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General solution of the Schrödinger equation with potential field quantization

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Abstract: A simple procedure has been found for the general solution of the time-independent Schrödinger equation (SE) with the help of quantization of potential area in one dimension without making any approximation. Energy values are not dependent on wave functions. So, to find the energy values, it is enough to find the classic turning points of the potential function. Two different solutions were obtained, namely, symmetric and antisymmetric in bound states. These normalized wave functions are always periodic. It is enough to take the integral of the square root of the potential energy function to find the normalized wave functions. If these calculations cannot be made analytically, they should then be performed by numerical methods. The SE has been solved for a particle in many one-dimension and the spherical symmetric central potential wells as examples. Their energies and normalized wave functions were found as examples. These solutions were also applied to the theories of scattering and alpha decay. The results obtained with the experimental values were compared with the calculated values. One was seen to be very fit.

Key words: Potential quantization, Schrödinger equation, infinitely high potential well, potential wells of any form, α-decay theory, scattering, energy values of states

1. Introduction

Mechanical total energy (E) is the sum of the kinetic energy (T) and potential energy (U). That is, $E = T + U$. If $E = U$ then $T = 0$. If $E > U$ then $T > 0$; If $E < U$ then $T < 0$ (it is not possible, classically). In quantum mechanics, the total energy is equal to the eigenvalue of the total energy operator $\hat{H}$ (Hamiltonian). One-dimension Hamiltonian is: $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U(x)$. This operator is hermitic and its eigenvalues are real numbers. The equation of eigenvalues of this Hamiltonian is given as follows (time-independent Schrödinger Equation, SE):

$$\hat{H}\psi(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U(x)\right]\psi(x) = E\psi(x) \tag{1}$$

The eigenvalue E can have two values: $-E$ and $+E, (E > 0)$. So, from Eq. (1), the following differential equation is obtained:

$$\frac{d^2\psi(x)}{dx^2} + \left[k^2 - {m_1}^2 U(x)\right]\psi(x) = 0, \tag{2}$$

where,

$$k^2 = -{m_1}^2 E \text{ for } -E \text{ and } k^2 = {m_1}^2 E \text{ for } +E;\ {m_1}^2 = 2m/\hbar^2 \tag{3}$$

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Although we succeed in solving the time-independent SE for some quantum mechanical problems in one dimension, an exact solution is not possible in complicated situations. We must then resort to approximation methods. For the calculation of the stationary states and energy eigenvalues, these include the variation method, the method of Nikiforov–Uvarov [1–4], the supersymmetric quantum mechanics, the supersymmetric Wentzel–Kramers–Brillouin (WKB) and the WKB approximations [5, 6], and perturbation theory. Perturbation theory is applicable if the Hamiltonian differs from an exactly solvable part by a small amount. The variation method [18,37] is appropriate for the calculation of the ground state energy if one has a qualitative idea of the form of the wave function. The WKB method [13,15] is applicable in the nearly classical limit. The exact general solution of the differential equation given in Eq. (2) has not been achieved yet. This is a very challenging problem for theoretical physicists.

In this study, we followed a simple procedure for the exact general solution of the time-independent SE in one dimension without making any approximation. We applied this simple procedure to various quantum mechanical problems: one-dimension potentials, spherically symmetric potentials, tunneling effect, scattering theory and presented some examples.

2. Solution of the time-independent SE in one dimension

Let us rewrite the time-independent SE in one dimension given in Eq. (2):

$$\frac{d^2\psi(x)}{dx^2}+\left[k^2-{m_1}^2U(x)\right]\psi(x)=0. \tag{4}$$

Here, E and $U(x)$ are respectively the total and effective potential energies of a particle of mass m. Two kinds of solutions to the SE correspond precisely to bound and scattering (unbound) states. When $E<U(-\infty)$ and $U(+\infty)$ the solutions correspond to bound states; when $E>U(-\infty)$ or $U(+\infty)$ the solutions correspond to scattering states [6]. In real life, many potential functions go to zero at infinity, in which case the criterion is simplified even further: when $E<0$, bound states occur; when $E>0$, scattering states occur. In this section, we will explore potentials that give rise to both kinds of states. We will solve Eq. (4) in two steps.

2.1. The first step

Inspired by the theorem given in [8,9], we consider the following integral function:

$$S(x) = \int U(x)\,dx. \tag{5}$$

If $f(x)=S(x)$ in this theorem, we get:

$$F(\varepsilon) = \int_{-\infty}^{+\infty} S(x)\,y(x-x_0,\varepsilon)\,dx. \tag{6}$$

From this function for $\varepsilon=0$, the following is obtained:

$$F(\varepsilon)=F(0) = \int_{-\infty}^{+\infty} S(x)\,\delta(x-x_0)\,dx = S(x_0)=S. \tag{7}$$

Now, let us take the potential and wave functions respectively as follows:

$$S=\int_{x_1}^{x_2} U(x)\,dx; U(x)=S(x_0)\,\delta(x-x_0)=S\delta(x-x_0) \text{ and } \psi(x)=F(x). \tag{8}$$

$\delta(x-x_0)$ is Dirac function. x_1 and x_2 are the roots of the equation $E=U(x)$ that is the apsis of the classical turning point of the potential function. If we take $x_0 = (x_1+x_2)/2$ and $d=x_2-x_1, (x_2>x_1)$, then we find $x_1 = x_0-d/2, x_2 = x_0+d/2$ (see Figure 1). If we take these functions, the SE given in Eq. (4) becomes as follows:

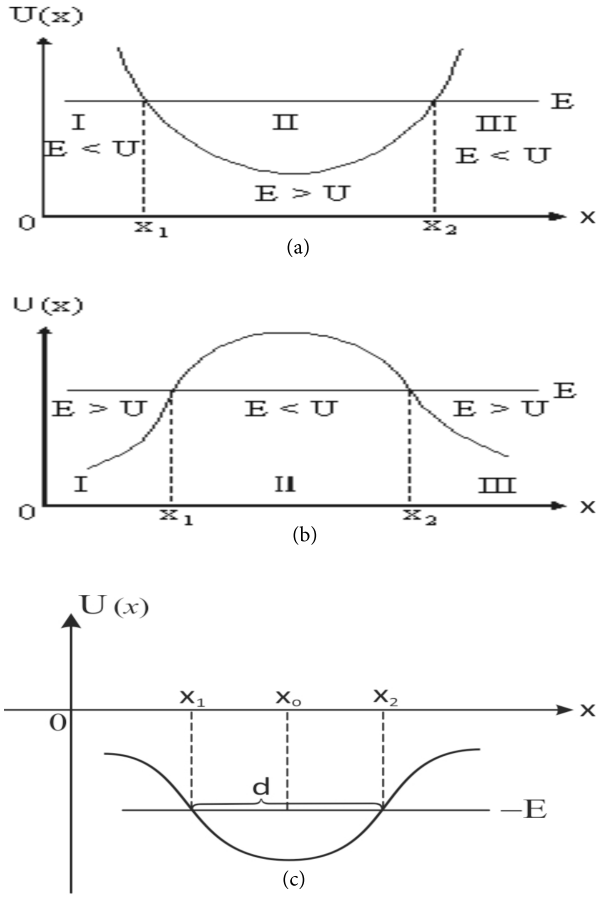

Figure 1. Regions relevant to a particle of energy E moving in a one dimensional potential field $U(x)$. **(a)**, in domains I and III, $E<U(x)$, (unbound state); in domain II, $E>U(x)$, (bound state). **(b)**, in domains I and III, $E>U(x)$, in domain II, $E<U(x)$, (unbound state). The roots of the equation $E=U(x)$ are turning points of the corresponding classical motion. **(c)**, regions for $U(x)<0$ and $E<0$.

$$\frac{d^2F(x)}{dx^2}+k^2F(x)=m_1{}^2S\delta(x-x_0)\,F(x). \tag{9}$$

To evaluate the behavior of $F(x)$ at $x=x_0$, let us integrate Eq. (9) over the interval $[x_1,x_2] = [x_0-d/2,x_0+d/2]$ and also consider the limit $d\to 0$, then we obtain the following:

$$F'(x_0+d/2)-F'(x_0-d/2)=m_1^2SF(x_0). \tag{10}$$

Eq. (10) shows that the derivation of F(x) is not continuous at the $x = x_0$ point [6,9], whereas the wave functions $F(x)$ should be continuous at the same point.

To solve the differential equation given in Eq. (9), we can perform the transformation of Fourier of the equation, and then we obtain the following function:

$$F(x) = Ae^{-k|(x-x_0)|}, \left[A = \frac{a^2}{k}\sqrt{\frac{\pi}{2}}, a^2 = -\frac{2m}{\hbar^2\sqrt{2\pi}}SF(x_0)\right]. \tag{11}$$

From Eq. (11), we get the following function:

$$F(x) = Ae^{k(x-x_0)} \text{ for } x < x_0 \text{ and } F(x) = Ae^{-k(x-x_0)} \text{ for } x > x_0 \tag{12}$$

Substituting the functions given in Eq. (12) into Eq. (10) and taking the limit $d \to 0$, the following values are obtained:

$$k = \frac{m}{\hbar^2}S \text{ or } E = \pm\frac{m}{2\hbar^2}S^2. \tag{13}$$

The same values given in Eq. (13) can also be obtained from Eq. (11). To find the constant A, the function $F(x)$ can be normalized to 1:

$$\int_{-\infty}^{x_0} AA^* e^{2k(x-x_0)}dx + \int_{x_0}^{+\infty} AA^* e^{-2k(x-x_0)}dx = 1.$$

From this equation, we obtain the coefficient of normalization as:

$$|A| = \sqrt{k} = \sqrt{mS}\Big/\hbar. \tag{14}$$

From Eq. (12), by the linear combinations of these functions, we also have the following functions:

$$F(x) = Ae^{k(x-x_0)} + Be^{-k(x-x_0)}, \tag{15a}$$

$$F(x) = \frac{1}{2}A\left[e^{k(x-x_0)} + e^{-k(x-x_0)}\right] = A\cosh\left[k(x-x_0)\right], \tag{15b}$$

$$F(x) = \frac{1}{2}A\left[e^{k(x-x_0)} - e^{-k(x-x_0)}\right] = A\sinh\left[k(x-x_0)\right]. \tag{15c}$$

2.2. The second step

Let us assume the wave function $\psi(x)$ to be $\psi(\mathbf{x}) = \mathbf{F}(\mathbf{x})\,\mathbf{e}^{\mathbf{iG(x)}}$, (we assume that G(x) is a real function). If we substitute this function into Eq. (4), we get:

$$F''(x) - F(x)G'^2(x) - k^2F(x) - m_1{}^2U(x)F(x) + i\left[2F'(x)G'(x) + F(x)G''(x)\right] = 0. \tag{16}$$

From the real and imaginary parts of Eq. (16), we can have the following two equations:

$$F''(x) - F(x)G'^2(x) - k^2F(x) - m_1{}^2U(x)F(x) = 0, \tag{17}$$

$$2F^{'}(x)G^{'}(x)+F(x)G^{''}(x)=0. \tag{18}$$

From Eq. (17), the following equations are obtained:

$$\text{For } F(x)=Ae^{k(x-x_0)}; F(x)\left[{m_1}^2U(x)+G^{'2}(x)\right]=0, \tag{19a}$$

$$\text{For } F(x)=Ae^{-k(x-x_0)}; F(x)\left[{m_1}^2U(x)+G^{'2}(x)\right]=0, \tag{19b}$$

$$\text{For } F(x)=Ae^{k(x-x_0)}+Be^{-k(x-x_0)}; F(x)\left[{m_1}^2U(x)+G^{'2}(x)\right]=0, \tag{19c}$$

$$\text{For } F(x)=A\cosh\left[k(x-x_0)\right]; F(x)\left[{m_1}^2U(x)+G^{'2}(x)\right]=0, \tag{19d}$$

$$\text{For } F(x)=A\sinh\left[k(x-x_0)\right]; F(x)\left[{m_1}^2U(x)+G^{'2}(x)\right]=0. \tag{19e}$$

From Eqs. (19a)–(19e), the following two equations are obtained:

$${m_1}^2U(x)+G^{'2}(x)=0, \tag{20a}$$

$$F(x)=0. \tag{20b}$$

From Eq. (20a), the function $G(x)$ is obtained as follows:

$$G(x)=\pm m_1\int\sqrt{-U(x)}dx=\pm im_1\int\sqrt{U(x)}dx=\pm iQ(x) \tag{21a}$$

$$Q(x)=m_1\int\sqrt{U(x)}dx. \tag{21b}$$

Thus, the wave function $\psi(x)$ is written as follows:

$$\psi(x)=F(x)e^{\pm iG(x)} \text{ or } \psi(x)=F(x-x_0)e^{\pm iG(x-x_0)}. \tag{22}$$

The functions given in Eq. (22) can also be written as follows:

$$\boldsymbol{\psi}(\mathbf{x})=\mathbf{F}(\mathbf{x})\left[\mathbf{A}e^{\mathbf{iG(x)}}+\mathbf{B}e^{-\mathbf{iG(x)}}\right] \tag{23a}$$

$$\boldsymbol{\psi}(\mathbf{x})=\mathbf{F}(\mathbf{x}-\mathbf{x_0})\left[\mathbf{A}e^{\mathbf{iG(x-x_0)}}+\mathbf{B}e^{-\mathbf{iG(x-x_0)}}\right] \tag{23b}$$

In the functions given in Eqs. (23a) and (23b):

(a) For $E>U(x)$; $k=m_1\sqrt{-E}, G(x)=m_1\int\sqrt{-U(x)}dx$,

(b) For $E<U(x)$; $k=m_1\sqrt{E}, G(x)=m_1\int\sqrt{U(x)}dx$.

