = 3$ (solid line), $k = 10$ (dashed line), $k = 15$ (dot dashed line) and $m = 1.0672$, $z_1 = \beta_0 = -1$, $z_2 = 1$.

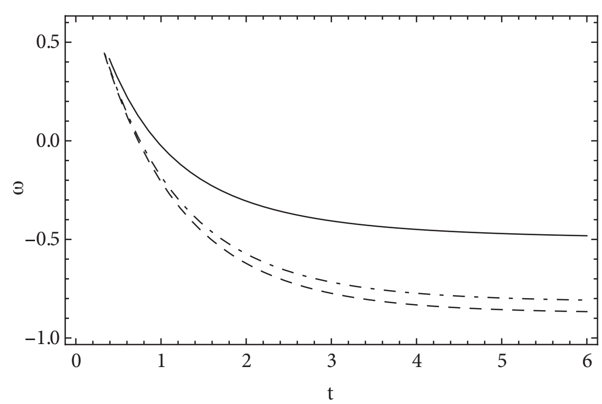

Figure 6. Plots of the effective EoS versus t for different values of the parameter k. We set $k = 3$ (solid line), $k = 10$ (dashed line), $k = 15$ (dot dashed line) and $m = 1.0672$, $z_1 = \beta_0 = -1$, $z_2 = 1$.

For the special case where the constants $z_1 = z_3 = 0$, the model also has important cosmological results. For this specific choice, the average scale factor, deceleration parameter, and effective EoS parameter reduce to the following forms:

$$a\,(t) = \left(-\frac{(m+2)\,z_2\beta_0}{k+1}\right)^{\frac{1}{3}} t^{\frac{k+1}{3}}, \;\; q = -1 + \frac{3}{k+1}, \; \omega = -1 + \frac{2}{k+1}\,. \tag{57}$$

From the above equations, quintessence models of dark energy (i.e. $-1 < \omega < -1/3$) can be achieved for the condition $k > 2$ while we have a phantom dark energy model ($\omega < -1$) for $k > -1$. In these conditions, our universe is both expanding and accelerating. Furthermore, for the interval $-1 < k < 2$, the model represents decelerating universe.

4.4. Case 3: $\mathbf{f(T,B) = -T + F(B)}$

Finally, we consider an interesting model that includes the torsion scalar plus a function of the boundary term. If $F(B)$ is linear in B, then the model reduces to the standard general relativity theory. By placing this model into the Noether symmetry equations we conclude that the vector field of Eq. (26) does not comprise its component β. Thus, Noether symmetry conditions of Eqs. (29)–(33) generate the following solutions for the vector field and the function $F(B)$:

$$\alpha = a_0 Y^{-(m+1)}, \qquad \gamma = -a_0\,(m+2)\,Y^{-(m+2)}B \tag{58}$$

$$F\,(B) = b_0 B + \frac{(2m+1)Bln(B)}{(m+2)^2} \tag{59}$$

where a_0 and b_0 are integration constants. Considering the solution of Eq. (58) allows us to do the following coordinate transformations:

$$z = \frac{Y^{m+2}}{a_0(m+2)}, \qquad u = Y^{m+2}B\,. \tag{60}$$

The Lagrangian in the transformed variables for the present model thus takes the suitable form

$$L = \frac{(2m+1)(2a_0\dot{u}\dot{z} - u^2)}{(m+2)^2 u} - \rho_{m0} \tag{61}$$

in which z is a cyclic variable. The Euler–Lagrange equations relative to the Lagrangian of Eq. (61) are

$$\frac{2(2m+1)a_0\dot{u}}{(m+2)^2 u} = I_0 \tag{62}$$

$$2a_0\ddot{z} + u = 0, \tag{63}$$

$$(2m+1)\left(2a_0\dot{u}\dot{z} - u^2\right) + (m+2)^2\rho_{m0}u = 0, \tag{64}$$

where I_0 is a constant of motion for the present model. The general solution of Eqs. (62)–(64) is

$$u\,(t) = u_0 e^{st}, \qquad z\,(t) = -\frac{u_0 e^{st}}{2s^2 a_0} + u_1 t + u_2 \tag{65}$$

with the constraint $\rho_{m0} + I_0 u_1 = 0$. Here, u_i are integration constants and we define $s = \frac{I_0(m+2)}{2a_0(2m+1)}$. Going back to the physical variable, one can find the solution in the following form:

$$Y\,(t) = \left[a_0(m+2)\left(-\frac{u_0 e^{st}}{2s^2 a_0} + u_1 t + u_2\right)\right]^{\frac{1}{m+2}} \tag{66}$$

The average scale factor is

$$a(t) = \left[a_0(m+2)\left(-\frac{u_0e^{st}}{2s^2a_0} + u_1t + u_2\right)\right]^{\frac{1}{3}} \tag{67}$$

