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Information-theoretic measure of the hyperbolical exponential-type potential

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Abstract: The approximate analytical solution of the 3-dimensional radial Schrödinger equation in the framework of the parametric Nikiforov–Uvarov method was obtained with a hyperbolical exponential-type potential. The energy eigenvalue equation and the corresponding wave function have been obtained explicitly. Using the integral method, we calculated Shannon entropy, information energy, Fisher information, and complexity measure. It was deduced that the complexity measure calculated using Shannon entropy with information energy and that calculated using Shannon entropy with Fisher information were similar.

Key words: Eigensolutions, Shannon entropy, Fisher information, complexity measure

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1. Introduction

In recent years, various quantum mechanical systems have been studied using information-theoretic measures of Shannon entropy and Fisher information [1]. Thus, a new uncertainty principle that originated from information theory has raised interest in quantum information theory. This uncertainty principle was recognized in quantum mechanics after the popular Heisenberg uncertainty principle, which demonstrated the impossibility of the simultaneous and precise measurement of position and momentum of a particle via a simple inequality relation [2]. The new uncertainty relation is based on probabilistic uncertainty measurement that currently exists as entropic uncertainty proposed in the concept of Shannon entropy in the form of

$$S(\gamma) + S(\rho) \geq D + D\log\pi, \tag{1}$$

where D is the spatial dimension and $S(\gamma)$ and $S(\rho)$ are the entropic systems for momentum and position spaces, respectively. Shannon entropy has received great attention and is widely reported by various authors in the context of information theory, among whom are Yañez-Navarro et al. [3], who investigated Shannon entropy for the position-dependent Schrödinger equation of a particle with a uniform solitonic mass density in the case of a trivial null potential. The authors observed that Shannon entropy in the momentum space decreases for narrower mass width, while in the position space Shannon entropy increases for narrower mass width. Dong and Draayer [4] studied Shannon entropy and standard deviation for a particle in a symmetrical square tangent potential well and deduced entropy squeezing in the position space. Najafizade et al. [5] investigated nonrelativistic Shannon information entropy for Kratzer potential. From their results, Shannon entropy in the

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position space decreases as the potential parameter increases, while in the momentum space Shannon entropy increases as the potential parameter increases. The authors [6] also studied nonrelativistic Shannon entropy for Killingbeck potential and examined the effects of some potential parameters on the stability of a system. Serrano et al. [7]

studied information-theoretic measures for the solitonic profile mass Schrödinger equation with a squared hyperbolic cosecant potential and deduced a decrease in Shannon entropy in the position

space for narrower mass width and an increase in the momentum space for narrower mass width. Yahya et al. [8] studied quantum information entropies for any 1-state P?schl–Teller-type potential and discovered that the Heisenberg principle and Cramer–Rao inequality hold. Idiodi and Onate [9] calculated entropy, Fisher information, and variance for the Frost–Musulin potential function. Onate et al. [10] studied the solutions of the 3-dimensional Schrödinger equation together with Shannon entropy and Fisher information under Eckart Manning–Rosen potential. In the literature, a report on complexity measure is lacking. This calls for further studies; hence, the motivation for these studies. In these studies, complexity measure will be considered in terms of Shannon entropy and information energy. It will also be considered in terms of Shannon entropy and Fisher information. The scheme of our work is as follows: in Section 2, we obtain the bound state solutions. In Section 3, we calculate the theoretical quantities. In Section 4, we discuss the results, and finally we give the conclusion in Section 5.

2. The solution of the 3-dimensional radial Schrödinger equation with a hyperbolical exponential-type potential

The radial Schrödinger equation for any quantum system is given by [11–14]:

$$\left[\frac{d^2}{dr^2}+\frac{2m\left(E_{n\ell}-V(r)\right)}{\hbar^2}-\frac{\ell(\ell+1)}{r^2}\right]U_{n\ell}(r)=0, \tag{2}$$

where $E_{n\ell}$ is the nonrelativistic energy of the system, m is the mass of the particle, $\hbar$ is the reduced Planck constant, $U_{n\ell}(r)$ is the wave function, and $V(r)$ is the interacting potential. In this study, the interacting potential is the hyperbolical exponential-type potential written in the form

$$V(r)=D\left[1-\frac{2\eta\left(1+e^{-2\delta\lambda}\right)}{1-e^{-2\delta\lambda}}+\frac{\eta^2\left(1+2e^{-2\delta\lambda}+e^{-4\delta\lambda}\right)}{\left(1-e^{-2\delta\lambda}\right)^2}\right], \tag{3}$$

where D characterizes the depth of the potential, η is the potential strength, λ is the screening parameter, and δ is the internuclear separation. To solve the radial Schrödinger equation given in Eq. (2) for the $\ell\neq 0$ state, it requires an approximation scheme to deal with the centrifugal barrier. Considering potential in Eq. (3), the following approximation scheme,

$$\frac{1}{r^2}\approx\frac{4\lambda^2e^{-2\delta\lambda}}{\left(1-e^{-2\delta\lambda}\right)^2}, \tag{4}$$

is valid for $\lambda<<1$. Now, substituting the approximation of Eq. (4) and potential of Eq. (3) into Eq. (2) and by defining a variable of the form $y=e^{-2\delta\lambda}$, Eq. (2) becomes

$$\left[\frac{d^2}{dy^2}+\frac{1-y}{y(1-y)}\frac{d}{dy}+\frac{-Py^2+Qy-R}{y^2\left(1-y\right)^2}\right]U_{n\ell}(y), \tag{5}$$

where we have used the following for mathematical simplicity:

$$P = m\left(\frac{D\left(1+2\eta+\eta^2\right)-E_{n\ell}}{2\lambda^2\hbar^2}\right), \tag{6}$$

$$Q = m\left(\frac{D\left(1-\eta^2\right)-E_{n\ell}}{\lambda^2\hbar^2}\right)-\ell(\ell+1), \tag{7}$$

$$R = m\left(\frac{D\left(1-2\eta+\eta^2\right)-E_{n\ell}}{2\lambda^2\hbar^2}\right). \tag{8}$$