As shown in Figure 1a, the total energy is equal to the potential energy at the points $x_1 = x_0-d/2$ and $x_2 = x_0+d/2$. So, the kinetic energy is zero at these points. Now, let us examine the behavior of the function in the interval $[x_0-d/2,x_0+d/2]$

Firstly, let us consider Eq. (20b):

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}.$$

Let us assume that the roots of this equation are $x_1 = x_0-d/2$ and $x_2 = x_0+d/2$. So, we can write the following:

$$F(x_1) = F(x_0-d/2) = 0; F(x_2) = F(x_0+d/2) = 0. \tag{24}$$

From Eq. (24), the following expressions can be obtained:

(a) For $F(x) = Ae^{k(x-x_0)}+Be^{-k(x-x_0)}; k = m_1\sqrt{-E} = im_1\sqrt{E} = iK, [K = m_1\sqrt{E} > 0]$

$$a_{11}A+a_{12}B = 0; a_{21}A+a_{22}B = 0;$$

$$a_{11} = e^{-idK/2}; a_{12} = e^{idK/2}; a_{21} = e^{idK/2}; a_{22} = e^{-idK/2}.$$

Since A and B have to nonzero values, the determinant of coefficients in this system of equations must be zero. Thus, the following equation can be written as follows:

$$det\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -2i\sin(dK) = 0. \tag{25}$$

Here, let us take $\mathbf{dK} = \mathbf{q}$. From Eq. (25), we have the following values:

$$\sin(dK) = \sin(q) = 0 \rightarrow q = n\pi, [n = 1, 2, 3, 4, \ldots \text{integer}].$$

So, we obtain the quantization condition of energy as follows:

$$\mathbf{Kd} = \sqrt{\frac{\mathbf{2m}}{\hbar^{\mathbf{2}}}|\mathbf{E}|}\mathbf{d} = \mathbf{q}, [\mathbf{q} = \mathbf{n}\pi, (\mathbf{n} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots \ integer \ numbers)] \tag{26}$$

Here, the coefficient B is obtained as $B = -A$. Thus the function $F(x)$ is written as follows:

$F(x) = A\left[e^{iK(x-x_0)}-e^{-iK(x-x_0)}\right] = 2iA \ \sin[K(x-x_0)] = B \ \sin[K(x-x_0)]$. The coefficient B in this function is found by normalizing the function in the interval $[x_1,x_2]$. One is found as $|B| = \sqrt{2/d}$. Thus, the functions $F(x)$ and $\psi(x)$ are written as follows:

$$\mathbf{F}(\mathbf{x}) = \sqrt{\mathbf{2/d}}\sin[\mathbf{K}(\mathbf{x}-\mathbf{x_0})]; \psi(\mathbf{x}) = \sqrt{\mathbf{2/d}}\sin[\mathbf{K}(\mathbf{x}-\mathbf{x_0})]\mathbf{e}^{\mathbf{iG}(\mathbf{x}-\mathbf{x_0})} \tag{27}$$

(b) For $F(x) = A\cosh[k(x-x_0)]; \ k = iK$

$$F(x_1) = F(x_0-d/2) = 0; F(x_2) = F(x_0+d/2) = 0;$$

$$F(x_1) = F(x_2) = A\cosh(dk/2) = A\cos(dK/2) = 0$$

$$dK/2 = q/2 = (2n-1)\pi/2 \rightarrow q = (2n-1)\pi, [n = 1, 2, 3, \ldots \text{integer}].$$

So, we obtain the quantization condition of energy as follows (symmetric case):

$$\mathbf{Kd} = \sqrt{\frac{\mathbf{2m}}{\hbar^{\mathbf{2}}}|\mathbf{E}|}\mathbf{d} = \mathbf{q}, \mathbf{q} = (\mathbf{2n}-\mathbf{1})\pi, (\mathbf{n} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots \ integer\ numbers)\,]. \tag{28}$$

The coefficient A in this function is found by normalizing the function in the interval $[x_1,x_2]$. That is, $|A| = \sqrt{2/d}$. Thus, the functions $F(x)$ and $\psi(x)$ are written as follows:

$$\mathbf{F}(\mathbf{x}) = \sqrt{\mathbf{2/d}}\ \mathbf{cos}[\mathbf{K}(\mathbf{x}-\mathbf{x_0})]; \boldsymbol{\psi}(\mathbf{x}) = \sqrt{\mathbf{2/d}}\ \mathbf{cos}[\mathbf{K}(\mathbf{x}-\mathbf{x_0})]\mathbf{e}^{\mathbf{iG(x-x_0)}}. \tag{29}$$

(c) For $F(x) = A\sinh[k(x-x_0)]$; $k = iK$;

$$F(x_1) = F(x_0-d/2) = 0; F(x_2) = F(x_0+d/2) = 0;$$

$$F(x_1) = -A\sinh(dk/2) = -iAsin(dK/2) = 0$$

$$F(x_2) = A\sinh(dk/2) = iAsin(dK/2) = 0$$

$$dK/2 = q/2 = n\pi \rightarrow q = 2n\pi, [n = 1, 2, 3, \ldots \ integer\ numbers].$$

So, we obtain the quantization condition of energy as follows (antisymmetric case):

$$\mathbf{Kd} = \sqrt{\frac{\mathbf{2m}}{\hbar^{\mathbf{2}}}|\mathbf{E}|}\mathbf{d} = \mathbf{q}, [\mathbf{q} = \mathbf{2n}\pi, (\mathbf{n} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots \ integer\ numbers)\,]. \tag{30}$$

The coefficient A in this function is found by normalizing the function in the range $[x_1,x_2]$. That is, $|A| = \sqrt{2/d}$. Thus, the functions $F(x)$ and $\psi(x)$ are written as follows:

$$\mathbf{F}(\mathbf{x}) = \sqrt{\mathbf{2/d}}\ \mathbf{sin}[\mathbf{K}(\mathbf{x}-\mathbf{x_0})]; \boldsymbol{\psi}(\mathbf{x}) = \sqrt{\mathbf{2/d}}\ \mathbf{sin}[\mathbf{K}(\mathbf{x}-\mathbf{x_0})]\,\mathbf{e}^{\mathbf{iG(x-x_0)}} \tag{31}$$

It is possible to combine Eqs. (28) and (30) in one equation as follows (general case):

$$\mathbf{Kd} = \sqrt{\frac{\mathbf{2m}}{\hbar^{\mathbf{2}}}|\mathbf{E}|}\mathbf{d} = \mathbf{q}, [\mathbf{q} = \mathbf{n}\pi, (\mathbf{n} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots \ integer\ numbers)]\,. \tag{26}$$

Now let us write the kinetic energy of the particle as follows: $T = \frac{p^2}{2m} = E - U(x)$. By integrating this equation from x_1 to x_2 and using Eq. (8), the following equation is obtained:
$\int_{x_1}^{x_2} T dx = \int_{x_1}^{x_2} \frac{p^2}{2m} dx = \int_{x_1}^{x_2} [E - U(x)]\, dx = \int_{x_1}^{x_2} E dx - \int_{x_1}^{x_2} U(x) dx = S_k$, and the following equation is written:

$$S_k = E(x_2-x_1) - S = Ed - S. \tag{32}$$

From Figure 1, we can observe that:

(a) For the case where $E > U(x)$, the kinetic energy is positive and $[E(x_2-x_1) - S] > 0$ (bound state).

(b) For the case where $E < U(x)$, the kinetic energy is imaginary and $[E(x_2-x_1) - S] < 0$ (unbound state).

(c) For the case where $E = U(x)$, the kinetic energy is zero and $[E(x_2-x_1) - S] = 0$ (ground state).

In addition, for the bound states, in the interval $[x_1,x_2]$, the kinetic energy is positive; outside this interval, the kinetic energy is imaginary. The minimum point of the potential corresponds to ground state. In ground state, the kinetic energy is zero, namely, $T=0$ or $S_k=0$. Thus, at the minimum point of the potential, we can write that:

$$E_0\left(x_2-x_1\right)-S=0 \text{ or } S=E_0\left(x_2-x_1\right)=E_0 d.$$

By substituting this value of S into Eq. (13), we get the ground state energy expression as follows:

$$E_0=-\frac{m}{2\hbar^2}S^2=-\frac{m}{2\hbar^2}{E_0}^2d^2\rightarrow E_0=-\frac{2\hbar^2}{md^2}. \tag{33}$$

Here, E_0 represents the ground state energy. The negative sign indicates that the state is bound and it can be omitted for the positive energies in the calculations.

3. Boundary conditions

Let us divide the potential field into three domains as shown in Figure 1 and represent the functions $\psi_1(x)$, $\psi_2(x)$, and $\psi_3(x)$ in each domain. The wave functions and their derivatives should be continuous. Because of these conditions, the above functions must satisfy the following conditions:

$$\begin{aligned}&\psi_1(x_1)=\psi_2(x_1);\quad \psi_1'(x_1)=\psi_2'(x_1);\quad \psi_2(x_2)=\psi_3(x_2),\\ &\psi_2'(x_2)=\psi_3'(x_2);\quad \lim_{x\to+\infty}\psi_1(x)\to 0 \text{ and } \lim_{x\to+\infty}\psi_3(x)\to 0.\end{aligned} \tag{34}$$

The normalization of the bound state function requires that the functions vanish at infinity. With these boundary and normalization conditions of the wave functions, we can find the integral constants, A, B and the energy E. As it is also seen above, in the bound states, we do not need the solutions of the SE. It is sufficient to know only the classical turning points, x_1 and x_2, of the potential function.

Quantization of the energy values can also be found by means of boundary conditions. Let's find them now. In bound states, let us apply the conditions given in Eq. (34) to the following functions:

$$Ae^{k(x-x_0)}+Be^{-k(x-x_0)};\quad A\cosh\left[k\left(x-x_0\right)\right];\quad B\sinh\left[k\left(x-x_0\right)\right].$$

(a) For the function $\mathbf{Ae^{k(x-x_0)}+Be^{-k(x-x_0)}}$

According to Figure 1a, in domain I, $E<U(x)$; in domain II, $E>U(x)$; in domain III, $E<U(x)$. According to Eqs. (23a) and (23b), the corresponding wave functions are written as follows:

$$\psi_1(x)=A_1e^{ik(x-x_0)}e^{-Q(x-x_0)} \tag{35a}$$

$$\psi_2(x)=[A_2e^{k(x-x_0)}+B_2e^{-k(x-x_0)}]e^{iQ(x-x_0)} \tag{35b}$$

$$\psi_3(x)=A_3e^{-ik(x-x_0)}e^{-Q(x-x_0)} \tag{35c}$$

Here, $Q\left(x-x_0\right)=m_1\int\sqrt{-U(x-x_0)}dx$, and $k=m_1\sqrt{-E}=im_1\sqrt{E}$

Boundary conditions: $\psi_1(x_1) = \psi_2(x_1), \psi_2(x_2) = \psi_3(x_2), \psi_1'(x_1) = \psi_2'(x_1), \psi_2'(x_2) = \lim_{x\to+\infty}\psi_3'(x_2)$, $\psi_1(x)\to 0$, and $\psi_3(x)\to 0$

From these conditions, four linear equations are obtained as follows:

$$\begin{bmatrix} a_{11}a_{12}a_{13}a_{14} \\ a_{21}a_{22}a_{23}a_2 \\ a_{31}a_{32}a_{33}a_{34} \\ a_{41}a_{42}a_{43}a_{44} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ B_2 \\ A_3 \end{bmatrix} = 0. \tag{36a}$$

For this system of equations to have a solution different from zero, the determinant of coefficients should vanish, namely:

$$det \begin{vmatrix} a_{11}a_{12}a_{13}a_{14} \\ a_{21}a_{22}a_{23}a_2 \\ a_{31}a_{32}a_{33}a_{34} \\ a_{41}a_{42}a_{43}a_{44} \end{vmatrix} = 0. \tag{36b}$$