For this model, we obtain the deceleration parameter

$$q = -1 + \frac{6sa_0\left[u_0e^{st}\left(u_1st + u_2s - 2u_1\right) + 2sa_0u_1^2\right]}{\left(u_0e^{st} + 2sa_0u_1\right)^2} \tag{68}$$

and the effective EoS parameter

$$\omega = -1 + \frac{4sa_0\left[u_0e^{st}\left(u_1st + u_2s - 2u_1\right) + 2sa_0u_1^2\right]}{\left(u_0e^{st} + 2sa_0u_1\right)^2} \tag{69}$$

Similar to the behavior of cosmological solutions in the previous model, the average scale factor with an initial condition $a(0) = 0$, shown in Figure 7, is a monotonically increasing function of time. Evolution of the deceleration parameter as a function of time for different values of anisotropy parameter m is depicted in Figure 8. It can be seen from this figure that our model shows the transition of q from the decelerating to the accelerating phase, and in the limit $t \to \infty$, its evolution becomes the de Sitter universe. Figure 9 shows the behavior of the effective EoS parameter with respect to cosmic time t for the different values of m. In this figure, we observe that crossing of the phantom divide line $\omega = -1$ can be addressed in this model described by the Noether symmetry solution.

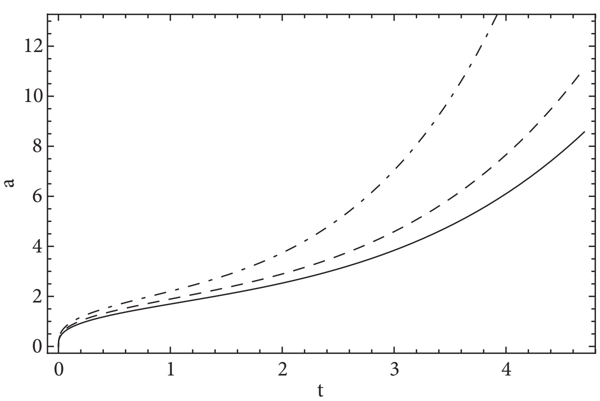

Figure 7. The behaviors of the average scale factor against t for different values of m. We set values $m = 1.00672$ (solid line), $m = 2.2$ (dashed line) $m = 4.5$ (dot dashed line) and $u_0 = -0.8$, $a_0 = I_0 = u_1 = 1$.

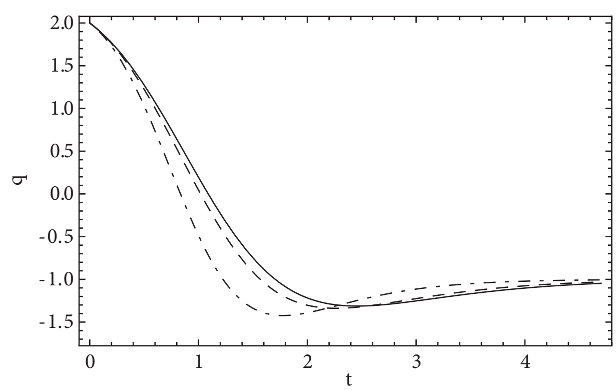

Figure 8. The behaviors of the deceleration parameter versus t for different values of m by taking values $m = 1.00672$ (solid line), $m = 2.2$ (dashed line) $m = 4.5$ (dot dashed line) and $u_0 = -0.8$, $a_0 = I_0 = u_1 = 1$.

5. Summary and conclusion

The modified theories of gravity that are constructed to describe the accelerated expansion of the universe are of great importance. One of these theories is the new generalization of teleparallel gravity including both functions of the torsion scalar and the boundary term in the form of $f(T,B)$ introduced by Bahamendo et al. [21]. In this work, we considered the cosmology constructed from $f(T,B)$ theory of gravity with anisotropy background. For this purpose, we considered the LRS Bianchi type I cosmological model in the presence of a matter-dominated

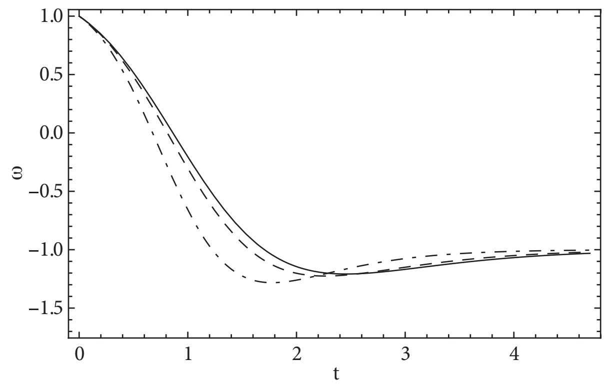

Figure 9. The behaviors of the effective EoS parameter versus t for the different value of m. We set values $m = 1.00672$ (solid line), $m = 2.2$ (dashed line) $m = 4.5$ (dot dashed line) and $u_0 = -0.8$, $a_0 = I_0 = u_1 = 1$.