To use the powerful and elegant parametric Nikiforov–Uvarov method, Eq. (5) is compared with the equation of the following form [15]:

$$\left(\frac{d^2}{ds^2}+\frac{c_1-c_2 s}{s(1-c_3 s)}\frac{d}{ds}+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{s^2(1-c_3 s)^2}\right)\psi(s)=0. \tag{9}$$

The parametric constants in Eq. (9) above are obtained as follows:

$$\left.\begin{array}{l} c_4=\frac{1-c_1}{2}, c_5=\frac{c_2-2c_3}{2}, c_6=c_5^2+\xi_1, c_7=2c_4c_5-\xi_2, c_8=c_4^2+\xi_3, \\ c_9=c_3\left(c_7+c_3c_8\right)+c_6, c_{10}=c_1+2c_4+2\sqrt{c_8}, c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right), \\ c_{12}=c_4+\sqrt{c_8}, c_{13}=c_5-\left(\sqrt{c_9}+c_3\sqrt{c_8}\right) \end{array}\right\}, \tag{10}$$

and we deduce the following:

$$\left.\begin{array}{l} c_1=c_2=c_3=1, c_4=0, c_5=-\frac{1}{2}, c_6=\frac{1}{4}+P, c_7=-Q, c_8=R, \\ c_9=\frac{1}{4}\left[(1+2\ell)^2+\frac{8mD\eta^2}{\lambda^2\hbar^2}\right], c_{10}=1+2\sqrt{R}, c_{11}=1+2\sqrt{R}+\sqrt{(1+2\ell)^2+\frac{8mD\eta^2}{\lambda^2\hbar^2}} \\ c_{12}=\sqrt{R}, c_{13}=-\frac{1}{2}\left[\sqrt{(1+2\ell)^2+\frac{8mD\eta^2}{\lambda^2\hbar^2}}+1\right] \end{array}\right\}. \tag{11}$$

In other to obtain the energy eigenvalue equation and the corresponding wave function, Tezcan and Sever [16–21] gave the following conditions:

$$nc_2-(2n+1)c_5+c_7+2c_3c_8+n(n-1)c_3+(2n+1)\sqrt{c_9}+\left(2\sqrt{c_9}+c_3(2n+1)\right)\sqrt{c_8}=0, \tag{12}$$

and

$$\psi_{n,\ell}(s)=N_{n,\ell}s^{c_{12}}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}}P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3 s), \tag{13}$$

where $P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}$ is a Jacobi polynomial. Substituting the parametric constants in Eq. (11) into Eqs. (12) and (13), respectively, we obtain the energy equation as

$$E_{n\ell}=D\left(1+\eta^2-2\eta\right)-\frac{2\lambda^2\hbar^2}{m}\left[\frac{n(n+1)+\ell(\ell+1)+\frac{1}{2}+\frac{2mD\eta(\eta-1)}{\lambda^2\hbar^2}+\left(n+\frac{1}{2}\right)\sqrt{(2\ell+1)^2+\frac{8mD\eta^2}{\lambda^2\hbar^2}}}{1+2n+\sqrt{(2\ell+1)^2+\frac{8mD\eta^2}{\lambda^2\hbar^2}}}\right]^2, \tag{14}$$

and the corresponding wave function as

$$U_{n,\ell}\left(y\right)=N_{n,\ell}y^{a}\left(1-y\right)^{b}P_{n}^{(2a,b)}\left(1-y\right), \tag{15}$$

where

$$a=\sqrt{\frac{m(D-E_{n,\ell}-2D\eta+D\eta^{2})}{2\lambda^{2}\hbar^{2}}}, \tag{16}$$

$$b=\sqrt{(2\ell+1)^{2}+\frac{8mD\eta^{2}}{\lambda^{2}\hbar^{2}}}. \tag{17}$$

To obtain the normalization constant $N_{n,\ell}$, we use the normalization condition:

$$\int_{0}^{\infty}\left|U_{n,\ell}(\delta)\right|^{2}d\delta=1. \tag{18}$$

$$-\frac{1}{2\lambda}\int_{1}^{0}\left|U_{n,\ell}(y)\right|^{2}\frac{dy}{y}=1.\quad y=e^{-2\delta\lambda}. \tag{19}$$

$$\frac{1}{2\lambda\left(\frac{1-s}{2}\right)}\int_{-1}^{1}\left|U_{n,\ell}(s)\right|^{2}ds=1,\quad 2s=1-y. \tag{20}$$

