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Soliton solutions of perturbed nonlinear Schrodinger equation with Kerr law nonlinearity via the modified simple equation method and the subordinary differential equation method

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Abstract: The objective of this paper was to obtain soliton solutions for a perturbed nonlinear Schrodinger equation with Kerr law nonlinearity using the modified simple equation method and subordinary differential equation method. These methods appear to be efficient and they can be applied in seeking exact solutions of many other nonlinear evolution equations encountered in science and engineering studies.

Key words: Fiber optics, group velocity dispersion, third order dispersion

1. Introduction

During the past few decades, studies on optical solitons have become popular among researchers in the physical sciences. Various models have been reported so far to describe the dynamics of solitons. The nonlinear Schrodinger (NLS) equation is one of the most important nonlinear evolution equations encountered in the study of nonlinear optics. Five types of nonlinearity are often discussed in the studies of such nonlinear evolution equations (NLEEs). They are: (i) parabolic law nonlinearity, (ii) Kerr law nonlinearity, (iii) power law nonlinearity, (iv) dual power law nonlinearity, and (v) log law nonlinearity. In this paper, we consider a nonlinear Schrodinger equation with perturbation terms to describe the propagation of optical solitons in nonlinear optical fibers exhibiting Kerr law nonlinearity. A medium that exhibits Kerr law nonlinearity is one in which the intensity of light passing through it depends on its refractive index. Here, a perturbed NLS equation with Kerr law nonlinearity [1–10] is written as

$$iu_t + u_{xx} + \alpha\left|u\right|^2 u + i\gamma_1 u_{xxx} + i\gamma_2\left|u\right|^2 u_x + i\gamma_3\left(\left|u\right|^2\right)_x u = 0. \tag{1}$$

Here, the independent variables x and t are the spatial and the temporal variables, respectively, and the dependent variable $u\left(x,t\right)$ is a complex valued function that represents a wave profile. Moreover, the first term represents a temporal evolution term, the second term represents the group velocity dispersion (GVD) term, the third term represents the nonlinear term dictated by Kerr law nonlinearity, the fourth term represents a third order dispersion (TOD) term, the fifth term represents a nonlinear dispersion term, and the sixth term represents another version of a nonlinear term. Furthermore, $i = \sqrt{-1}$ is the imaginary number and $\alpha, \gamma_1, \gamma_2, \gamma_3$ are constants. Eq. (1) has important applications in various fields of physics such as plasma physics, semiconductor physics, and solid mechanics.

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The rest of the paper is arranged as follows: In Section 2, the modified simple equation (MSE) method and the subordinary differential equation (sub-ODE) method are described successively; in Section 3, the perturbed NLS equation is reduced to a nonlinear ordinary differential equation (NLODE); and in Sections 4 and 5, the MSE and the sub-ODE methods are successively applied for finding soliton solutions to a perturbed nonlinear Schrodinger equation with Kerr law nonlinearity. In Section 6, a brief conclusion is presented.

2. Outlines of the modified simple equation and sub-ODE methods

In this section, the MSE and the sub-ODE methods are successively outlined as follows.

2.1. Modified simple equation (MSE) method

In this subsection, we outline the MSE method [11,12] for finding exact solutions of nonlinear partial differential equations (NLPDEs). Let us consider an NLPDE for $u\left(x,t\right)$ in the form of

$$P\left(u,u_t,u_x,u_{tt},u_{tx},u_{xx},...\right)=0, \tag{2}$$

where P is a polynomial in $u\left(x,t\right)$ and its various partial derivatives with respect to the independent variables x and t. Here, $u_t=\frac{\partial u}{\partial t},u_{tx}=\frac{\partial^2 u}{\partial t\partial x}$, etc. In order to solve $u\left(x,t\right)$ from Eq. (2) by the MSE method, it is required to perform the following steps:

Step I. The first step is to introduce the transformations

$$u\left(x,t\right)=U\left(\xi\right),\xi=x-vt, \tag{3}$$

where v is a constant, generally the constant speed of wave propagation.