universe, and due to highly nonlinear and complicated field equations, we used a physical assumption $X = Y^m$. The Noether symmetry approach is well known to be an important method for solving dynamical equations. Here, we discussed the Noether symmetry equations for two interesting cases of the $f(T, B)$ gravity theory. The first case was $f(T, B) = b_0 B^k T^{(1-k)/2}$, where b_0 and k are arbitrary real numbers. By introducing cyclic variables, we obtained some exact cosmological solutions of the corresponding field equations using the Noether symmetry approach. The second interesting case we were interested in was the form $f(T, B) = -T + F(B)$, where $F(B)$ is only the function of B. We determined the explicit form of $F(B)$ and solved the field equations via the Noether symmetry method. We also presented some cosmological parameters for the two cases and depicted the graphical behaviors of the models. The main and interesting feature of these solutions is that they describe an accelerating expansion of the universe. We also stress that phantom divide crossing can be realized in the second case but it is not crossed in the first case.

Acknowledgments

I am grateful to Dr Timur Şahin for the fruitful discussions. This work was supported by the Scientific Research Projects Unit of Akdeniz University.

References

[1] Riess, A. G.; Filippenko, A. V.; Challis, P.; Clocchiatti, A.; Diercks, A.; Garnavich, P. M.; Gilliland, R. L.; Hogan, C. J.; Jha, S.; Kirshner, R. P. et al. *Astron. J.* **1998**, *116*, 1009-1038.

[2] Perlmutter, S.; Aldering, G.; Goldhaber, G.; Knop, R. A.; Nugent, P.; Castro, P. G.; Deustua, S.; Fabbro, S.; Goobar, A.; Groom, D. E. et al. *Astrophys. J.* **1999**, *517*, 565-586.

[3] Spergel, D. N.; Verde, L.; Peiris, H. V.; Komatsu, E.; Nolta, M. R.; Bennett, C. L.; Halpern, M.; Hinshaw, G.; Jarosik, N.; Kogut, A. et al. *Astrophys. J. Suppl. S.* **2003**, *148*, 175-194.

[4] Netterfield, C. B.; Ade, P. A. R.; Bock, J. J.; Bond, J. R.; Borrill, J.; Boscaleri, A.; Coble, K.; Contaldi, C. R.; Crill, B.P.; de Bernardis, P. et al. *Astrophys. J.* **2002**, *571*, 604-614.

[5] Carroll, S. M. *Living Rev. Rel.* **2001**, *4*, 1-56.

[6] Ratra, B.; Peebles, P. J. E. *Phys. Rev. D* **1988**, *37*, 3406-3427.

[7] Guo, Z. K.; Piao, Y. S.; Zhang, X.; Zhang, Y. Z. *Phys. Lett. B* **2005**, *608*, 177-182.

[8] Caldwell, R. R. *Phys. Lett. B* **2002**, *545*, 23-29.

[9] de Souza, R. C.; Kremer, G. M. *Classical Quant. Grav.* **2008**, *25*, 225006.

[10] Kucukakca, Y. *Eur. Phys. J. C* **2014**, *74*, 3086.

[11] Gecim, G.; Kucukakca, Y.; Sucu, Y. *Adv. High Energy Phys.* **2015**, *2015*, 567395.

[12] Sen, A. *J. High Energy Phys.* **2002**, *04*, 048.

[13] Nojiri, N.; Odintsov, S. D. *Phys. Rev. D* **2003**, *68*, 123512.

[14] Capozziello, S.; de Laurentis, M. *Phys. Rep.* **2011**, *509*, 167-321.

[15] Olmo, G. J. *Phys. Rev. Lett.* **2005**, *95*, 261102.

[16] de la Cruz-Dombriz, A.; Dobado, A.; Maroto, A. L. *Phys. Rev. D* **2009**, *80*, 124011.

[17] Bengochea, G. R.; Ferraro, R. *Phys. Rev. D* **2009**, *79*, 124019.

[18] Linder, E. *Phys. Rev. D* **2010**, *81*, 127301.

[19] Myrzakulov, R. *Eur. Phys. J. C* **2011**, *71*, 1752.

[20] Li, B.; Sotiriou, T. P.; Barrow, J. D. *Phys. Rev. D* **2011**, *83*, 064035.

[21] Bahamonde, S.; Böhmer, C. G.; Wright, M. *Phys. Rev. D* **2015**, *92*, 104042.

[22] Bahamonde, S.; Zubair, M.; Abbas, G. *Phys. Dark Universe* **2018**, *19*, 78-90.

[23] Bahamonde, S.; Capozziello, S. *Eur. Phys. J. C* **2017**, *77*, 107.