Substituting for $U_{n,\ell}(s)$ in Eq. (20), we have

$$\frac{N_{n\ell}^{2}}{2\lambda}\int_{-1}^{1}\left(\frac{1-s}{2}\right)^{d-1}\left(\frac{1+s}{2}\right)^{2b}\left(P_{n}^{(d,b)}(s)\right)^{2}ds=1, \tag{21}$$

where $d=2a$. Using the integral of the form

$$\int_{-1}^{1}\left(\frac{1-p}{2}\right)^{x}\left(\frac{1+p}{2}\right)^{y}\left(P_{n}^{(x,y)}(p)\right)^{2}dp=\frac{2\Gamma(x+n+1)\Gamma(y+n+1)}{n!x\Gamma(x+y+2n+1)\Gamma(x+y+n+1)}, \tag{22}$$

the normalization constant is obtained as

$$N_{n,\ell}=\sqrt{\frac{n!\lambda(2a-1)\Gamma(2b+2a+2n)\Gamma(2a+2b+n)}{\Gamma(2a+n)\Gamma(2b+n+1)}}. \tag{23}$$

3. Theoretical quantities and the hyperbolical exponential-type potential

In this section, theoretical quantities such as Shannon entropy, Onicescu information energy, and Fisher information are calculated. The results of these three quantities can also be used to calculate the complexity measures. In the context of this study, the square of the wave function is taken as the probability density function, i.e. $\rho(\delta) = |U_{n,\ell}(\delta)|^2$. The wave function given in Eq. (15) can also be written in the form of a hypergeometric function. Thus, the probability density function is written as follows:

$$\rho(\delta) = N_{n\ell}^2 e^{-4a\delta\lambda} \left(1 - e^{-2\delta\lambda}\right)^{2b} {}_2F_1\left(-n, n + 2(a+b); 2a+1, e^{-2\delta\lambda}\right)^2, \tag{24}$$

$$\gamma(\delta) = N_{n\ell}^2 e^{-4a\delta\lambda} \left(1 - e^{-2\delta\lambda}\right)^{2b} \left[P_n^{(2a,b)}\left(1 - e^{-2\delta\lambda}\right)\right]^2. \tag{25}$$

3.1. Shannon entropy

The Shannon entropy for position space and momentum space respectively is given as

$$S(\rho) = \int_0^\infty 4\pi\rho(\delta) In\rho(\delta) d\delta, \tag{26}$$

$$S(\gamma) = \int_0^\infty 4\pi\gamma(\delta) In\gamma(\delta) d\delta. \tag{27}$$

Substituting Eq. (24) into Eq. (26), the Shannon entropy in position space is obtained as

$$S(\rho) = -\frac{2\pi\aleph}{\lambda} \int_1^0 \rho(y) \frac{dy}{y}, \quad y = e^{-2\delta\lambda}, \tag{28}$$

where $\aleph = In\left((0.9048)^{2a} (0.0952)^{2b} {}_2F_1\left(-n, n+2(a+b); 2a+1, e^{-2\delta\lambda}\right)^2\right)$.

$$S(\rho) = \frac{2\pi\aleph}{\lambda} \int_0^1 \rho(z) \frac{dz}{1-z}, \quad z = 1 - y. \tag{29}$$

Defining

$${}_2F_1\left(-n, n+x+y; x+1, 1-c\right) = -\frac{\Gamma(x+n)}{\Gamma(x+1)}, \tag{30}$$

together with an integral of the form

$$\int_0^1 y^\alpha (1-y)^\beta {}_2F_1\left(-n, n+2(\alpha+\beta); 2\alpha+1, 1-y\right)^2 dy = \frac{n!\Gamma(\alpha+1)^2 \Gamma(\beta+n+1)}{\beta\Gamma(\alpha+n+1)\Gamma(\alpha+\beta+n+2)}, \tag{31}$$

Shannon entropy for the position space in Eq. (29) becomes

$$S(\rho) = \begin{bmatrix} \frac{2\pi(n!)^2\Gamma(2b+2a+2n)\Gamma(2b+2a+n)\Gamma(2b+1)^2\Gamma(2a+n+1)}{\Gamma(2b+n+1)^2(2b+2a+n+1)\Gamma(2a+n)} \\ \times In\left[(0.9048)^{2a} (0.0952)^{2b} \left(\frac{\Gamma(2a+n)}{\Gamma(2a+1)}\right)^2\right] \end{bmatrix}. \tag{32}$$

Substituting Eq. (25) into Eq. (27), the Shannon entropy in momentum space is obtained as

$$S(\gamma)=-\frac{2\pi\aleph_1}{\lambda}\int_1^0 \gamma(y)\frac{dy}{y},\quad y=e^{-2\delta\lambda}, \tag{33}$$

where $\aleph_1=(0.9048)^{2a}(0.0952)^{2b}\left(P_n^{(2a,b)}\left(e^{-2\delta\lambda}\right)\right)^2$.

$$S(\gamma)=\frac{2\pi\aleph_1}{\lambda}\int_0^1 \gamma(s)\frac{2ds}{1-s},\quad s=1-2y. \tag{34}$$