Using the above transformations, Eq. (2) is reduced to an NLODE of the following form:

$$Q\left(U,U^{'},U^{''},U^{'''},...\right)=0, \tag{4}$$

where Q is a polynomial in $U\left(\xi\right)$ and its derivatives while the primes denote derivatives with respect to ξ such that $U^{'}=\frac{dU}{d\xi},U^{''}=\frac{d^2U}{d\xi^2}$, etc.

Step II. In this step, the solution of Eq. (4) is assumed in the form of

$$U\left(\xi\right)=\sum_{j=0}^{N}A_j\left[\frac{\Phi^{'}\left(\xi\right)}{\Phi\left(\xi\right)}\right]^j\left(\text{with}\Phi^{'}\left(\xi\right)=\frac{d\Phi}{d\xi}\right), \tag{5}$$

where A_j is a constant to be determined later such that $A_N\neq 0$ and $\Phi\left(\xi\right)$ is a function to be determined later.

Step III. In this step, the possible value of the integer N is to be determined through a balancing of degrees between the highest order derivative term and the nonlinear term of highest degree appearing in Eq. (2) or (4). Defining the degree of $U\left(\xi\right)$ as$D\left\{U\left(\xi\right)\right\}=N$ the degrees of other expressions are expressed as

$$D\left(\frac{d^qU}{d\xi^q}\right)=N+q,D\left(U^p\right)=Np,D\left\{U^p\left(\frac{d^qU}{d\xi^q}\right)^s\right\}=Np+\left(N+q\right)s,\text{etc.}$$

From such balancing of degrees, the value of the integer N can be determined.

Step IV. In this step, the value of N determined in Step III above is to be substituted in Eq. (5) to obtain the appropriate form of $U\left(\xi\right)$, and then the newly found expression for $U\left(\xi\right)$ is to be substituted into

Eq. (4). As a consequence of this substitution, we will get an equation involving a polynomial in $\frac{1}{\Phi(\xi)}$. Then we are to equate the coefficient of $\Phi^{-j}(\xi)\,(j=0,1,2,3,...)$ to zero. Thus, we will obtain a system of equations. From such a system of equations, we can solve $A_j\,(j=0,1,2,3,...)\,;\Phi'(\xi)$ and $\Phi(\xi)$ Substituting the values of $A_j,(j=0,1,2,...)\,,N,$ and all the expressions for $\Phi(\xi)$ and $\Phi'(\xi)$ into Eq. (5), we will obtain the solution of Eq. (4) and hence of Eq. (2).

2.2. The sub-ODE method or the auxiliary equation method

In this method, the elliptic equation

$$\left(\frac{d\Psi}{d\xi}\right)^2 = C_0 + C_1\Psi + C_2\Psi^2 + C_3\Psi^3 + C_4\Psi^4, \tag{6}$$

where $C_i(i=1,2,3,4)$ are constants, is an important auxiliary equation and it is widely used in solving many NLPDEs. In this paper, we investigate quasi-rational function solutions of Eq. (6) for the cases of $C_0 = C_1 = 0$.

With $C_0 = C_1 = 0$, Eq. (6) can be written as

$$\left(\frac{d\Psi}{d\xi}\right)^2 = A\Psi^2 + B\Psi^3 + C\Psi^4 \tag{7}$$

where $A = C_2, B = C_3, C = C_4$ are constants.

Using symbolic computation software like Mathematica or Maple, one can obtain a set of quasi-rational function solutions of the auxiliary elliptic equation (Eq. (7)) as listed below.