[24] Copi, C. J.; Huterer, D.; Schwarz, D. J.; Starkman, G. D. *Phys. Rev. D* **2007**, *75*, 023507.

[25] Pitrou, C.; Pereira, T. S.; Uzan, J. P. *J. Cosmol. Astropart. P.* **2008**, *2008*, 004.

[26] Akarsu, Ö.; Kılınç, C. B. *Gen. Relat. Gravit.* **2010**, *42*, 119-140.

[27] Sharif, M.; Shamir, M. F. *Classical Quant. Grav.* **2009**, *26*, 235020.

[28] Singh, C. P.; Kumar, S. *Int. J. Mod. Phys. D* **2006**, *15*, 419-438.

[29] Koivisto, T.; Mota, D. F. *Astrophys. J.* **2008**, *679*, 1-5.

[30] Adhav, K. S. *Astrophys. Space Sci.* **2012**, 339, 365-369.

[31] Sharif, M.; Zubair, M. *Astrophys. Space Sci.* **2014**, *349*, 457-465.

[32] Kucukakca, Y.; Camci, U.; Semiz, İ. *Gen. Relat. Gravit.* **2012**, *44*, 1893-1917.

[33] Capozziello, S.; de Ritis, R.; Rubano, C.; Scudellaro, P. *Riv. Nuovo Cimento* **1996**, *19*, 1-114.

[34] Capozziello, S.; de Ritis, R. *Phys. Lett. A* **1993**, *177*, 1-7.

[35] Camci, U.; Kucukakca, Y. *Phys. Rev. D* **2007**, *76*, 084023.

[36] Paliathanasis, A.; Tsamparlis, M.; Basilakos, S.; Capozziello, S. *Phys. Rev. D* **2014**, *89*, 063532.

[37] Belinchon, J. A.; Harko, T.; Mak, M. K. *Astrophys. Space Sci.* **2016**, *361*, 52.

[38] Kucukakca, Y. *Eur. Phys. J. C* **2013**, *73*, 2327.

[39] Tajahmad, B. *Eur. Phys. J. C* **2017**, *77*, 211.

[40] Capozziello, S.; de Felice, A. *J. Cosmol. Astropart. Phys.* **2008**, *2008*, 016.

[41] Vakili, B. *Phys. Lett. B* **2008**, *664*, 16-20.

[42] Kucukakca, Y.; Camci, U. *Astrophys. Space Sci.* **2012**, *338*, 211-216.

[43] Kucukakca, Y. *Astrophys. Space Sci.* **2016**, *361*, 80.

[44] Wei, H.; Guo, X. J.; Wang, L.F. *Phys. Lett. B* **2012**, *707*, 298-304.

[45] Basilakos, S.; Capozziello, S.; de Laurentis, M.; Paliathanasis, A.; Tsamparlis, M. *Phys. Rev. D* **2013**, 88, 103526.

[46] Sharif, M.; Nawazish, I. *Gen. Relat. Gravit.* **2017**, *49*, 76.

[47] Shamir, M. F.; Ahmad, M. *Mod. Phys. Lett. A* **2017**, *32*, 1750086.

[48] Shamir, M. F.; Kanwal, F. *Eur. Phys. J. C* **2017**, *77*, 286.

[49] Capozziello, S.; de Laurentis M.; Dialektopoulos, K. F. *Eur. Phys. J. C* **2016**, *76*, 629.

[50] Sharif, M.; Fatima, I. *J. Exp. Theor. Phys.+* **2016**, *122*, 104-112.

[51] Shamir, M. F.; Ahmad, M. *Eur. Phys. J. C* **2017**, *77*, 55.

[52] Tajahmad, B. *Eur. Phys. J. C* **2017**, *77*, 510.

[53] Gecim, G.; Kucukakca, Y. arXiv:1708.07430[gr-qc] 2017.

[54] Momeni, D.; Myrzakulov, R.; Gudekli, E. *Int. J. Geom. Methods M.* **2015**, *12*, 1550101.

[55] Sharif, M.; Nawazish, I. *J. Exp. Theor. Phys.+* **2015**, *120*, 49-56.

[56] Aghamohammadi, A. *International Journal of Astronomy and Astrophysics* **2017**, *4*, 1-8.

[57] Maluf, J. W. *Ann. Phys.* **2013**, *525*, 339-357.

[58] Collins, C. B. *Phys. Lett. A* **1977**, *60*, 397-398.

[59] Bali, R.; Kumawat, P. *Phys. Lett. B* **2008**, *665*, 332-337.

[60] Amirhashchi, H.; Pradhan, A.; Saha, B. *Astrophys. Space Sci.* **2011**, *333*, 295-303.

[61] Sharif, M.; Zubair, M. *Astrophys. Space Sci.* **2012**, *339*, 45-51.