Defining a relation of the form

$$P_n^{(\alpha,\beta)}(x)=\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)}\times\sum_{m=0}^{n}\begin{pmatrix} n \\ m \end{pmatrix}\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)}\left(\frac{s-1}{2}\right)^m, \tag{35}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}={}^{x}C_y=\frac{\Gamma(x+1)}{\Gamma(x-y+1)\Gamma(y+1)}, \tag{36}$$

and a function of the form $\frac{(1+x)}{2}=1-\frac{(1-x)}{2}$ together with an integral of the form

$$\int_{-1}^{1}(1-p)^{\alpha-1}(1+p)^{\beta}\left[P_n^{(\alpha,\beta)}(p)\right]^2.dp=\frac{2^{\alpha+\beta}\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{n!\alpha\Gamma(n+\alpha+\beta+1)}, \tag{37}$$

the Shannon entropy for momentum space in Eq. (34) becomes

$$S(\gamma)=\left[\begin{array}{l} \frac{2^{2a+2b+1}\pi(2a-1)\Gamma(2a+2b+2n)\Gamma(2a+2b+n)\Gamma(2a+n+1)}{2a\Gamma(2a+n)\Gamma(2a+2b+n+1)}\times \\ In\left[\begin{array}{l} (0.9048)^{2a}(0.0952)^{2b}\frac{\Gamma(2a+n+1)}{n!\Gamma(2a+2b+n+1)}\times \\ \sum_{m=0}^{n}\frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m+1)}\frac{\Gamma(2a+2b+n+1)}{\Gamma(2a+1)}(1-n)^m \end{array}\right] \end{array}\right]. \tag{38}$$

3.2. Onicescu information energy

Onicescu information energy was proposed to measure the information content of a quantum system. It is also a possible measure of electron correlation energy in atoms and molecules [22]. For an electron density $\rho(\delta)$, the Onicescu information energy in position space is given by

$$E(\rho)=\int_0^\infty 4\pi\rho(\delta)^2 d\delta. \tag{39}$$

By substituting Eq. (24) into Eq. (39), we have

$$E(\rho)=-\frac{2\pi}{\lambda}\int_1^0 \rho(y)^2\frac{1}{y}dy. \tag{40}$$

$$E(\rho)=\frac{2\pi}{\lambda}\int_{0}^{1}\rho(z)^2\frac{1}{1-z}dz. \tag{41}$$

Using the integral given in Eq. (31), we have Onicescu information energy in position space as

$$E(\rho)=2\pi\lambda(n!)^4\left[\frac{\Gamma(2a+2b+2n)\Gamma(2a+2b+n)\Gamma(2a+n+1)\Gamma\left(2b+1\right)^2}{\Gamma(2a+n)\Gamma\left(2b+n+1\right)^2\Gamma(2a+2b+n+1)}\right]^2. \tag{42}$$

The Onicescu information for momentum space is given by

$$E(\gamma)=\int_{0}^{\infty}4\pi\gamma(\delta)^2d\delta. \tag{43}$$

Substituting Eq. (25) into Eq. (43), we have

$$E(\gamma)=-\frac{2\pi}{\lambda}\int_{1}^{0}\gamma(y)^2\frac{1}{y}dy, \tag{44}$$

$$E(\gamma)=\frac{2\pi}{\lambda}\int_{-1}^{1}\gamma(s)^2\frac{2}{1-s}ds. \tag{45}$$

Using the integral in Eq. (37), the Onicescu information energy for momentum space is obtained as

$$E(\gamma)=\frac{2^{2b+2a-1}\pi\lambda}{n!}\left(\frac{(2a-1)\Gamma(2a+2b+2n)\Gamma(2a+2b+n)\Gamma(2a+n+1)}{a\Gamma(2a+n)\Gamma(2a+2b+n+1)}\right)^2. \tag{46}$$

3.3. Fisher information

Fisher information is basically used to predict the localization of a particle in a system. Fisher information for the position space is given by

$$I(\rho)=\int_{0}^{\infty}\frac{1}{\rho(\delta)}\left[\frac{d\rho(\delta)}{d\delta}\right]^2d\delta. \tag{47}$$

Substituting the probability density function, we easily have

$$I(\rho)=-\frac{1}{2\lambda}\int_{1}^{0}\frac{1}{\rho(y)}\left[\frac{d\rho(y)}{dy}\right]^2\frac{1}{y}dy, \tag{48}$$

$$I(\rho)=\frac{1}{2\lambda}\int_{0}^{1}\frac{1}{\rho(z)}\left[\frac{d\rho(z)}{dz}\right]^2\frac{1}{1-z}dz. \tag{49}$$

$$I(\rho)=\frac{N_{n\ell}^{2}}{2\lambda}\left[\begin{array}{l} 4b^{2}\int\limits_{0}^{1}z^{2b-2}\left(1-z\right)^{2a-1}{}_{2}F_{1}\left(-n,n+2(a+b);2a+1,1-z\right)^{2}dz- \\ 2b(2a-1)\int\limits_{0}^{1}z^{2b-1}\left(1-z\right)^{2a-2}{}_{2}F_{1}\left(-n,n+2(a+b);2a+1,1-z\right)^{2}dz- \\ 2b\int\limits_{0}^{1}z^{2b-1}\left(1-z\right)^{2a-2}{}_{2}F_{1}\left(-n,n+2(a+b);2a+1,1-z\right)^{2}dz+ \\ (2a-1)\int\limits_{0}^{1}z^{2b}\left(1-z\right)^{4a-3}{}_{2}F_{1}\left(-n,n+2(a+b);2a+1,1-z\right)^{2}dz+ \\ (2a-1)^{2}\int\limits_{0}^{1}z^{2b}\left(1-z\right)^{2a-3}{}_{2}F_{1}\left(-n,n+2(a+b);2a+1,1-z\right)^{2}dz+ \\ \int\limits_{0}^{1}z^{2b}\left(1-z\right)^{2a-3}{}_{2}F_{1}\left(-n,n+2(a+b);2a+1,1-z\right)^{2}dz \end{array}\right]. \tag{50}$$