If $A > 0$, Eq. (7) will have hyperbolic function solutions such as

$$\Psi_1 = \frac{\pm 2A}{\mp B + \sqrt{B^2 - 4AC}\cosh\left(\sqrt{A}\xi\right)} \text{for} B^2 - 4AC > 0, \tag{8}$$

and

$$\Psi_2 = \frac{\pm 2A}{\mp B + \sqrt{4AC - B^2}\sinh\left(\sqrt{A}\xi\right)} \text{for} B^2 - 4AC < 0. \tag{9}$$

If $A < 0 \text{and} B^2 - 4AC > 0$, Eq. (7) will have trigonometric function solutions such as

$$\Psi_3 = \frac{\pm 2A}{\mp B + \sqrt{B^2 - 4AC}\cos\left(\sqrt{-A}\xi\right)} \tag{10}$$

and

$$\Psi_4 = \frac{\pm 2A}{\mp B + \sqrt{B^2 - 4AC} sin\left(\sqrt{-A}\xi\right)}. \tag{11}$$

If $A = 0$, Eq. (7) will have the following rational function solution:

$$\Psi_5 = \frac{4B}{-4C + B^2\xi^2}. \tag{12}$$

3. Reduction of perturbed NLS equation to an NLODE

To reduce the perturbed NLS equation of Eq. (1) to an NLODE, let us introduce the transformations

$$u\left(x,t\right)=U\left(\xi\right)e^{i\left(kx-\omega t\right)},\xi=x-vt \tag{13}$$

where ω and k are constants and v is the constant speed of soliton propagation.

From Eq. (13), we obtain

$$\begin{aligned}
iu_t &= \left(-iv\frac{dU}{d\xi}+\omega U\right)e^{i(kx-\omega t)},\\
u_{xx} &= \left(\frac{d^2U}{d\xi^2}+2ik\frac{dU}{d\xi}-k^2U\right)e^{i(kx-\omega t)},\\
\alpha\left|u\right|^2u &= \alpha U^3e^{i(kx-\omega t)},\\
i\gamma_1u_{xxx} &= \left(i\gamma_1\frac{d^3U}{d\xi^3}-3\gamma_1k\frac{d^2U}{d\xi^2}-3i\gamma_1k^2\frac{dU}{d\xi}+\gamma_1k^3U\right)e^{i(kx-\omega t)}\\
i\gamma_2\left|u\right|^2u_x &= \left(i\gamma_2U^2\frac{dU}{d\xi}-\gamma_2kU^3\right)e^{i(kx-\omega t)}\\
i\gamma_3\left(\left|u\right|^2\right)_xu &= 2i\gamma_3U^2\frac{dU}{d\xi}e^{i(kx-\omega t)}.
\end{aligned}$$

Substituting the above relations into Eq. (1) and splitting the resulting equation into real and imaginary parts, we obtain the following:

The real part:

$$\left(1-3\gamma_1k\right)\frac{d^2U}{d\xi^2}+\left(\omega-k^2+\gamma_1k^3\right)U+\left(\alpha-\gamma_2k\right)U^3=0 \tag{14}$$

and the imaginary part:

$$\gamma_1\frac{d^3U}{d\xi^3}+\left(2k-v-3\gamma_1k^2\right)\frac{dU}{d\xi}+\left(\gamma_2+2\gamma_3\right)U^2\frac{dU}{d\xi}=0. \tag{15}$$

Integrating both sides of Eq. (15) with respect to ξ and choosing the integration constant as zero, we obtain

$$\gamma_1\frac{d^2U}{d\xi^2}+\left(2k-v-3\gamma_1k^2\right)U+\frac{1}{3}\left(\gamma_2+2\gamma_3\right)U^3=0. \tag{16}$$

From Eqs. (14) and (16), we obtain the constraint conditions

$$\frac{\gamma_1}{1-3\gamma_1k}=\frac{2k-v-3\gamma_1k^2}{\omega-k^2+\gamma_1k^3}=\frac{\gamma_2+2\gamma_3}{3\left(\alpha-\gamma_2k\right)}. \tag{17}$$

Now, instead of solving Eqs. (14) and (16) both, we have to solve either one of the two. Here, let us solve Eq. (16) only.