If it is taken as follows

$$x_1 = x_0-d/2, x_2 = x_0+d/2, Q'(-d/2) = Q'(d/2) = k, k = iK, dK = q$$

from Eqs. (36a) and (36b), following equation is obtained:

$$e^{[-iq-(1-i)Q(-d/2)+(1+i)Q(d/2)}\left[-1+e^{2iq}\right] = 0. \tag{37}$$

For the equality in Eq. (37) to be realized, it has to be $q = n\pi, (n = 1, 2, 3, \ldots)$ Thus, we obtain the quantization condition of energy as follows:

$$\mathbf{Kd} = \sqrt{\frac{\mathbf{2m}}{\hbar^{\mathbf{2}}}|\mathbf{E}|}\mathbf{d} = \mathbf{q}, [\mathbf{q} = \mathbf{n}\pi, (\mathbf{n} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots \ integers)] \tag{38}$$

(b) For the function $cosh[\mathbf{k}(\mathbf{x}-\mathbf{x_0})]$

According to Figure 1a, in domain I, $E < U(x)$; in domain II, $E > U(x)$; in domain III, $E < U(x)$. According to the equations given in Eqs. (23a) and (23b), the corresponding wave functions are written as follows:

$$\psi_1(x) = A_1\cosh[ik(x-x_0)]e^{iiG(x-x_0)}, \psi_2(x) = A_2\cosh[k(x-x_0)]e^{iG(x-x_0)};$$

$$\psi_3(x) = A_3\cosh[-ik(x-x_0)]e^{iiG(x-x_0)}, x_1 = x_0-d/2 \text{ and } x_2 = x_0+d/2$$

According to the conditions given in Eq. (34), it can be written that:

$$\psi_1(x_1) = \psi_2(x_1) \text{ or } \psi_1(x_1)-\psi_2(x_1) = 0, \tag{39a}$$

$$\psi_2(x_2) = \psi_3(x_2) \text{ or } \psi_2(x_2)-\psi_3(x_2) = 0. \tag{39b}$$

From Eqs. (39a) and (39b), two linear equations are obtained as follows:

$$a_{11}A_1+a_{12}A_3 = 0; a_{21}A_1+a_{22}A_3 = 0 \tag{39c}$$

$$a_{11} = e^{-G(-d/2)}\cos(dk/2), \quad a_{12} = 0; \quad a_{21} = 0, \quad a_{22} = -e^{G(d/2)}\cos(dk/2).$$

For this system of equations to have a solution different from zero, the determinant of coefficients should vanish, namely:

$$det\left|\begin{array}{c} a_{11}a_{12} \\ a_{21}a_{22} \end{array}\right| = 0 \rightarrow -\frac{1}{2}e^{-G(-d/2)+G(d/2)}[1+\cos(dk)] = 0. \tag{40}$$

In Eq. (40), $-\frac{1}{2}e^{-G(-d/2)+G(d/2)} \neq 0, k = m_1\sqrt{-E} = im_1\sqrt{E} = iK, \left(E > 0, K = m_1\sqrt{E}\right)$. So, $1+\cos(dk) = 1+\cos(idK) = 1+\cos(dK) = 0$. From the last equation, we have that: $dK = q = n\pi, (n = 1, 3, 5, 7, \ldots \ odd \ integer)$. This is also written as follows:

$$\mathbf{dK = q = (2n-1)\,\pi, (n = 1, 2, 3, \ldots integer)} \tag{41}$$

The solution of the system of equations in (39c) gives: $A_1 = 0$ and $A_3 = 0$. The coefficient $A_2 = A$ is found by the normalization of the function, $\psi_2(x) = A_2\cosh[k(x-x_0)]\,e^{iG(x-x_0)}$ namely:

$$\int_{x_0-d/2}^{x_0+d/2} A\cosh[k(x-x_0)]\,e^{iG(x-x_0)}A^*\cosh[k(x-x_0)]\,e^{-iG(x-x_0)}dx =$$

$$\frac{AA^*[dK+\sin(dK)]}{2K} = \frac{AA^*d}{2} = \frac{d|A|^2}{2} = 1 \rightarrow |A| = \sqrt{2/d} = \sqrt{2K/q}.$$

So, the normalized wave functions in the bound state are as follows:

$$\boldsymbol{\psi}(\mathbf{x}) = Acos[\mathbf{K(x-x_0)}]\,\mathbf{e^{iG(x-x_0)}} \text{ or } \psi(\mathbf{x}) = Acos[\mathbf{Kx}]\,\mathbf{e^{iG(x)}}. \tag{42}$$

(c) For the function $sinh[\mathbf{k(x-x_0)}]$

According to Figure 1a, in domain I, $E < U(x)$; in domain II, $E > U(x)$; in domain III, $E < U(x)$ According to Eqs. (23a) and (23b), the corresponding wave functions are written as follows:

$$\psi_1(x) = A_1\sinh[ik(x-x_0)]\,e^{iiG(x-x_0)}; \psi_2(x) = A_2\sinh[k(x-x_0)]\,e^{iG(x-x_0)}$$

$$\psi_3(x) = A_3\sinh[-ik(x-x_0)]\,e^{iiG(x-x_0)}; \quad x_1 = x_0-d/2 \text{ and } x_2 = x_0+d/2.$$

According to the conditions given in Eq. (34) we have the following:

$$\psi_1(x_1) = \psi_2(x_1) \text{ or } \psi_1(x_1)-\psi_2(x_1) = 0, \tag{43a}$$

$$\psi_2(x_2) = \psi_3(x_2) \text{ or } \psi_2(x_2)-\psi_3(x_2) = 0. \tag{43b}$$

From Eqs. (43a) and (43b), the following two linear equations are obtained:

$$a_{11}A_1+a_{12}A_3 = 0; \quad a_{21}A_1+a_{22}A_3 = 0 \tag{44}$$

$$a_{11} = -ie^{-G(-d/2)}\sin(dk/2); \quad a_{12} = 0; \quad a_{21} = 0; \quad a_{22} = ie^{G(d/2)}\sin(dk/2).$$

For this system of equations to have a solution different from zero, the determinant of coefficients should vanish, namely:

$$det\left|\begin{array}{c} a_{11}a_{12} \\ a_{21}a_{22} \end{array}\right| = 0 \rightarrow e^{-G(-d/2)+G(d/2)}\sin^2(dk/2) = 0. \tag{45}$$

In Eq. (45), $e^{-G(-d/2)+G(d/2)} \neq 0, k = m_1\sqrt{-E} = im_1\sqrt{E} = iK, \left(E > 0, K = m_1\sqrt{E}\right)$. So, $\sin^2(dk/2) = \sin^2(idK/2) = \sin^2(dK/2) = 0$. From the last equation, we have the following value:

$$\mathbf{dK = q = 2n\pi, (n = 1, 2, 3, \ldots integer)} \tag{46}$$

The solution of the system of equations in Eq. (44) gives $A_1 = 0$ and $A_3 = 0$. The coefficient $A_2 = A$ is found by the normalization of the function $\psi_2(x) = A_2 \sinh[k(x-x_0)] e^{iG(x-x_0)}$ namely:

$$\int_{x_0-d/2}^{x_0+d/2} A \sinh[k(x-x_0)] e^{iG(x-x_0)} A^* \sinh[k(x-x_0)] e^{-iG(x-x_0)} dx =$$

$$\frac{AA^*[-dK + \sin(dK)]}{2K} = -\frac{AA^* d}{2} = 1 \rightarrow |A| = \sqrt{2/d} = \sqrt{2K/q}.$$

So, the normalized wave functions in the bound state are as follows:

$$\psi(x) = A \sin[K(x-x_0)] e^{iG(x-x_0)} \text{ or } \psi(x) = A \sin[Kx] e^{iG(x)}. \tag{47}$$

Eqs. (41) and (46) can be combined as follows:

$$\mathbf{dK = q = n\pi, (n = 1, 2, 3, 4, \ldots integer)}. \tag{48}$$

So, in bound states, the normalized wave functions are:

$$\psi(x) = \sqrt{2/d} \cos[K(x-x_0)] e^{iG(x-x_0)} \text{ and } \psi(x) = \sqrt{2/d} \sin[K(x-x_0)] e^{iG(x-x_0)}, \tag{49a}$$

$$\text{Or, } \psi(x) = \sqrt{2/d} \cos[Kx)] e^{iG(x)} \text{ and } \psi(x) = \sqrt{2/d} \sin[Kx)] e^{iG(x)}, \tag{49b}$$

The energy:

$$E_q = \frac{\hbar^2}{2m}\frac{q^2}{d^2} = M_h \frac{q^2}{d^2}, (M_h = \frac{\hbar^2}{2m}). \tag{49c}$$

Now, in bound states, let us see the relations between the potential areas (see Figure 2):

(a) According to the partial integration, $\int udv = uv - \int vdu$ can be written as follows:
$S = S_p = \int_{x_1}^{x_2} U(x)\, dx = [xU(x)]_{x_1}^{x_2} - \int_{x_1}^{x_2} xU'(x)\, dx = x_2U(x_2) - x_1U(x_1) - S_t$; $S_t = \int_{x_1}^{x_2} xU'(x)\, dx$;
$U(x_1) = U(x_2) = E_q; d = x_2 - x_1$.

Thus, the following can be written:

$$S_p = Ed - S_t \text{ or } Ed = S_p + S_t. \tag{50a}$$

(b) Total energy=kinetic energy+potential energy; $E = T + U(x)$. From here, with the following integration:

$$\int_{x_1}^{x_2} Edx = \int_{x_1}^{x_2} Tdx + \int_{x_1}^{x_2} U(x)dx \rightarrow Ed = S_k + S_p. \tag{50b}$$

If Eqs. (50a) and (50b) are compared, it is seen that $S_k = S_t = \int_{x_1}^{x_2} xU'(x)\, dx$.

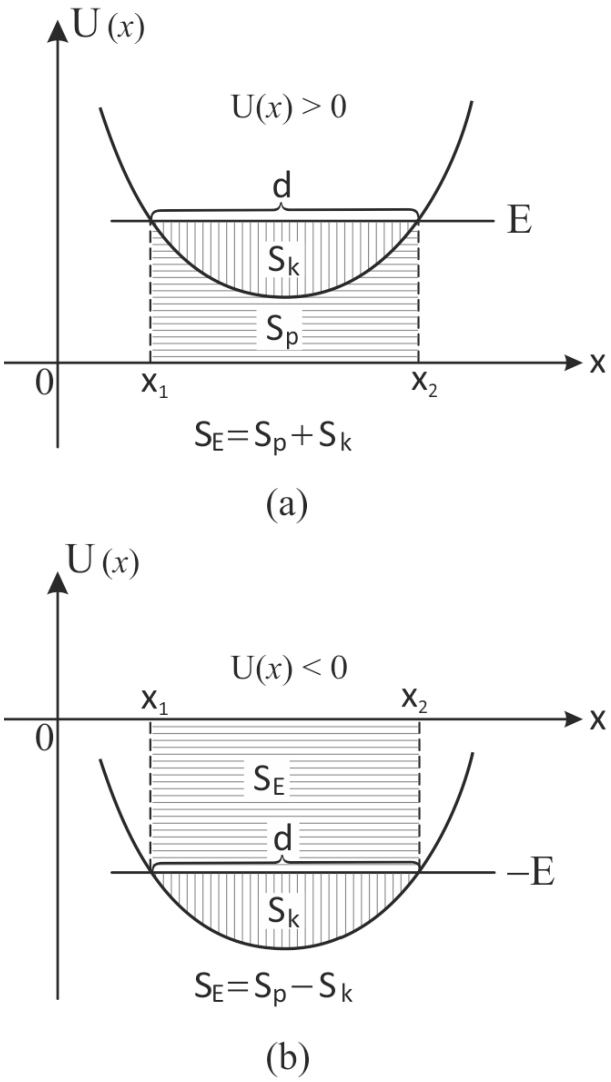

Figure 2. Potential area in the bound state: **(a)**, $U(x) > 0$ and $E > 0$; **(b)**, $U(x) < 0$ and $E < 0$.