Using the integral in Eq. (31), Fisher information for the position space is obtained as

$$I(\rho)=\left[\begin{array}{l} \frac{2b^{2}(n!)^{2}\Gamma(2b-1)^{2}\Gamma(2a+2b+2n)\Gamma(2a+2b+n)}{\Gamma(2a+n)\Gamma(2b+n-1)\Gamma(2a+2b+n-1)}- \\ \frac{b(2a-1)(n!)^{2}\Gamma(2b)^{2}\Gamma(2a+2b+2n)\Gamma(2a+2b+n)}{(2a-2)\Gamma(2b+n)\Gamma(2a+2b+n-1)\Gamma(2a+n+1)} \\ -\frac{b(n!)^{2}\Gamma(2b)^{2}\Gamma(2a+2b+2n)\Gamma(2a+2b+n)}{(2a-2)\Gamma(2b+n)\Gamma(2a+2b+n-1)\Gamma(2a+n+1)}+ \\ \frac{(2a-1)(n!)^{2}\Gamma(4a+n-1)\Gamma(2b+1)^{2}\Gamma(2a+2b+2n)\Gamma(2a+2b+n)}{(4a-3)\Gamma(2a+n)\Gamma(2b+n+1)\Gamma(2a+2b+n-1)\Gamma(2a+n+1)}+ \\ \frac{(2a-1)^{2}(n!)^{2}\Gamma(2a+n-1)\Gamma(2b+1)^{2}\Gamma(2a+2b+2n)\Gamma(2a+2b+n)}{(2a-3)\Gamma(2a+n)\Gamma(2b+n+1)\Gamma(2a+2b+n-1)\Gamma(2a+n+1)} \\ +\frac{(n!)^{2}\Gamma(2a+n-1)\Gamma(2b+1)^{2}\Gamma(2a+2b+2n)\Gamma(2a+2b+n)}{(2a-3)\Gamma(2a+n)\Gamma(2b+n+1)\Gamma(2a+2b+n-1)\Gamma(2a+n+1)} \end{array}\right]. \tag{51}$$

Fisher information for the position space is given by

$$I(\gamma)=\int\limits_{0}^{\infty}\frac{1}{\gamma(\delta)}\left[\frac{d\gamma(\delta)}{d\delta}\right]^{2}d\delta. \tag{52}$$

Substituting the probability density function in Eq. (25) into Eq. (52), we easily have

$$I(\gamma)=-\frac{1}{2\lambda}\int\limits_{1}^{0}\frac{1}{\gamma(y)}\left[\frac{d\gamma(y)}{dy}\right]^{2}\frac{1}{y}dy, \tag{53}$$

$$I(\gamma)=\frac{1}{4\lambda}\int\limits_{-1}^{1}\frac{1}{\gamma(s)}\left[\frac{d\gamma(s)}{ds}\right]^{2}\frac{2}{1-s}ds. \tag{54}$$

$$I(\gamma)=\frac{N_{n\ell}^2}{4\lambda}\left[\begin{array}{l} 4b^2\int\limits_{-1}^{1}\left(\frac{1+s}{2}\right)^{2b-2}\left(\frac{1-s}{2}\right)^{d-1}\left[P_n^{(d,b)}(s)\right]^2ds-2b(d-1)\int\limits_{-1}^{1}\left(\frac{1+s}{2}\right)^{2b-1}\left(\frac{1-s}{2}\right)^{d-2}\left[P_n^{(d,b)}(s)\right]^2ds \\ +\frac{4b}{s}\int\limits_{-1}^{1}\left(\frac{1+s}{2}\right)^{2b-1}\left(\frac{1-s}{2}\right)^{d-1}\left[P_n^{(d,b)}(s)\right]^2ds-(d-1)\frac{2}{s}\int\limits_{-1}^{1}\left(\frac{1+s}{2}\right)^{2b}\left(\frac{1-s}{2}\right)^{d-2}\left[P_n^{(d,b)}(s)\right]^2ds+ \\ (d-1)^2\int\limits_{-1}^{1}\left(\frac{1+s}{2}\right)^{2b}\left(\frac{1-s}{2}\right)^{d-3}\left[P_n^{(d,b)}(s)\right]^2ds+\frac{4}{s^2}\int\limits_{-1}^{1}\left(\frac{1+s}{2}\right)^{2b}\left(\frac{1-s}{2}\right)^{d-1}\left[P_n^{(d,b)}(s)\right]^2ds \end{array}\right]. \tag{55}$$

Using the integral in Eq. (37), Fisher information for the position space is obtained as

$$I(\gamma)=2^{2a+2b-2}\times$$

$$\left[\begin{array}{l} \frac{b^2(2a-1)\Gamma(2a+n+1)\Gamma(2b+n-1)\Gamma(2a+2b+2n)\Gamma(2a+2b+n)}{2a\Gamma(2a+n)\Gamma(2a+2b+n-1)\Gamma(2b+n+1)}-\frac{(2a-1)^2\Gamma(2a+2b+2n)}{s(2a-2)} \\ +\frac{(2a-1)^3\Gamma(2a+n-1)\Gamma(2a+2b+2n)\Gamma(2a+2b+n)}{4(2a-3)\Gamma(2a+n)\Gamma(2a+2b+n-1)}+\frac{2b\Gamma(2b+n)\Gamma(2a+n+1)\Gamma(2a+2b+2n)\Gamma(2a+2b+n)}{s\Gamma(2a+2b+n-1)\Gamma(2a+n)\Gamma(2b+n+1)} \\ +\frac{4\Gamma(2a+n+1)\Gamma(2a+2b+2n)\Gamma(2a+2b+n)}{s^2\Gamma(2a+n)\Gamma(2b+2a+n+1)}-\frac{b(2a-1)^2\Gamma(2b+n)\Gamma(2a+1)\Gamma(2a+2b+2n)\Gamma(2a+2b+n)}{2(2a-2)\Gamma(2a+2b+n-2)\Gamma(2a+n)\Gamma(2b+n+1)} \end{array}\right]. \tag{56}$$