We write Eq. (16) as

$$\frac{d^2U}{d\xi^2}+b_1U+b_2U^3=0, \tag{18}$$

where

$$b_1=\frac{2k-v-3\gamma_1k^2}{\gamma_1}, b_2=\frac{\gamma_2+2\gamma_3}{3\gamma_1}. \tag{19}$$

Thus, if we solve $U\left(\xi\right)$ from Eq. (18), then we can obtain the solution of Eq. (2) using Eq. (13).

4. Application of the MSE method in solving perturbed NLS equation

From Eq. (18), balancing degrees between the highest order derivative term and the term having the highest nonlinearity, we obtain

$N+2=3N\text{yielding}N=1$. Therefore, using Eq. (5), we assume the solution of Eq. (18) as

$$U\left(\xi\right)=A_0+A_1\frac{\Phi^{'}\left(\xi\right)}{\Phi\left(\xi\right)}, A_1\neq 0. \tag{20}$$

Now, we have

$$\frac{d^2U}{d\xi^2}=A_1\left[\frac{\Phi^{'''}\left(\xi\right)}{\Phi\left(\xi\right)}-3\frac{\Phi^{'}\left(\xi\right)\Phi^{''}\left(\xi\right)}{\Phi^2\left(\xi\right)}+2\left\{\frac{\Phi^{'}\left(\xi\right)}{\Phi\left(\xi\right)}\right\}^3\right]. \tag{21}$$

Substituting Eqs. (20) and (21) into Eq. (18), we obtain

$$A_1\left[\frac{\Phi^{'''}}{\Phi}-3\frac{\Phi^{'}\Phi^{''}}{\Phi^2}+2\left(\frac{\Phi^{'}}{\Phi}\right)^3\right]+b_1\left(A_0+A_1\frac{\Phi^{'}}{\Phi}\right)+b_2\left(A_0^3+3A_0^2A_1\frac{\Phi^{'}}{\Phi}+3A_0A_1^2\frac{\Phi^{'2}}{\Phi^2}+A_1^3\frac{\Phi^{'3}}{\Phi^3}\right)=0. \tag{22}$$

From Eq. (22), equating the coefficients of $\Phi^{-j}\left(j=0,1,2,3\right)$ to zero, we obtain

$$\Phi^0: b_1A_0+b_2A_0^3=0. \tag{23}$$

$$\Phi^{-1}: A_1\Phi^{'''}+A_1b_1\Phi^{'}+3A_0^2A_1b_2\Phi^{'}=0. \tag{24}$$

$$\Phi^{-2}: -3A_1\Phi^{'}\Phi^{''}+3A_0A_1^2b_2\Phi^{'2}=0. \tag{25}$$

$$\Phi^{-3}: 2A_1\Phi^{'3}+b_2A_1^3\Phi^{'3}=0. \tag{26}$$

From Eq. (23), we obtain

$$A_0=\pm\sqrt{\frac{-b_1}{b_2}}\left[\text{neglecting}A_0=0\right]. \tag{27}$$

From Eq. (26), we obtain

$$A_1=\pm\sqrt{\frac{-2}{b_2}}. \tag{28}$$

Substituting these values of A_0 and A_1 into Eqs. (24) and (25), division of the two resulting equations yields

$$\frac{\Phi^{'''}}{\Phi^{''}} = \sqrt{2b_1}. \tag{29}$$

Integration of Eq. (29) with respect to ξ yields

$$\Phi^{''} = C_0 e^{\left(\sqrt{2b_1}\right)\xi} \tag{30}$$

where C_0 is an integration constant.

Integrating Eq. (30) further, we obtain

$$\Phi^{'} = C_1 + \frac{C_0}{\sqrt{2b_1}} e^{\left(\sqrt{2b_1}\right)\xi} \tag{31}$$

and

$$\Phi = C_2 + C_1\xi + \frac{C_0}{2b_1} e^{\left(\sqrt{2b_1}\right)\xi} \tag{32}$$

where C_1 and C_2 are integration constants.