(c) According to Eq. (8): $E = -\frac{m}{2\hbar^2}S^2 = -\frac{m}{2\hbar^2}S_p{}^2 \rightarrow S_p = \sqrt{\frac{2\hbar^2}{m}|E|}$

$$Kd = q \rightarrow \sqrt{\frac{2m}{\hbar^2}|E|}d = q \rightarrow |E| = \frac{\hbar^2}{2m}\frac{q^2}{d^2} = M_h\frac{q^2}{d^2} \rightarrow S_p = 2M_h\frac{q}{d}. \tag{50c}$$

(d) From (50a) and (50b), we obtain the following:

$$S_k = M_h\frac{q}{d}(q-2). \tag{50d}$$

Thus from Eqs. (50a)–(50d), let us rewrite these potential areas as follows:

$$S_p = \int_{x_1}^{x_2} U(x)\,dx = 2M_h\frac{q}{d}\ (Potential\ energy\ field), \tag{51a}$$

$$S_k = \int_{x_1}^{x_2} xU'(x)\,dx = M_h\frac{q}{d}(q-2)\ (Kinetic\ energy\ field), \tag{51b}$$

$$S_E = S_p + S_k = M_h\frac{q^2}{d}\ (Total\ energy\ field), \tag{51c}$$

For $q=2$, $S_k=0$, the ground state occurs; for $q=n\pi, (n=1,2,3,\ldots)$, the excited states occur. Eqs. (51a)–(51c) indicate that potential areas are quantized.

4. Summary

It is possible to summarize the results above as follows. We can write the general solutions of the SE in one dimension as follows:

$$\psi(x) = Ae^{kx}e^{\pm iG(x)} \text{ or } \psi(x) = Ae^{k(x-x_0)}e^{\pm iG((x-x_0)}, \tag{52a}$$

$$\psi(x) = Ae^{-kx}e^{\pm iG(x)} \text{ or } \psi(x) = Ae^{-k(x-x_0)}e^{\pm iG((x-x_0)}, \tag{52b}$$

$$\psi(x) = \left[Ae^{kx}+Be^{-kx}\right]e^{\pm iG(x)} \text{ or } \psi(x) = \left[Ae^{k(x-x_0)}+Be^{-k(x-x_0)}\right]e^{\pm iG(x-x_0)}, \tag{52c}$$

$$\psi(x) = Acosh(kx)\,e^{\pm iG(x)} \text{ or } \psi(x) = Acosh[k(x-x_0)]e^{\pm iG(x-x_0)}, \tag{52d}$$

$$\psi(x) = A\sinh(kx)\,e^{\pm iG(x)} \text{ or } \psi(x) = A\sinh[k(x-x_0)]e^{\pm iG(x-x_0)}, \tag{52e}$$

$$\psi(x) = Ae^{kx\pm iG(x)}+Be^{-kx\mp iG(x)} \text{ or } \psi(x) = Ae^{k(x-x_0)\pm iG(x-x_0)}+Be^{-k(x-x_0)\mp iG(x-x_0)}, \tag{52f}$$

In these functions, we have the following values:

$$\text{(a) For } E>U(x);\ k=m_1\sqrt{-E}, G(x) = m_1\int\sqrt{-U(x)}dx \tag{53a}$$

$$\text{(b) For } E<U(x);\ k=m_1\sqrt{E}, G(x)=m_1\int\sqrt{U(x)}dx. \tag{53b}$$

$m_1 = \sqrt{2m}\big/\hbar = \sqrt{2m/\hbar^2}$, [$m$ mass or reduced mass of particle, $\hbar = h/(2\pi)$, h Planck constant]; x_1 and $x_2 (x_2>x_1)$ are, depending on E, the roots of the equation $E=U(x)$; and $x_0 = (x_1+x_2)/2; d = x_2-x_1$.

This solution is similar to that of the WKB approach but not exactly the same. There is approximation in the WKB method, but there is no approach in the method we have given here. Our procedure gives exact results. Those who are familiar with the WKB approach can easily see the differences.

In bound states, the normalized wave functions are as follows [G is taken as real function):

$$\boldsymbol{\psi}(\mathbf{x}) = \mathbf{A}\cos[\mathbf{Kx}]\,\mathbf{e}^{\mathbf{iG(x)}} \text{ or } \psi(\mathbf{x}) = \mathbf{A}\cos[\mathbf{K}(\mathbf{x}-\mathbf{x_0})]\,\mathbf{e}^{\mathbf{iG(x-x_0)}}, \tag{54a}$$

$$\boldsymbol{\psi}(\mathbf{x}) = \mathbf{B}\sin[\mathbf{Kx}]\,\mathbf{e}^{\mathbf{iG(x)}} \text{ or } \psi(\mathbf{x}) = \mathbf{B}\sin[\mathbf{K}(\mathbf{x}-\mathbf{x_0})]\,\mathbf{e}^{\mathbf{iG(x-x_0)}}, \tag{54b}$$

$$A = B = \sqrt{2/d} = \sqrt{2K/q}; \quad k = m_1\sqrt{|E|} = \sqrt{\frac{2m}{\hbar^2}}\sqrt{|E|}; \quad G(x) = m_1 \int \sqrt{|U(x)|}dx.$$

x_1 and x_2 are the roots of the equation: $|E| = |U(x)|$ or $|E| = |U(x-x_0)|$. From this, we have the following:

$$x_0 = (x_1+x_2)/2; \quad d = x_2-x_1; \quad x_1 = x_0-d/2; \quad x_2 = x_0+d/2; \quad E = U(x); \quad |E| = |U(x_1)| = |U(x_2)|$$

$$\text{or } |E| = |U(-d/2)| = |U(d/2)|; \quad 2|E| = |U(-d/2)| + |U(d/2)|. \tag{54c}$$

$$\mathbf{Kd = q \rightarrow \sqrt{\frac{2m}{\hbar^2}|E|}d = q \rightarrow |E| = \frac{\hbar^2}{2m}\frac{q^2}{d^2} = M_h\frac{q^2}{d^2}; \quad [M_h = \frac{\hbar^2}{2m}]} \tag{55}$$

For $q = 2$, ground state occurs; for $q = n\pi, (n = 1, 2, 3, \ldots)$, excited states occur. The energy values are also given by the following formulas:

$$E = -\frac{m}{2\hbar^2}S^2 = -\frac{m}{2\hbar^2}{S_p}^2; \quad S_p = \sqrt{\frac{2\hbar^2}{m}|E|}; \quad S = S_p(x_0, d) = 2M_h\frac{q}{d}; \quad S_k(x_0, d) = M_h\frac{q}{d}(q-2). \tag{56}$$

4.1. Practical procedure to find the energy values

As seen in Eq. (55), the total energy values depend on d; so the d value should be calculated. It can be calculated by one of the equations given in Eq. (54c), but to find the values $of d$ and energy E, the practical procedure can be given as follows:

First, solving the equation $|U(x)| = M_h q^2/y, (y = d^2)$, the following values are found: $x_1, x_2, d = x_2 - x_1$, $d_2(y) = d * d$. Then, the equation $y = d_2(y)$ is solved, and the y value is found. So, the energy value is obtained as follows:

$$|E| = M_h\frac{q^2}{y} = M_h\frac{q^2}{d^2} = \frac{\hbar^2}{2m}\frac{q^2}{d^2}. \tag{57}$$

For $q = 2$, the ground state occurs; for $q = n\pi, (n = 1, 2, 3, \ldots)$ the excited states occur.

5. Examples of one dimensional potentials

5.1. Potential: $\mathbf{U(x) = a|x|^p, (a > 0, p > 0)}$

5.1.1. Energy

According to the practical procedure above:

$$E = U(x) = a|x|^p = M_h\frac{q^2}{y} \rightarrow x_1 = -\left[\frac{M_h}{a}\frac{q^2}{y}\right]^{1/p}; \quad x_2 = \left[\frac{M_h}{a}\frac{q^2}{y}\right]^{1/p}$$

$$d = x_2 - x_1 = 2\left[\frac{M_h}{a}\frac{q^2}{y}\right]^{1/p}; \quad d_2(y) = d * d = 4\left[\frac{M_h}{a}\frac{q^2}{y}\right]^{2/p}; \quad y = d_2(y)$$

$$y = \left[\frac{a2^{-p}}{M_h q^2}\right]^{-2/(p+2)}; \quad E_q = M_h\frac{q^2}{y} = M_h q^2\left[\frac{a2^{-p}}{M_h q^2}\right]^{2/(p+2)} \tag{5.1.1}$$

For $q=2$, the ground state occurs; for $q=n\pi,(n=1,2,3,\ldots)$, the excited states occur. In the formula given in (5.1.1), if $p=2$ and $a=\frac{1}{2}m\omega^2$, it is found that $E_q=\frac{q}{4}\hbar\omega$. This is the energy of the simple harmonic oscillator. We have $E_0=\frac{1}{2}\hbar\omega$ ground state energy for $q=2$; $E_q=\frac{n\pi}{2}\hbar\omega,(n=1,2,3,..)$ excited state energy for $q=n\pi$. The well-known energy of the simple harmonic oscillator is $E_n=\left(n+\frac{1}{2}\right)\hbar\omega,(n=0,1,2,\ldots)$. The ground state energy $E_0=\frac{1}{2}\hbar\omega$ is the same but the excited energy $E_q=E_n=\frac{n\pi}{2}\hbar\omega$ is different [8]. This difference is observed in the experimentally measured energy spectra of nuclear nuclei, but this phenomenon cannot be explained by harmonic oscillatory models. This event can be explained here. We think that the latter is more accurate because there is no approximation in our solutions.

For the same potential, the ground state energy obtained from the supersymmetric quantum mechanics (SQ) is as follows [10]:

$$E_0=\left[\frac{0.8862\Gamma\left(\frac{3}{2}+\frac{1}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)}\frac{\hbar^2}{2m}a^{\frac{2}{p}}\right]^{p/(p+2)}. \tag{5.1.2}$$

From Eq. (5.1.2), for $p=2$ and $a=\frac{1}{2}m\omega^2$, we find that $E_0=0.999985\frac{1}{2}\hbar\omega\approx\frac{1}{2}\hbar\omega$. In this potential, for $p=1$ (V-Form potential), $E_0\approx 0.794\left[\frac{\hbar^2a^2}{m}\right]^{1/3}$ is obtained from the formula given in Eq. (5.1.1); $E_0\approx 0.763\left[\frac{\hbar^2a^2}{m}\right]^{1/3}$ from the SQ [10]; $E_0\approx 0.813\left[\frac{\hbar^2a^2}{m}\right]^{1/3}$ from the variation method [11]; and $E_0\approx 0.885\left[\frac{\hbar^2a^2}{m}\right]^{1/3}$ from the WKB method [11]. These three values are approximately the same.

5.1.2. Wave functions

$$G(x)=m_1\int\sqrt{|U(x)|}dx=\sqrt{\frac{2m}{\hbar^2}}\int\sqrt{U(x)}dx=Q(x)=\sqrt{\frac{2m}{\hbar^2}}\sqrt{a}\frac{2|x|^{(p+2)/2}}{p+2},$$

$$\psi(x)=A\cos[Kx]\,e^{iG(x)}=\sqrt{2K/q}\cos[Kx]\,e^{iQ(x)}=\sqrt{2/d}\cos[Kx]\,e^{iQ(x)} \tag{5.1.3a}$$

$$\psi(x)=B\sin[Kx]\,e^{iG(x)}=\sqrt{2K/q}\sin[Kx]\,e^{iQ(x)}=\sqrt{2/d}\sin[Kx]\,e^{iQ(x)}. \tag{5.1.3b}$$

In the case of simple harmonic oscillator, $Q(x)=\sqrt{\frac{2m}{\hbar^2}}\sqrt{a}\frac{2|x|^{(p+2)/2}}{p+2}=\frac{m\omega}{2\hbar}x^2$. The well-known wave function of the harmonic oscillator is as follows:

$$\psi_n(\rho)=A_ne^{-\rho^2/2}H_n(\rho);\quad[\rho=\sqrt{\frac{m\omega}{\hbar}}x] \tag{5.1.4}$$

Here, $H_n(\rho)$ is Hermit polynomials. If Eqs. (5.1.3a) and (5.1.3b) are expanded in series, polynomials are obtained. So, the well-known function given in Eq. (5.1.4) of harmonic oscillator is an approximate function.

5.2. Infinitely high square potential well or finite square potential well

We consider a particle of mass m captured in a box limited by $0\le x\le a$. The corresponding potential is given:

$U(x)=0$ for $0<x<a$; $U(x)=\infty$ for $x<0$ and $x>a$.

5.2.1. Energy

The turning points of this potential are given by the following equation: $U(x) = E$. From this equation, we can find the classical turning points of the potential function as follows: $x_1 = 0$ and $x_2 = a$; $x_0 = (x_1+x_2)/2 = a/2, d = x_2 - x_1 = a$. By substituting this value of d into the equation $dK = q$, and then solving for $|E|$, we can have the energy value as $E_q = \frac{\hbar^2}{2m}\frac{q^2}{a^2} = M_h\frac{q^2}{a^2}$; for $q = 2$, the ground state occurs; for $q = n\pi, (n = 1, 2, 3, \ldots)$, the excited states occur. The well-known energy is $E_n = \frac{\hbar^2}{2m}\frac{(n\pi)^2}{a^2}$ [8]. It is seen that the two energy values are the same.