3.4. Complexity measure C

In this study, we consider the complexity measure in two different forms. In the first form, complexity measure C_{SE} is considered in terms of the Shannon entropy and information energy. In the second form, complexity C_{SF} is considered in terms of the Shannon entropy and Fisher information. The complexity is calculated for the ground state in both cases. In terms of Shannon entropy and information energy,

$$C_{SE}=S(\rho)E(\rho)+S(\gamma)E(\gamma)+S(\rho)E(\gamma)+S(\gamma)E(\rho). \tag{57}$$

Substituting Eqs. (32), (38), (42), and (46) into Eq. (57), we have

$$C_{SE}=9.872\lambda^2\left[\begin{array}{l} \left[\frac{\Gamma(2a+2b)^2\Gamma(2a+1)}{\Gamma(2a)\Gamma(2a+2b+1)}\right]^3\left[2+\frac{2^{2a+2b-1}\Gamma(2a-1)^2}{a^2}\right]\times \\ \left[\begin{array}{l} \frac{2^{2a+2b}(2a-1)In\left[(0.9048)^{2a}(0.0952)^{2b}\frac{\Gamma(2a+1)}{\Gamma(2a+2b+1)}\times\sum\limits_{m=0}^{n=m}\frac{\Gamma(2a+2b+1)}{\Gamma(2a+1)}\right]}{2a}+ \\ In\left[(0.9048)^{2a}(0.0952)^{2b}\left(\frac{\Gamma(2a)}{\Gamma(2a+1)}\right)^2\right] \end{array}\right] \end{array}\right]. \tag{58}$$

In terms of the Shannon entropy and Fisher information,

$$C_{SF}=\frac{S(\rho)+S(\gamma)}{I(\rho)+I(\gamma)}. \tag{59}$$

Substituting Eqs. (32), (38), (51), and (56) into Eq. (59), we have

$$C_{SF}=\frac{\frac{2\pi\Gamma(2b+2a)^{2}\Gamma(2a+1)}{(2b+2a+1)\Gamma(2a)}\left[\begin{array}{l}\frac{2^{2a+2b}(2a-1)In\left[(0.9048)^{2a}(0.0952)^{2b}\frac{\Gamma(2a+1)}{\Gamma(2a+2b+1)}\times\sum_{m=0}^{n=m}\frac{\Gamma(2a+2b+1)}{\Gamma(2a+1)}\right]}{2a}\\ +In\left[(0.9048)^{2a}\left(0.0952\right)^{2b}\left(\frac{\Gamma(2a)}{\Gamma(2a+1)}\right)^{2}\right]\end{array}\right]}{\left[\begin{array}{l}\frac{2^{2a+2b}b^{2}(2a-1)\Gamma(2a+1)\Gamma(2b-1)[\Gamma(2a+2b)\Gamma]^{2}}{8a\Gamma(2a)\Gamma(2a+2b-1)\Gamma(2b+1)}-\frac{2^{2a+2b}(2a-1)^{2}\Gamma(2a+2b)}{4s(2a-2)}\\ +\frac{2^{2a+2b}(2a-1)^{3}\Gamma(2a-1)[\Gamma(2a+2b)\Gamma]^{2}}{16(2a-3)\Gamma(2a)\Gamma(2a+2b-1)}+\frac{2^{2a+2b}b\Gamma(2b)\Gamma(2a+1)[\Gamma(2a+2b)\Gamma]^{2}}{2s\Gamma(2a+2b-1)\Gamma(2a)\Gamma(2b+1)}\\ +\frac{2^{2a+2b}\Gamma(2a+1)[\Gamma(2a+2b)\Gamma]^{2}}{s^{2}\Gamma(2a)\Gamma(2b+2a+1)}-\frac{2^{2a+2b}b(2a-1)^{2}\Gamma(2b)\Gamma(2a+1)[\Gamma(2a+2b)\Gamma]^{2}}{8(2a-2)\Gamma(2a+2b-2)\Gamma(2a)\Gamma(2b+1)}\end{array}\right]+\left[\begin{array}{l}\frac{2b^{2}\Gamma(2b-1)^{2}[\Gamma(2a+2b)\Gamma]^{2}}{\Gamma(2a)\Gamma(2b-1)\Gamma(2a+2b-1)}+\frac{(2a-1)\Gamma(4a-1)\Gamma(2b+1)[\Gamma(2a+2b)\Gamma]^{2}}{(4a-3)\Gamma(2a)\Gamma(2a+2b-1)\Gamma(2a+1)}\\ -\frac{b(2a-1)\Gamma(2b)[\Gamma(2a+2b)\Gamma]^{2}}{(2a-2)\Gamma(2a+2b-1)\Gamma(2a+1)}+\frac{\Gamma(2a-1)\Gamma(2b+1)[\Gamma(2a+2b)\Gamma]^{2}}{(2a-3)\Gamma(2a)\Gamma(2a+2b-1)\Gamma(2a+1)}\\ -\frac{b\Gamma(2b)[\Gamma(2a+2b)\Gamma]^{2}}{(2a-2)\Gamma(2a+2b-1)\Gamma(2a+1)}+\frac{(2a-1)^{2}\Gamma(2a-1)\Gamma(2b+1)[\Gamma(2a+2b)\Gamma]^{2}}{(2a-3)\Gamma(2a)\Gamma(2a+2b-1)\Gamma(2a+1)}\end{array}\right]}. \tag{60}$$

4. Discussion

To test the accuracy of the energy equation, we computed numerical results for 2p, 3p, 3d, 4p, 4d, and 4f. These were compared with the results of two other methods as shown in the Table. Our results showed an

Table. Comparison of bound state energy eigenvalues for various n and ℓ with $\eta = 0.1$ and $D = 10$ for 2p, 3p, 3d, 4p, 4d, and 4f.