Now, substituting A_0, A_1, Φ, and $\Phi^{'}$ as obtained above into Eq. (20), we obtain

$$U(\xi) = \pm\sqrt{\frac{-b_1}{b_2}} \pm \sqrt{\frac{-2}{b_2}} \frac{C_1 + \frac{C_0}{\sqrt{2b_1}} e^{\left(\sqrt{2b_1}\right)\xi}}{C_2 + C_1\xi + \frac{C_0}{2b_1} e^{\left(\sqrt{2b_1}\right)\xi}}. \tag{33}$$

Substituting Eq. (33) into Eq. (13), we obtain the solution of Eq. (1) as

$$u(x,t) = \left[\pm\sqrt{\frac{-b_1}{b_2}} \pm \sqrt{\frac{-2}{b_2}} \left\{ \frac{C_1 + \frac{C_0}{\sqrt{2b_1}} e^{\left(\sqrt{2b_1}\right)(x-vt)}}{C_2 + C_1(x-vt) + \frac{C_0}{2b_1} e^{\left(\sqrt{2b_1}\right)(x-vt)}} \right\}\right] e^{i(kx-\omega t)} \tag{34}$$

with the constraint conditions given in Eq. (17).

In Eq. (34), if we choose $C_1 = 0, C_2 = \frac{C_0}{2b_1}$, we obtain

$$\begin{aligned} u(x,t) &= \left[\pm\sqrt{\frac{-b_1}{b_2}} \pm \sqrt{\frac{-2}{b_2}} \left\{ \frac{\sqrt{2b_1} e^{\sqrt{2b_1}(x-vt)}}{1 + e^{\sqrt{2b_1}(x-vt)}} \right\}\right] e^{i(kx-\omega t)} \\ &= \sqrt{\frac{-b_1}{b_2}} \left[\pm 1 \pm \frac{2e^{\sqrt{\frac{b_1}{2}}(x-vt)}}{e^{-\sqrt{\frac{b_1}{2}}(x-vt)} + e^{\sqrt{\frac{b_1}{2}}(x-vt)}}\right] e^{i(kx-\omega t)} \\ &= \sqrt{\frac{-b_1}{b_2}} \left[\pm 1 \pm \frac{\cosh\left\{\sqrt{\frac{b_1}{2}}(x-vt)\right\} + \sinh\left\{\sqrt{\frac{b_1}{2}}(x-vt)\right\}}{\cosh\left\{\sqrt{\frac{b_1}{2}}(x-vt)\right\}}\right] e^{i(kx-\omega t)} \\ &= \sqrt{\frac{-b_1}{b_2}} \left[\pm 1 \pm \left\{1 + \tanh\left(\sqrt{\frac{b_1}{2}}(x-vt)\right)\right\}\right] e^{i(kx-\omega t)}. \end{aligned} \tag{35}$$

Now, substituting b_1 and b_2 from Eq. (19) into Eq. (35), Eq. (1) is found to have the solutions

$$u_1(x,t) = \pm i\sqrt{\frac{3\left(2k-v-3\gamma_1 k^2\right)}{\gamma_2+2\gamma_3}}\left[2+\tanh\left\{\sqrt{\frac{2k-v-3\gamma_1 k^2}{2\gamma_1}}(x-vt)\right\}\right]e^{i(kx-\omega t)} \tag{36}$$

$$u_2(x,t) = \pm i\sqrt{\frac{3\left(2k-v-3\gamma_1 k^2\right)}{\gamma_2+2\gamma_3}}\tanh\left\{\sqrt{\frac{2k-v-3\gamma_1 k^2}{2\gamma_1}}(x-vt)\right\}e^{i(kx-\omega t)} \tag{37}$$

if $\gamma_1\left(2k-v-3\gamma_1 k^2\right)>0$.