5.2.2. Wave functions

Here, $U(x) = 0$. So, $G(x) = m_1\int\sqrt{|U(x)|}dx = 0$,

$$\psi(x) = A\cos[Kx]\,e^{iG(x)} = \sqrt{2/a}\cos[Kx] = \sqrt{2K/q}\cos[Kx],$$

$$\psi(x) = B\sin[Kx]\,e^{iG(x)} = \sqrt{2/a}\sin[Kx] = \sqrt{2K/q}\sin[Kx],$$

$$\psi(x) = A\cos\left[K(x-\frac{a}{2})\right] = \sqrt{2/a}\cos\left[K(x-\frac{a}{2})\right] = \sqrt{2K/q}\cos\left[K(x-\frac{a}{2})\right],$$

$$\psi(x) = B\sin\left[K(x-\frac{a}{2})\right] = \sqrt{2/a}\sin\left[K(x-\frac{a}{2})\right] = \sqrt{2K/q}\sin\left[K(x-\frac{a}{2})\right].$$

5.3. Trigonometric potential well

We consider the potential energy of the particle $U(x)$ so that $U(x) = U_0 cotg^2(\pi x/a)$, $[U_0 > 0, a > 0$ and $0 < x < a]$.

5.3.1. Energy

The roots of the equation

$$E_q = U(x) = M_h\frac{q^2}{y} \rightarrow U_0 cotg^2[\pi x/a] = M_h\frac{q^2}{y};$$

$$x_1 = -\frac{1}{\pi}arccotg\left[\sqrt{\frac{M_h}{U_0}\frac{q^2}{y}}\right]; x_2 = \frac{1}{\pi}arccotg\left[\sqrt{\frac{M_h}{U_0}\frac{q^2}{y}}\right]; d = x_2 - x_1; y = d * d$$

$$y = \frac{4}{\pi^2}arccotg^2\left[\sqrt{\frac{M_h}{U_0}\frac{q^2}{y}}\right]; \quad E_q = M_h\frac{q^2}{d^2} = M_h\frac{q^2}{y}. \tag{5.3.1}$$

Eq. (5.3.1) is not solved analytically. So, it may be solved by the numerical method for the exact values. We have for $q = 2$, the ground state; for $q = n\pi, (n = 1, 2, 3, \ldots)$, the excited states.

5.3.2. Wave functions

$$G(x) = m_1\int\sqrt{|U(x)|}dx = m_1\int\sqrt{U(x)}dx = m_1\int\sqrt{U_0 cotg^2(\pi x/a)}dx = Q(x), Q(x) = \sqrt{\frac{2ma^2U_0}{\hbar^2\pi^2}}\ln\left[\sin(\frac{\pi x}{a})\right],$$

$$\psi(x) = A\cos[Kx)]\,e^{iG(x)} = \sqrt{2K/q}\cos[Kx]\,e^{iQ(x)} = \sqrt{2/d}\cos[Kx]\,e^{iQ(x)},$$

$$\psi(x) = B\sin[Kx)]\,e^{iG(x)} = \sqrt{2K/q}\sin[Kx]\,e^{iQ(x)} = \sqrt{2/d}\sin[Kx]\,e^{iQ(x)}.$$

5.4. Infinitely high parabolic potential well

We consider the potential energy of the particle, $U(x)$, so that $U(x) = U_0\left(\frac{a}{x}-\frac{x}{a}\right)^2$, $[U_0> 0, a > 0,\ x > 0]$.

5.4.1. Energy

The positive roots of the equation $E = U(x) = U_0\left(\frac{a}{x}-\frac{x}{a}\right)^2 = M_h\frac{q^2}{y}$ are as follows:

$$x_1 = \sqrt{\frac{a^2\left[M_hq^2+2yU_0-q\sqrt{M_h}\sqrt{M_hq^2+4yU_0}\right]}{2yU_0}}; x_2 = \sqrt{\frac{a^2\left[M_hq^2+2yU_0+q\sqrt{M_h}\sqrt{M_hq^2+4yU_0}\right]}{2yU_0}}$$

$$d_2 = d^2 = (x_2-x_1)^2 = \frac{a}{2}\left[\sqrt{\frac{\left[M_hq^2+2yU_0+q\sqrt{M_h}\sqrt{M_hq^2+4yU_0}\right]}{yU_0}}-\sqrt{\frac{\left[M_hq^2+2yU_0-q\sqrt{M_h}\sqrt{M_hq^2+4yU_0}\right]}{yU_0}}\right]^2.$$

The root of the equation $d_2 = y$ is $y = \sqrt{a^2\frac{M_hq^2}{U_0}}$ and $E_q = M_h\frac{q^2}{d^2} = M_h\frac{q^2}{y} = \sqrt{\frac{M_hU_0}{a^2}}q \rightarrow E_q = \sqrt{\frac{\hbar^2U_0}{2ma^2}}q$.

We have ground state for $q = 2$; the excited states for $q = n\pi, (n = 1, 2, 3, \ldots)$ [8]. For $m = 1 and \hbar = 1$, this energy value is $E_n = \frac{\pi}{2}n\sqrt{\frac{2U_0}{a^2}}$. For this potential, the energy values obtained from the supersymmetric WKB [12] and standard WKB [14], respectively, are as follows:

$$E_n = \sqrt{\frac{2U_0}{a^2}}\left[2n+\sqrt{1+8a^2U_0}\right]-2U_0 \text{ and } E_n = \sqrt{\frac{2U_0}{a^2}}\left[2n+1+\sqrt{2a^2U_0}\right]-2U_0.$$

5.4.2. Wave functions

$$G(x) = m_1\int\sqrt{|U(x)|}dx = m_1\int\sqrt{U(x)}dx = m_1\int\sqrt{U_0\left(\frac{a}{x}-\frac{x}{a}\right)^2}dx = m_1\int\sqrt{U_0}\left(\frac{a}{x}-\frac{x}{a}\right)dx = Q(x);$$

$$Q(x) = \sqrt{\frac{2mU_0}{\hbar^2}}\left[a\ln(x)-\frac{x^2}{2a}\right]$$

$$\psi(x) = A\cos[Kx]\,e^{\pm iQ(x)};\quad \psi(x) = B\sin[Kx]\,e^{\pm iQ(x)};\quad |A| = |B| = \sqrt{2/d} = \sqrt{2K/q}, [\delta = \sqrt{{E_q}^2-4ab}].$$

5.5. Potential $\mathbf{U(x) = ax^2+b/x^2}$

We consider the potential energy of the particle as $U(x) = ax^2+b/x^2$, where a and b are positive constants.

5.5.1. Energy

The roots of the equation $U(x) = ax^2+b/x^2 = M_h\frac{q^2}{y} = E_q$ are as follows:

$$x_1 = -\sqrt{\frac{E_q+\delta}{2a}}, x_2 = \sqrt{\frac{E_q-\delta}{2a}}, x_3 = -\sqrt{\frac{E_q-\delta}{2a}}, x_4 = \sqrt{\frac{E_q+\delta}{2a}}; [\delta = \sqrt{{E_q}^2-4ab}].$$

From these grandeurs, we get

$$d_{31} = x_3 - x_1 = -\sqrt{\frac{E_q+\delta}{2a}} + \sqrt{\frac{E_q-\delta}{2a}}; {d_{31}}^2 = E_q/a - 2\sqrt{b/a};$$

$$d_{42} = x_4 - x_2 = \sqrt{\frac{E_q+\delta}{2a}} - \sqrt{\frac{E_q-\delta}{2a}}; {d_{42}}^2 = E_q/a - 2\sqrt{b/a}.$$

From these equations, we can easily obtain $d_{31} = -d_{42} = d, d^2 = E_q/a - 2\sqrt{b/a}$.

From the solution of the equation $E_q = M_h \frac{q^2}{d^2}$, we get the positive energy value as follows:

$$E_q = \sqrt{ab} + \sqrt{ab + aM_q q^2} = \sqrt{ab} + \sqrt{ab + a\frac{\hbar^2}{2m}q^2} \tag{5.5.1}$$

For $q = 2$, the ground state occurs; for $q = n\pi, n = 1, 2, 3, \ldots)$, the excited states occur.
For $m = 1$ and $\hbar = 1$, this energy becomes:

$$E_q = \sqrt{ab} + \sqrt{ab + \frac{a}{2}\pi^2 n^2}, (n = 1, 2, 3, \ldots). \tag{5.5.2a}$$

The energies obtained from the supersymmetric WKB and standard WKB [8,12–14] respectively are as follows:

$$E_n = \sqrt{2a}\left[2n + 1 + \sqrt{\frac{1}{4} + 2b}\right], (n = 0, 1, 2, 3, \ldots), \tag{5.5.2b}$$

$$E_n = \sqrt{2a}\left[2n + 1 + \sqrt{2b}\right], (n = 0, 1, 2, 3, \ldots). \tag{5.5.2c}$$

It can be assumed that the values given in Eqs. (5.5.1) and (5.5.2a) are more accurate because there is no approximation.

5.5.2. Wave functions

$$G(x) = m_1 \int \sqrt{|U(x)|}dx = m_1 \int \sqrt{U(x)}dx = m_1 \int \sqrt{ax^2 + b/x^2}dx$$

$$= m_1 \frac{1}{2}\left\{\sqrt{b + ax^4} + \sqrt{b}\left[\ln\left(x^2\right) - \ln\left(b + \sqrt{b}\sqrt{b + ax^4}\right)\right]\right\} = Q(x),$$

$$Q(x) = \sqrt{\frac{m}{2\hbar^2}}\left\{\sqrt{b + ax^4} + \sqrt{b}\left[\ln\left(x^2\right) - \ln\left(b + \sqrt{b}\sqrt{b + ax^4}\right)\right]\right\},$$

$\psi(x) = A\cos[Kx]\, e^{iQ(x)}$ and $\psi(x) = B\sin[Kx]\, e^{iQ(x)}$; $|A| = |B| = \sqrt{2/d} = \sqrt{2K/q}$.

6. The radial Schrödinger equation for spherical symmetric potentials

The Schrödinger equation (SE) in three dimensions is given as follows:

$$\Delta\Psi\left(\vec{r}\right)+\frac{2m}{\hbar^2}\left[E-V(\vec{r})\right]\Psi\left(\vec{r}\right)=0. \tag{58}$$

Here, E and V are the total and potential energies, respectively, m is the mass or reduced mass of particle. Spherical polar coordinates $x=r\sin\left(\theta\right)\cos\left(\phi\right), y=r\sin\left(\theta\right)\sin\left(\phi\right),$ and $z=r\cos\left(\theta\right)$ are appropriate for the symmetry of the problem. The SE given in Eq. (58), expressed in these coordinates, is as follows:

$$\left[\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right]\Psi\left(r,\theta,\phi\right)+\frac{1}{r^2}\hat{L}^2\left(\theta,\phi\right)\Psi\left(r,\theta,\phi\right)+\frac{2m}{\hbar^2}\left[E-V(r,\theta,\phi)\right]\Psi\left(r,\theta,\phi\right)\ =\ 0. \tag{59a}$$

Here,

$$\hat{L}^2\left(\theta,\phi\right)\ =\ \frac{\partial^2}{\partial\theta^2}+cotg\left(\theta\right)\frac{\partial}{\partial\theta}+\frac{1}{sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}. \tag{59b}$$

The potential energy of a particle which moves in a central, spherically symmetric field of force depends only upon the distance r between the particle and the force center. Thus, the potential energy should be $V(r,\theta,\phi)=V(r)$. Solution of Eqs. (59a) and (59b) can be found by the method of separation of variables. To apply this method, the solution is assumed to be in the following form:

$$\Psi\left(r,\theta,\phi\right)=R\left(r\right)Y\left(\theta,\phi\right)\ \text{ or }\ \Psi\left(r,\theta,\phi\right)=R\left(r\right)|jm>. \tag{60}$$

In Eq. (60), $R(r)$ is independent of the angles and $Y(\theta,\phi)$ or $|jm>$ is independent of r. Substituting Eq. (60) into Eq. (59a) and rearranging it, the following two equations are obtained:

$$\frac{\partial^2R(r)}{\partial r^2}+\frac{2}{r}\frac{\partial R(r)}{\partial r}+\left\{\frac{2m}{\hbar^2}\left[E-V\left(r\right)\right]-\frac{C}{r^2}\right\}R\left(r\right)=0, \tag{61}$$

$$\hat{L}^2\left(\theta,\phi\right)Y\left(\theta,\phi\right)+CY\left(\theta,\phi\right)=0, \tag{62}$$

where C is constant. Eq. (62) is independent of the total energy E and of the potential energy $V\left(r\right)$ and, therefore, the angular dependence of the wave functions is determined by the property of spherical symmetry, and admissible solutions of Eq. (62) are valid for every spherically symmetric system regardless of the special form of the potential function. The solutions of Eq. (62) can be found in any quantum mechanics and mathematical physics textbook [15–18] and the solutions are known as spherical harmonic functions, $Y_{l\mu}\left(\theta,\phi\right), where C=l\left(l+1\right), (l=0,1,2,3,\ldots)$ are positive integer numbers and $\mu=-l,-l+1,-l+2,\ldots0,1,2,\ldots,l$. Eq. (61) is the radial SE. Substituting $C=l\left(l+1\right)$ and $F\left(r\right)=rR(r)$ values into Eq. (61), the radial wave equation is obtained as follows:

$$\frac{\partial^2F(r)}{\partial r^2}+\frac{2m}{\hbar^2}\left[E-U\left(r\right)\right]F\left(r\right)=0. \tag{63}$$

Here, $U\left(r\right)=V\left(r\right)+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$ is the effective potential energy. Eq. (63) is a one dimensional differential equation and is the same as Eq. (2). In Eq. (2), the free variable is x, while in Eq. (63) the free variable is r. So, the solution procedure of one dimensional differential equation has been given above in Eqs. (52a)–(57). Now let us give some examples.