State	λ	Present result	[23]	[24]
2p	0.10	2.61556	2.61935	2.61874
	0.15	3.89830	3.90645	3.90544
	0.20	4.99062	5.00457	5.00533
	0.25	5.86611	5.88725	5.88594
3p	0.10	4.73223	4.73638	4.73540
	0.15	6.03829	6.04649	6.04543
	0.20	6.90394	6.91733	6.91663
	0.25	7.46417	7.48358	7.48400
3d	0.10	3.61747	3.62769	3.62699
	0.15	5.27263	5.29510	5.29404
	0.20	6.43684	6.47598	6.47492
4p	0.10	5.99970	6.00390	6.00287
	0.15	7.10812	7.11589	7.11526
	0.20	7.70634	7.71860	7.71903
4d	0.10	5.32177	5.33216	5.33129
	0.15	6.71441	6.73642	6.73583
	0.20	7.53672	7.54331	7.54480
3f	0.10	4.67061	4.69058	4.68965
	0.15	6.38708	6.43112	4.42992
	0.20	7.35782	7.43334	7.43397

excellent agreement with the existing results. The comparison of the results with those of Lucha and Sch?berl [22] shows the validity of the approximation. In Figure 1, we examine the ground state energy eigenvalue $E_{0,\ell}$ against the potential parameter η with $\mu = \hbar = \ell = 1, \lambda = 0.2$, and $D = 9$. It is observed that the energy eigenvalue is higher at the lower values of the potential parameter but decreases monotonically as the potential parameter increases. However, at $\eta \geq 1$,the eigenvalue energy tends to be stable. Thus, a particle under this system becomes less attractive as it remains stable at this point. In Figure 2, we plot eigenvalue energy and energy stored against the potential parameter with $\mu = \hbar = \ell = 1, D = 9$, and $\lambda = 0.2$ at the ground state. It is observed that for the five values of the potential parameter, the eigenvalue is almost zero. However, at $\eta = 0.01$,, the two energies are almost equal. The energy eigenvalue is pulled down completely by the attractive nature of the interacting particles. The attractive force is greater on the information energy at $0.01 \leq \eta \leq 0.03$ where there is an almost steady and gentle increase in the information energy. At $\eta \geq 0.04$, the force becomes much less on the information energy and thus there is a sharp rise in the information energy. The particles have gained more energy and as such, they are capable of moving very fast. In Figure 3, we plot the complexity measure of the Shannon entropy and information energy against the potential depth. In Figure 4, we plot the complexity measure of Shannon entropy and Fisher information against the potential depth. As can be seen from the two figures, the complexity measures follow the same trend, except that the complexity measure of Shannon entropy and information energy has higher values. In both figures, there is a sharp rise from zero level to the maximum. A sharp decrease from the maximum is also observed in both figures. Finally, the two figures have positive shapes. In Figures 5 and 6, we examine Shannon entropy in position space and momentum space respectively against the potential strength of the hyperbolical exponential-type potential. It is observed that the Shannon entropy in position space decreases as the potential strength increases, but in the momentum space, Shannon entropy increases as the potential strength increases. Hence, Shannon entropy verified the Heisenberg uncertainty principle for hyperbolical exponential-type potential.

5. Conclusion

In this work, we have studied the bound state solutions of the nonrelativistic 3-dimensional radial Schrödinger equation with the hyperbolical exponential-type potential model. By employing a suitable approximation

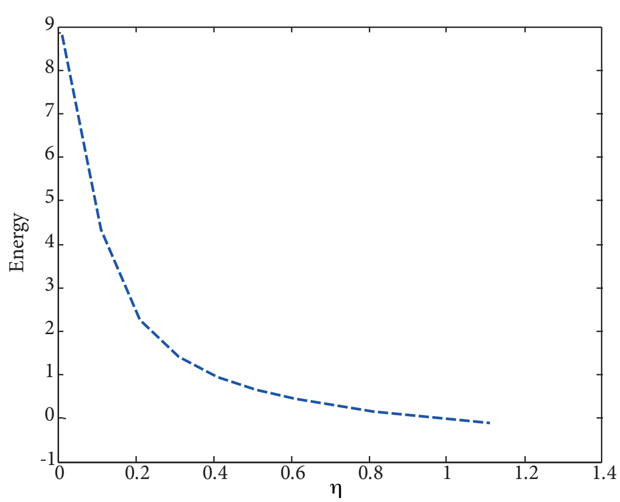

Figure 1. Energy eigenvalue $E_{n,\ell}$ against potential parameter η.