In similar fashions, if we choose $C_1=0, C_2=-\frac{C_0}{2b_1}$ in Eq. (34), we obtain

$$u_3(x,t) = \pm i\sqrt{\frac{3\left(2k-v-3\gamma_1 k^2\right)}{\gamma_2+2\gamma_3}}\left[2+\coth\left\{\sqrt{\frac{2k-v-3\gamma_1 k^2}{2\gamma_1}}(x-vt)\right\}e^{i(kx-\omega t)}\right] \tag{38}$$

and

$$u_4(x,t) = \pm i\sqrt{\frac{3\left(2k-v-3\gamma_1 k^2\right)}{\gamma_2+2\gamma_3}}\coth\left\{\sqrt{\frac{2k-v-3\gamma_1 k^2}{2\gamma_1}}(x-vt)\right\}e^{i(kx-\omega t)} \tag{39}$$

if $\gamma_1\left(2k-v-3\gamma_1 k^2\right)>0$.

5. Application of the sub-ODE method (or auxiliary equation method) in solving perturbed NLS equation

In this section, the sub-ODE method is applied in finding soliton solutions of a perturbed NLS equation. Multiplying both sides of Eq. (16) by $\frac{dU}{d\xi}$, we obtain

$$\gamma_1\frac{dU}{d\xi}\frac{d^2U}{d\xi^2}+\left(2k-v-3\gamma_1 k^2\right)U\frac{dU}{d\xi}+\frac{1}{3}\left(\gamma_2+2\gamma_3\right)U^3\frac{dU}{d\xi}=0.$$

Integrating both sides of the above equation with respect to ξ and choosing the integration constant as zero, we have

$$\frac{\gamma_1}{2}\left(\frac{dU}{d\xi}\right)^2+\frac{\left(2k-v-3\gamma_1 k^2\right)}{2}U^2+\frac{1}{12}\left(\gamma_2+2\gamma_3\right)U^4=0.$$

This equation can be written as

$$\left(\frac{dU}{d\xi}\right)^2=AU^2+BU^3+CU^4=0, \tag{40}$$

where

$$A=\frac{v-2k+3\gamma_1 k^2}{\gamma_1}, B=0, C=-\frac{\left(\gamma_2+2\gamma_3\right)}{6\gamma_1} \tag{41}$$

Here,

$$B^2-4AC=\frac{2}{3}\frac{\left(v-2k+3\gamma_1 k^2\right)}{\gamma_1}\frac{\left(\gamma_2+2\gamma_3\right)}{\gamma_1}. \tag{42}$$

Using Eqs. (8) and (42), we can solve $U(\xi)$ from Eq. (40), and if we substitute this $U(\xi)$ into Eq. (13), we will get solutions of Eq. (1) as

$$u_5(x,t) = \pm\sqrt{\frac{6\left(v-2k+3\gamma_1 k^2\right)}{\gamma_1\left(\gamma_2+2\gamma_3\right)}}\sec h\left\{\left(\sqrt{\frac{v-2k+3\gamma_1 k^2}{\gamma_1}}\right)(x-vt)\right\}e^{i(kx-\omega t)} \tag{43}$$

if

$$\gamma_1\left(v-2k+3\gamma_1 k^2\right) > 0 \text{and} \gamma_1\left(\gamma_2+2\gamma_3\right) > 0,$$

and

$$u_6(x,t) = \pm\sqrt{\frac{6\left(2k-v-3\gamma_1 k^2\right)}{\gamma_1\left(\gamma_1+2\gamma_3\right)}}\cos ech\left\{\left(\sqrt{\frac{v-2k+3\gamma_1 k^2}{\gamma_1}}\right)(x-vt)\right\}e^{i(kx-\omega t)} \tag{44}$$

if

$$\gamma_1\left(v-2k+3\gamma_1 k^2\right) > 0 \text{and} \gamma_1\left(\gamma_2+2\gamma_3\right) < 0.$$

6. Conclusion

In this paper, we applied the MSE method and the auxiliary equation method (sub-ODE method) in finding some soliton solutions to a perturbed nonlinear Schrodinger equation, and we can conclude that these methods are efficient and powerful tools for finding exact solutions of many nonlinear evolution equations generally encountered in many areas of nonlinear science and engineering. In these two methods, we see that computation is simpler in the case of the modified simple equation method and, therefore, this method is more advantageous.

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