6.1. Coulomb type central potential well

The potential energy of hydrogen-like atom is $V(r) = -Ze^2/r$. If the centrifugal potential function is added to this potential $V(r)$, we have the following effective potential function: $U(r) = -\frac{a}{r} + \frac{b}{r^2}$, $\left[a = Ze^2 \text{ and } b = \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]$

6.1.1. Energy

The classical turning points of this effective potential are given by the following equation: $-\frac{a}{r} + \frac{b}{r^2} = -|E| = -M_h\frac{q^2}{y}$. From this equation, we can find the classical turning points of this effective potential function and d_2 as follows [21]:

$$r_1 = \frac{ay - \sqrt{a^2y^2 - 4bM_hyq^2}}{2M_hq^2}; r_2 = \frac{ay + \sqrt{a^2y^2 - 4bM_hyq^2}}{2M_hq^2}; d_2 = (r_2 - r_1)^2 = \frac{y\left[ya^2 - 4bM_hq^2\right]}{{M_h}^2q^4}.$$

The root of the equation $d_2 = y$ is $y = \frac{M_hq^2\left[M_hq^2 + 4b\right]}{a^2}$, and $|E_q| = M_h\frac{q^2}{d^2} = M_h\frac{q^2}{y}$;

$$E_q = -\frac{2me^4Z^2}{\hbar^2\left[4l + 4l^2 + q^2\right]} = -\frac{me^4}{2\hbar^2}\frac{Z^2}{\left[l(l+1) + q^2/4\right]} = -E_0\frac{Z^2}{\left[l(l+1) + q^2/4\right]},\ E_0 = \frac{me^4}{2\hbar^2} \tag{6.1.1}$$

We have the ground state for $q = 2$; the excited states for $q = n\pi, (n = 1, 2, 3, \ldots)$. For the hydrogen atom; $Z = 1$ atom number, m electron mass, e electron charge and in the ground state, $l = 0$. From Eq. (6.1.1), $E_0 = -13.6eV$ is obtained. This is the well-known ground state energy of the hydrogen atom. The well-known excited state energy of hydrogen-like atoms is given as follows: $E_n = -E_0\frac{Z^2}{n^2}, (n = 1, 2, 3, \ldots)$. There is no obvious $l = 0$ quantum number in this formula, but in Eq. (6.1.1) this number of quantum is clearly visible [19].

6.1.2. Wave functions

$$G(r) = m_1\int\sqrt{|U(r)|}dr = \sqrt{\frac{2m}{\hbar^2}}\int\sqrt{-U(r)}dr = \sqrt{\frac{2m}{\hbar^2}}\int\sqrt{\frac{a}{r} - \frac{b}{r^2}}dr = Q(r);$$

$$Q(r) = 2\sqrt{\frac{2m}{\hbar^2}}\left[\sqrt{ar - b} - \sqrt{b}\arctan(\sqrt{\frac{ar - b}{b}})\right],$$

$F(r) = A\cos\left[Kr\right]e^{iQ(r)}$ and $F(r) = \sqrt{\frac{2}{d}}\cos\left[Kr\right]e^{iQ(r)} = \sqrt{\frac{2K}{q}}\cos\left[Kr\right]e^{iQ(r)}$, $F(r) = B\sin\left[\frac{q}{d}r\right]e^{iQ(r)}$, and $F(r) = \sqrt{\frac{2}{d}}\sin\left[\frac{q}{d}r\right]e^{iQ(r)} = \sqrt{\frac{2K}{q}}\sin\left[\frac{q}{d}r\right]e^{iQ(r)}$,

$$\Psi(r, \theta, \phi) = R(r)Y(\theta, \phi) = \frac{F(r)}{r}Y(\theta, \phi) = \frac{F(r)}{r}|jm>. \tag{6.1.2}$$

In Eq. (6.1.2), m is not mass, it is magnetic quantum number. The known wave function of the hydrogen atom is an exponential function including Laguerre polynomials. Our results are more accurate because there is no approach.

6.2. Spherical symmetric square well (infinitely high or finite)

Consider a particle of mass m captured in a box limited by $0 \leq r \leq a$. The corresponding central potential can be given by $V(r) = 0$ for $0 \leq r \leq a; V(r) = \infty$ for $r < 0$ and $r > a$. With this potential, the effective potential is as follows:

$$U(r) = \frac{b}{r^2}; [b = \frac{\hbar^2}{2m} l(l+1)].$$

6.2.1. Energy

With this effective potential, the equation $U(r) = \frac{b}{r^2} = \frac{M_h q^2}{y}$ can be written. From this equation, the classical turning points and some grandeur are found as follows [19]: $r_1 = \frac{1}{q}\sqrt{\frac{by}{M_h}}; r_2 = a;\ d_2 = (r_2 - r_1)^2 = \left[a - \frac{1}{q}\sqrt{\frac{by}{M_h}}\right]^2$.

From the solution of the equation $d_2 = y$; we can obtain $y = \frac{M_h a^2 q^2}{\left[\sqrt{b} \pm \sqrt{M_h} q\right]^2}$ and the energy value as follows:

$$E_q = \frac{\left[\sqrt{b} \pm \sqrt{M_h} q\right]^2}{a^2} = \frac{\hbar^2}{2ma^2}\left[\sqrt{l(l+1)} \pm q\right]^2. \tag{6.2.1}$$

For $q = 2$, the ground state occurs; for $q = n\pi, (n = 1, 2, 3, \ldots)$ the excited states occur.

The known allowed energies are given as follows [9]:

$$E_{nl} = \frac{\hbar^2}{2ma^2}{\beta_{nl}}^2. \tag{6.2.2}$$

Here, β_{nl} is the nth zero of the l^{th} spherical Bessel functions. Eq. (6.2.1) can be written as follows:

$$E_q = E_{nl} = \frac{\hbar^2}{2ma^2}\left[\sqrt{l(l+1)} \pm n\pi\right]^2. \tag{6.2.3}$$

Some values of β_{nl} and energies calculated according to Eqs. (6.2.2) and (6.2.3) [with sign + in (6.2.3)] are given in the Table 1.

Table 1. Some energy values of the infinitely high spherical symmetric square well (Unit $\hbar^2/(2ma^2)$).

nl	β_{nl}	E_{nl} From (6.2.2)	E_{nl} From (6.2.3)
1 s	3.142	9.872	9.870
1 p	4.493	20.187	20.755
1 d	5.763	33.212	31.260
2 s	6.283	39.476	39.478
2 p	7.725	59.676	59.250
2 d	9.095	82.719	76.260

6.2.2. Wave functions

$$G(r) = m_1 \int \sqrt{|U(r)|}dr = \sqrt{\frac{2m}{\hbar^2}} \int \sqrt{|U(r)|}dr = m_1 \int \sqrt{\frac{b}{r^2}}dr = \sqrt{\frac{2m}{\hbar^2}} \int \sqrt{\frac{b}{r^2}}dr,$$

$$G(r) = m_1 \int \sqrt{\frac{b}{r^2}}dr = \sqrt{\frac{2m}{\hbar^2}}\sqrt{b}\ln(r) = \sqrt{l(l+1)}\ln(r) = Q(r),$$

$$F(r) = A\cos[Kr]\, e^{iG(r)} = A\cos[Kr]\, e^{iQ(r)};$$

$$F(r) = B\sin[Kr]\, e^{iG(r)} = \sin[Kr)]\, e^{iQ(r)}; \quad F(r) = A\cos\left[\frac{q}{d}r\right] e^{iG(r)} = A\cos\left[\frac{q}{d}r\right] e^{iQ(r)};$$

$$F(r) = B\sin\left[\frac{q}{d}r\right] e^{iG(r)} = \sin\left[\frac{q}{d}r\right] e^{iQ(r)}; \quad |A| = |B| = \sqrt{2/d} = \sqrt{2K/q}$$

$$\Psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi) = \frac{F(r)}{r}Y(\theta,\phi).$$

6.3. Three dimensional isotropic harmonic oscillator potential

The potential energy of three dimensional isotropic harmonic oscillators is given by $V(r) = \frac{1}{2}m\omega^2 r^2$. With this potential, the effective potential $U(r)$ is as follows:

$$U(r) = \frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2 l(l+1)}{2m}\frac{1}{r^2} = ar^2 + \frac{b}{r^2}; \quad [a = \frac{1}{2}m\omega^2 \text{ and } b = \frac{\hbar^2 l(l+1)}{2m}].$$

6.3.1. Energy

From this effective potential, the positive roots of the equation $E_q = M_h\frac{q^2}{y} = U(r)$ are $r_1 = \sqrt{\frac{E_q - \delta}{2a}}; r_2 = \sqrt{\frac{E_q+\delta}{2a}}$; $[E_q = M_h\frac{q^2}{y}; \delta = \sqrt{{E_q}^2 - 4ab}]$.

With these roots, $d = r_2 - r_1 = \sqrt{\frac{E_q+\delta}{2a}} - \sqrt{\frac{E_q-\delta}{2a}}; d_2 = y = d * d = d^2$.

By substituting this value of d_2 into the equation $y = M_h\frac{q^2}{y}$, and then solving for E_q, we can obtain the appropriate energy as follows [19]:

$$E_q = \frac{1}{2}\hbar\omega[\sqrt{l(l+1)} + \sqrt{l(l+1)+q^2}]. \tag{6.3.1}$$

For $q = 2$, the ground state occurs; for $q = n\pi, (n = 1, 2, 3, \ldots)$, the excited states occur.

The known energy values are given as follows [20]:

$$E_{nl} = (2n + l + 3/2)\hbar\omega, (n = 0, 1, 2, 3, \ldots.). \tag{6.3.2}$$

Some values of the energies calculated according to Eqs. (6.3.1) and (6.3.2) are given in Table 2.

Table 2. Some energy values of the isotropic harmonic oscillator (unit $\hbar\omega$).

nl	E_{nl} [according to (6.3.2)]	E_{nl} [according to (6.3.1)]
1 s	3.500	1.571
1 p	4.500	2.430
1 d	5.500	3.217
2 s	5.500	3.142
2 p	6.500	3.927
2 d	7.500	4.597

6.3.2. Wave functions

$$G(r) = m_1 \int \sqrt{|U(r)|}dr = \sqrt{\frac{2m}{\hbar^2}} \int \sqrt{U(r)}dr = \sqrt{\frac{2m}{\hbar^2}} \int \sqrt{ar^2 + \frac{b}{r^2}}dr = Q(r),$$

$$Q(r) = m_1 \frac{1}{2} \left\{ \sqrt{ar^4+b} - \sqrt{b} \ln \left[2\frac{\sqrt{b}+\sqrt{ar^4+b}}{r^2} \right] \right\},$$

$$F(r) = A\cos[Kr]\, e^{iG(r)} = A\cos[Kr]\, e^{iQ(r)} = A\cos\left[\frac{q}{d}r\right] e^{iQ(r)},$$

$$F(r) = B\sin[Kr]\, e^{iG(r)} = B\sin[Kr]\, e^{iQ(r)} = B\sin\left[\frac{q}{d}r\right] e^{iQ(r)},$$

$$|A| = |B| = \sqrt{2/d} = \sqrt{2K/q}; \quad \Psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi) = \frac{F(r)}{r} Y(\theta,\phi).$$

The known radial wave functions are as follows [20]: $R_{nl}(\rho) = A\rho^{l+1} e^{-\frac{1}{2}\rho^2} L_n^{l+1/2}\left(\rho^2\right)$. Here, $\rho = \sqrt{m\omega/\hbar}r$ and L are the Laguerre polynomials.