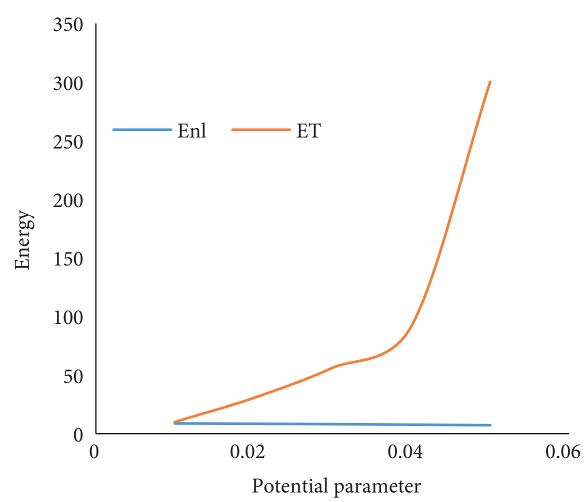

Figure 2. Variation of the energy eigenvalue (blue color) and total information energy (red color) against potential parameter η.

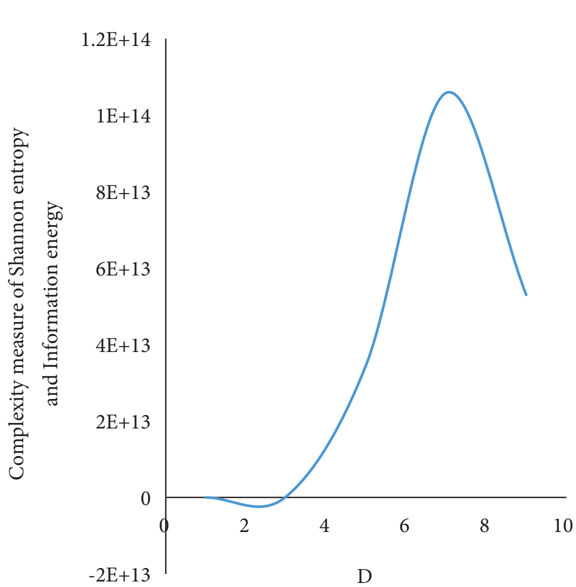

Figure 3. Complexity measure of Shannon entropy and information energy against potential depth D.

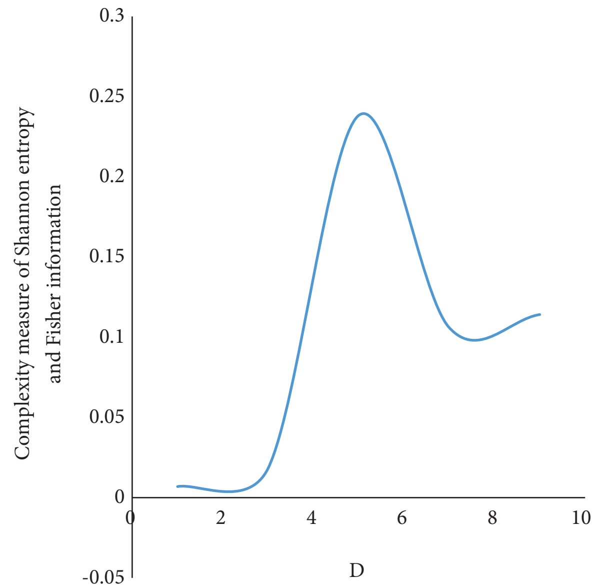

Figure 4. Complexity measure of Shannon entropy and Fisher information against potential depth D.

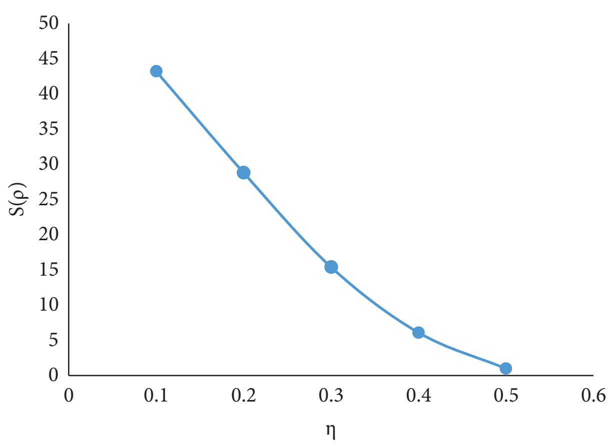

Figure 5. Shannon entropy in position space $S(\rho)$ against potential strength.

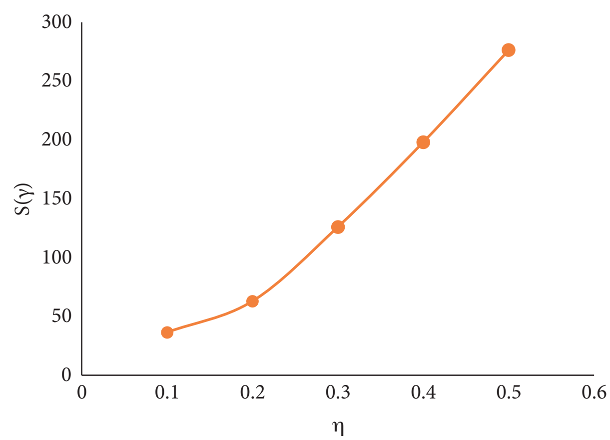

Figure 6. Shannon entropy in momentum space $S(\gamma)$ against potential strength.

scheme to deal with the centrifugal barrier, we obtained the energy equation and the wave function in a closed and compact form. Finally, we calculated the information-theoretical quantities using the integral method. The complexity measure was calculated in terms of Shannon entropy-information and Shannon entropy-Fisher information. The two complexity measures are similar.

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