6.4. Three dimensional isotropic harmonic oscillator with spin-orbit coupling (shell model in nuclear physics)

The simple potential energy of a three dimensional isotropic harmonic oscillator is given by $V_0(r) = \frac{1}{2}m\omega^2 r^2$. The spin$-$orbit interaction potential must be added to this simple potential. The spin$-$orbit interaction potential is given as follows:

$$V_{lsj}(r) = -\frac{\hbar^2}{2m^2c^2}\frac{1}{r}\frac{dV_0(r)}{dr}\vec{ls}; \left\{ \vec{l}.\vec{s} = \frac{1}{2}\left[j(j+1) - l(l+1) - s(s+1)\right] \right\}$$

Here l, s, and j are the orbital, spin, and total angular momentum quantum numbers of a particle, respectively. It is possible to find the derivation of this expression in any quantum mechanics textbook. If $V_{lsj}(r)$ is calculated, the following value is found:

$V_{lsj}(r) = -\frac{\hbar^2\omega^2}{4mc^2}\left[j(j+1) - l(l+1) - s(s+1)\right] = -C_{lsj.}$ If $s = 1/2$ (for fermions) is accepted, $V_{lsj}(r) = -\frac{\hbar^2\omega^2}{4mc^2}\left[j(j+1) - l(l+1) - \frac{3}{4}\right] = -C_{lsj}$ is obtained. Thus, the harmonic oscillator potential and effective potential with spin-orbit coupling becomes as follows:

$$V(r) = V_0(r) + V_{lsj}(r); \quad U(r) = ar^2 - C_{lsj} + b/r^2; \quad \left[a = \frac{1}{2}m\omega^2, b = \frac{\hbar^2}{2m}l(l+1) \right].$$

6.4.1. Energy

From this effective potential, the positive roots of the equation $E_q = M_h \frac{q^2}{y} = U(r)$ are as follows:

$$r_1 = \sqrt{\frac{(E_q + C_{lsj}) - \delta}{2a}}; \quad r_2 = \sqrt{\frac{(E_q + C_{lsj}) + \delta}{2a}}; \quad [\delta = \sqrt{(E_q + C_{lsj})^2 - 4ab}].$$

With these roots, $d = r_2 - r_1 = \sqrt{\frac{E_q+\delta}{2a}} - \sqrt{\frac{E_q-\delta}{2a}}; d_2 = y = d * d = d^2$. By substituting this value of d_2 into the equation $y = M_h \frac{q^2}{y}$, and then solving for E_q, we can obtain the appropriate energy as follows:

$$E_q = \frac{1}{2}\hbar\omega \left[\sqrt{l(l+1)} - C_j + \sqrt{\left(\sqrt{l(l+1)} - C_j\right)^2 + q^2}\right], [C_j = C_{lsj}/(\hbar\omega)] \tag{6.4.1}$$

For $q = 2$, the ground state occurs; for $q = n\pi, (n = 1, 2, 3, \ldots)$, the excited states occur.

The known energy values (with first order perturbation) are as follows [7,9,20]:

$$E_{nl} = (2n + l + 3/2)\hbar\omega - \frac{C_0}{4\hbar\omega}\left[j(j+1) - l(l+1) - s(s+1)\right]\hbar\omega. \tag{6.4.2}$$

Here, $n = 0, 1, 2, \ldots$, integer numbers and C_0 is a positive parameter. In Table 3, some energy values calculated according to Eqs. (6.4.1) and (6.4.2) are given, with the following values:

$$C_0 = 0.015\hbar\omega, s = 1/2 \text{ and } C_j = \frac{C_0}{4\hbar\omega}\left[j(j+1) - l(l+1) - 3/4\right]\hbar\omega.$$

Table 3. Some energy values of the isotropic harmonic oscillator with spin-orbit coupling.

States	According to (6.4.2) (unit $\hbar\omega$)	According to (6.4.1) (unit $\hbar\omega$)
$1d_{5/2}$	3.493	3.211
$1f_{5/2}$	4.515	4.083
$1f_{7/2}$	4.489	4.061
$1g_{7/2}$	5.519	4.986
$1g_{9/2}$	5.485	4.955

6.4.2. Wave functions

$$G(r) = m_1 \int \sqrt{|U(r)|}dr = \sqrt{\frac{2m}{\hbar^2}} \int \sqrt{U(r)}dr = Q(r)$$

$$Q(r) = \sqrt{\frac{2m}{\hbar^2}} \left\{ \frac{1}{2}\sqrt{ar^4 - C_{lsj}r^2 + b} - \frac{C_{lsj}}{4\sqrt{a}} \ln\left[2\sqrt{ar^4 - C_{lsj}r^2 + b} + \frac{2ar^2 - C_{lsj}}{\sqrt{a}}\right] - \right.$$

$$\left. \frac{\sqrt{b}}{2} \ln\left[2\sqrt{b(ar^4 - C_{lsj}r^2 + b)} - C_{lsj}r^2 + 2b\right] + \sqrt{b}\ln(r) \right\}$$

$$F(r) = A\cos[Kr]\, e^{iG(r)} = A\cos[Kr]\, e^{iQ(r)} = A\cos\left[\frac{q}{d}r\right] e^{iQ(r)}$$

$$F(r) = B\sin[Kr]\, e^{iG(r)} = B\sin[Kr]\, e^{iQ(r)} = \sin\left[\frac{q}{d}r\right] e^{iQ(r)}$$

$$|A| = |B| = \sqrt{2/d} = \sqrt{2K/q} = \sqrt{q/d}; \quad \Psi(r,\theta,\phi) = R(r)\, Y(\theta,\phi) = \frac{F(r)}{r} Y(\theta,\phi)$$

The known radial wave functions are [20]: $R_{nl}(\rho) = A\rho^{l+1} e^{-\frac{1}{2}\rho^2} {L_n}^{l+1/2}(\rho^2)$

Here $\rho = \sqrt{m\omega/\hbar}r$ and L is the Laguerre polynomial.

6.5. Three axial deformed harmonic oscillator potential (Nilsson model in the nuclear physics)

It was examined in detail in [36].

6.6. Periodic potential of arbitrary form

It was examined in detail in [35] with examples.

6.7. Trigonometric and Hyperbolic Pöschl–Teller potential wells

It was examined in detail in [34].

6.8. Saxon–Woods type central potential

Saxon–Woods type potential function is given as $V_1(r) = -\frac{V_0}{1+e^{[(r-R_0)/a]}}$; [a, V_0,$R_0 parameters$]. The spin-orbit interaction potential energy term can be added to this potential. The spin-orbit interaction potential energy term is given as follows: $V_{lsj}(r) = -\frac{\hbar^2}{4\mu^2c^2}\frac{1}{r}\frac{dV_1(r)}{dr}\left[j(j+1) - l(l+1) - s(s+1)\right]$, [$\mu is$ the reduced mass]. If this function is calculated, the following potential is found:

$$V_{lsj} = -C_{lsj}\frac{1}{r}\frac{e^{r/a}}{\left[1+e^{\left[\frac{r-R_0}{a}\right]}\right]^2}; \quad \{C_{lsj} = \frac{\hbar^2c^2V_0e^{-R_0/a}}{4\mu^2c^4a}\left[j(j+1) - l(l+1) - s(s+1)\right].$$

The centrifugal energy is given by $V_c(r) = \frac{\hbar^2 l(l+1)}{2\mu}\frac{1}{r^2}$. The Coulomb potential $V_{cp}(r)$ for charged particles must also be gotten if they exist. So, the effective potential can be obviously written as $U(r) = V_1(r) + V_{lsj} + V_c(r) + V_{cp}(r)$ or $U(r) = -\frac{V_0}{1+e^{\left[\frac{r-R_0}{a}\right]}} - C_{lsj}\frac{1}{r}\frac{e^{r/a}}{\left[1+e^{\left[\frac{r-R_0}{a}\right]}\right]^2} + \frac{\hbar^2 l(l+1)}{2\mu}\frac{1}{r^2} + V_{cp}(r)$.

The classical turning points of this effective potential cannot be found analytically and, therefore, it must be found numerically. The classical turning points are the roots of the equation $U(r) = -|E_q| = -M_h q^2/y$. Let these roots be $r_1(y)$ and $r_2(y)$. With these roots, first, $d^2 = [r_2(y) - r_1(y)]^2 = d_2(y)$ is found, and after solving the equation $d_2(y) = y$, the following is found: y and $E_q = M_h q^2/y$.

6.8.1. Numeric calculations

Let us calculate the energy values of $Cu(29,68)$. We have taken the potentials as follows:

$$V_{cp}(r) = (Z-1)\, e^2 \frac{3{R_{co}}^2 - r^2}{2{R_{co}}^3}, \ [Coulomb\ potential\ in\ the\ sphere]$$

$$V(r) = -\frac{V_0}{1+e^{(r-R_0)/a_0}},\ (Saxon-Woods\ potentials)$$

$$\alpha(L,S,J) = 0.5\left[J(J+1) - L(L+1) - S(S+1)\right]$$

$$V_{LSJ}(r) = -\frac{\hbar^2}{2\mu}\frac{1}{r}\alpha(L,S,J)\frac{V_{s0}}{a_{s0}}\frac{e^{(r-R_{s0})/a_{s0}}}{\left[1+e^{(r-R_{s0})/a_{s0}}\right]^2},\ (spin-orbit\ potential)$$

For protons, $U(r) = V(r) + V_{LSJ}(r) + V_c(r) + V_{cp}(r)$; for neutrons, $U(r) = V(r) + V_{LSJ}(r) + V_c(r)$. The parameters in these potentials have been calculated by the method of Volya [21,22] and their values are $a_{s0} = a_0 = 0.662$; $R_{co} = R_0 = 5.142885$; $V_0 = 47.655271$; $V_{so} = 28.422020$; $R_{so} = 4.726557$. The values of d are obtained by solving the following equation:

$$U(-d/2) + U(d/2) = -2\frac{\hbar^2}{2\mu}\frac{q^2}{d^2},$$

and the energy values have been calculated with the following formula:

$$E_q = -\frac{\hbar^2}{2\mu}\frac{q^2}{d^2}.$$

The results are seen in the Table 4.

Table 4. A few energy values of the Cu (29,68) nucleus with Saxon–Woods potential (unit MeV).

States	$E_q(MeV)(neutron)$	$E_q(MeV)(proton)$
1 s 1/2	−45.6256	−33.9536
1 p 3/2	−15.1995	−11.3653
1 p 1/2	−15.1987	−11.3644
1 d 5/2	−6.5021	−4.9028
1 d 3/2	−6.5011	−4.9015
2 s 1/2	−45.6061	−33.9536
1 f 7/2	−3.4830	−2.6484
2 p 3/2	−25.1702	−18.8825

6.9. Relativistic Dirac equation in a central potential

Consider a Dirac particle (spin is 1/2) of mass m captured in a central potential well, $V(r)$. With this potential, Dirac Hamiltonian can be written as follows [8,9]: $H_D = \vec{\alpha}\vec{p} + \beta m + V(r)$. Here, we have the following relativistic units: $\hbar = c = 1$, $\vec{p} = -i\vec{\nabla}$; $\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$, and β are hermitical 4-operators acting on the spin variables alone. Including the position vector $\vec{r}$; $\vec{\alpha} = d\vec{r}/dt(velocity)$. The Hamiltonian H_D and radial equation of Schrödinger can be brought to the form below [23]:

$$\frac{d^2F(r)}{dr^2} + [\alpha - U(r)]F(r) = 0;\quad \alpha = E^2 - m^2;\quad U(r) = U_{re}(r) + U_j(r) + U_L(r) \pm U_{rez}(r)$$

$$U_{re}(r) = 2EV(r) - V(r)^2;\ U_j(r) = (j+1/2)^2\Big/r^2;\ U_L(r) = L(L+1)/r^2;$$

$$U_{rez}(r) = \sqrt{(j+1/2)^2\Big/r^4 - V'(r)^2}.$$

For $\alpha > U(r)\, boundstate : k = i\sqrt{\alpha}; G(r) = \int \sqrt{U(r)}dr$. r_1 and r_2 are the roots of the following equation:

$$[\alpha - U(r)] = 0 \text{ or } \left(E^2 - m^2\right) - U(r) = 0 \text{ and } d = r_2 - r_1;\ (r_1 < r_2).$$

The bound state energies are given by the solution of the following equation:

$$Kd = \sqrt{|\alpha|}d = d\sqrt{|E^2 - m^2|} = d\sqrt{m^2 - E^2} = q, [m > |E|].$$

For $q = 2$, the ground state occurs; for $q = n\pi, (n = 1, 2, 3, \ldots)$, the excited states occur.

Now let us find the function $F(r)$ in bound states. In bound states, always $E > U(r)\, thatis, \alpha > U(r)$ Therefore:

$$K = \sqrt{|\alpha|} = \sqrt{|E^2 - m^2|} = \sqrt{m^2 - E^2} = K > 0,$$

$$G(r) = i\int \sqrt{U(r)}dr = i\int \sqrt{-|U(r)|}dr = Q(r),$$

$$F(r) = A\cos[Kr]\, e^{iG(r)} = A\cos[Kr]\, e^{iQ(r)} = A\cos\left[\frac{q}{d}r\right] e^{iQ(r)},$$

$$F(r) = B\sin[Kr]\, e^{iG(r)} = B\sin[Kr]\, e^{iQ(r)} = \sin\left[\frac{q}{d}r\right] e^{iQ(r)}.$$

The application to the atoms of hydrogen-like was given in detail in [23].

7. Transmission coefficient for an arbitrary form potential barrier

7.1. Determination of the wave functions

Let us consider the solution as follows:

$$F(r) = Ae^{kr \pm iG(r)} + Be^{-kr \mp iG(r)} \tag{64}$$

Here, **(a)** For the case where $E > U(r), k = im_1\sqrt{E}, G(r) = im_1 \int \sqrt{U(r)}dr$,

(b) For the case where $E < U(r), k = m_1\sqrt{E}, G(r) = m_1 \int \sqrt{U(r)}dr$.

Let us divide the potential into three domains, as seen in Figure 3. In region I, $E > U_1$; in region II, $E < U_2$; and in region III, $E > U_3$. Now, consider that a particle with total energy E comes from the left as in Figure 3 and hits the barrier at the point r_1.

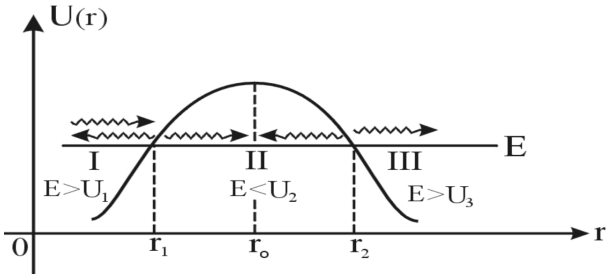

Figure 3. The unbounded state (potential barrier) and tunneling.

According to the function given in Eq. (64), the wave functions can be obtained for these three regions as follows:

$$F_1(r) = A_1 e^{iKr \pm Q_1(r)} + B_1 e^{-iKr \mp Q_1(r)},$$

$$F_2(r) = A_2 e^{Kr \mp iQ_2(r)} + B_2 e^{-Kr \pm iQ_2(r)}, \tag{65}$$

$$F_3(r) = A_3 e^{iKr \pm Q_3(r)}.$$

Here, $k = m_1\sqrt{E}$,$Q_p(r) = m_1 \int \sqrt{U_p(r)}dr, (p = 1, 2, 3)$.

In the calculations of the above functions, the fact that the waves travel both from left to right and right to left in region I and only from left to right in region III was taken into account. Since there is no wave coming from right to left in region III, the coefficient B_3 must be zero.

7.2. Calculation of the transmission coefficient T

To calculate the transmission coefficient T, the coefficients A_i and B_i in the functions given in Eq. (65) must be found. In order to find these coefficients, the following boundary conditions are used:

$$F_1(r_1) = F_2(r_1); F_1'(r_1) = F_2'(r_1); \quad F_2(r_2) = F_3(r_2); F_2'(r_2) = F_3'(r_2)$$

$$E = U_1(r_1) = U_2(r_1); E = U_2(r_2) = U_3(r_2); m_1\sqrt{E} = m_1\sqrt{U_p(r_1)} = m_1\sqrt{U_p(r_2)} = K;$$

$$m_1\sqrt{E} = m_1\sqrt{U_p(r_1)} = m_1\sqrt{U_p(r_2)} = K$$

$$Q_p(r) = m_1 \int \sqrt{U_p(r)}dr \rightarrow Q_p'(r) = m_1\sqrt{U_p(r)}; \quad Q_p'(r_1) = Q_p'(r_2) = K.$$

In the functions given in Eq. (65), there are five unknown coefficients. Four of them can be found in term of A_1. All of the coefficients have been calculated in this study, but here, only the coefficient A_3 is sufficient. According to the signs of the exponential terms in Eq. (65), two expressions for A_3 can be found.

For the lower part of signs:

$$A_3 = \frac{2exp\left[(1+i)Kr_1 + (1-i)Kr_2 - Q_1(r_1) + iQ_2(r_1) + iQ_2(r_2) + Q_3(r_2)\right]}{exp\left[2Kr_1 + 2iQ_2(r_1)\right] + exp\left[2Kr_2 + 2iQ_2(r_2)\right]} A_1.$$

For the upper part of signs:

$$A_3 = \frac{2exp\left[(1+i)Kr_1 + (1-i)Kr_2 + Q_1(r_1) + iQ_2(r_1) + iQ_2(r_2) - Q_3(r_2)\right]}{exp\left[2Kr_1 + 2iQ_2(r_2)\right] + exp\left[2Kr_2 + 2iQ_2(r_1)\right]} A_1.$$

The transmission coefficient is defined as follows:

$$T = \frac{A_3 {A_3}^*}{A_1 {A_1}^*} = \frac{2}{\cosh[2Kd] + \cos[2P]} \tag{66}$$

$[P = Q_2(r_2) - Q_2(r_1) = \sqrt{\frac{2m}{\hbar^2}} \int_{r_1}^{r_2} \sqrt{U_2(r)}dr\ ]$

$$Q_2(r) = m_1 \int \sqrt{U_2(r)}dr = \sqrt{\frac{2m}{\hbar^2}} \int \sqrt{U_2(r)}dr.$$

In the literature [6], the transmission coefficient T (or the barrier penetration probability) which is calculated by the WKB method is known as follows:

$$T = e^{-2g}, [g = \sqrt{\frac{2m}{\hbar^2}} \int_{r_1}^{r_2} \sqrt{U_2(r) - E} dr]. \tag{67}$$

In Eqs. (66) and (67), r_1 and r_2 are abscises of the points that the particle hits and leaves the potential barrier, respectively.

7.3. Application to cold emission

7.3.1. Calculation of transmission coefficient

Cold emission of electrons from a metal surface is the basis of an important device known as scanning tunneling microscope (STM). An STM consists of a very sharp conducting probe which is scanned over the surface of a metal (or any other solid conducting medium). A large voltage difference is applied between the probe and the surface. The surface electric field strength immediately below the probe tip is proportional to the applied potential difference, and inversely proportional to the spacing between the tip and the surface. Electrons tunneling between the surface and the probe tip cause a weak electric current. The magnitude of this current is proportional to the tunneling probability T. It follows that the current is an extremely sensitive function of the surface electric field strength and, hence, of the spacing between the tip and the surface (assuming that the potential difference is held constant). An STM can thus be used to construct a very accurate contour map of the surface under investigation. In fact, STMs are capable of achieving sufficient resolution to image individual atoms.

Suppose that a cold metal surface is subject to a large uniform external electric field of strength $\in$, which is directed such that it accelerates electrons away from the surface. The electrons just below the surface of a metal can be regarded as being in a potential well of depth W, where W is called the work function of the surface. Adopting a simple one dimensional treatment of the problem let the metal lie at $x < 0$, and the surface at $x = 0$. The applied electric field is shielded from the interior of the metal. Hence, the energy E, for example, of an electron just below the surface is unaffected by the field. In the absence of the electric field, the potential barrier just above the surface is simply $U(x) - E = W$. The electric field modifies this to $U(x) - E = W - e \in x$. The potential barrier is sketched in Figure 4. It can be seen in Figure 4 that an electron just below the surface of the metal is confined by a triangular potential barrier which extends from $x = x_1 to x_2$, where $x_1 = 0$ and $x_2 = W/(e \in)$. In Eq. (66), if it is put that:

$$d = x_2 - x_1 = W/(e \in); m_1 = \sqrt{\frac{2m}{\hbar^2}}; \quad k = \sqrt{\frac{2m}{\hbar^2} E} = m_1 \sqrt{E} = m_1 \sqrt{-|E|} = i m_1 \sqrt{W}$$

$$Q_2(x) = m_1 \int \sqrt{-e \in x} = i \frac{2\sqrt{2me \in}}{3\hbar} x^{3/2} = i \sqrt{\frac{8me \in}{9\hbar^2}} x^{3/2}.$$

With these values, the following transmission coefficient is obtained:

$$T = T_{new} = \frac{2}{\cosh[2Kd] + \cos[2Q_2(x_2)]} = \frac{2}{cos\left[\frac{2\sqrt{2m}}{\hbar e \in} W^{3/2}\right] + cosh\left[\frac{4\sqrt{2m}}{3\hbar e \in} W^{3/2}\right]}. \tag{68}$$

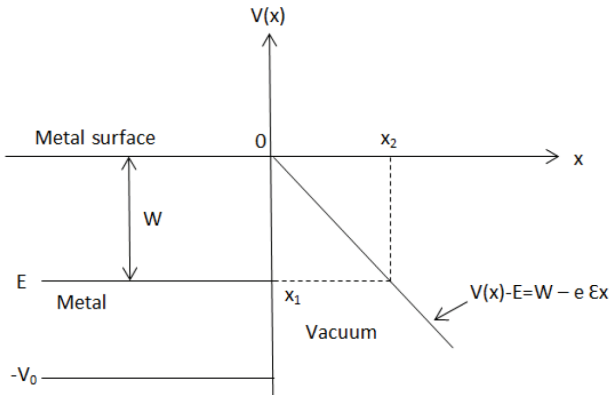

Figure 4. The potential barrier for an electron in a metal surface subject to an external electric field.

Here, in calculation of Eq. (68), the following is used:

$$\cosh(iy) = \cos(y) \text{ and } \cos(iy) = \cosh(y).$$

Using the WKB approximation [6], the probability of such an electron tunneling through the barrier and consequently being emitted from the surface is calculated as follows:

$$T_{wkb} = exp\left[-2m_1 \int_{x_1}^{x_2} \sqrt{U(x) - E}\right] dx = exp\left[-2m_1 \int_{x_1}^{x_2} \sqrt{W - e \in x}\right] dx$$

$$T_{wkb} = exp\left[-\frac{4\sqrt{2m}}{3\hbar e \in} W^{3/2}\right]. \tag{69}$$

The above result given in Eq. (69) is known as the Fowler–Northeim formula. This formula is the result of the WKB approximation. The formula given in Eq. (68) is an exact formula because there is no approximation.

7.3.2. Numerical calculations and comparison of Eqs. (68) and (69)

The barrier penetration probabilities or the transmission coefficients T have been calculated from Eqs. (68) and (69). In the calculations, the electron mass, $mc^2 = 0.511003 MeV$; the electron charge, $e = 1.19999\,(MeV.fm)^{1/2}$ have been taken. The obtained values for the different metals are seen in Table 5. In Table 5, it can be seen that the new method is more appropriate than the classical WKB method. In the calculations of the current-voltage characteristics of a diode in semiconductor physics, it is expected to have better results $[\hbar c = 197.329 MeV.fm]$.

7.4. Application to alpha decay in atom nuclei and calculation of half-life

7.4.1. Calculation of half-life formula

An α-particle is the nucleus of a helium atom. It consists of two protons and two neutrons. In the process ofα-decay of nuclei, an α-particle is assumed to move in a spherical region determined by the daughter nucleus. The central feature of this one-body model is that the α-particle is preformed inside the parent nucleus. The theory does not prove that the α-particle is preformed but it proves that it behaves as if it is [24]. Figure 5 shows a plot, suitable for the purposes of the theory, of the potential energy between the α-particle and

Table 5. Comparison of the transmission coefficients values calculated with the classical and the new method.

Metals	Work function W (eV) [4]	Transmission coefficient T_{new} from equation (68) Electric field $\in (V/cm)$			Transmission coefficient T_{wkb} from equation (69) Electric field $\in (V/cm)$		
		5×10^6	5×10^7	1×10^7	5×10^6	5×10^7	1×10^7
Na	2.46	5.12448×10^{-23}	0.0206	1.43171×10^{-11}	1.28112×10^{-23}	0.0051	3.57928×10^{-12}
Al	4.08	5.07471×10^{-49}	0.000052	1.42474×10^{-24}	1.26868×10^{-49}	0.000013	3.56185×10^{-25}
Cu	4.70	1.40147×10^{-60}	3.60174×10^{-6}	2.36768×10^{-30}	3.50368×10^{-61}	9.00435×10^{-7}	5.91919×10^{-31}
Zn	4.31	3.25862×10^{-53}	0.000020	1.14169×10^{-26}	8.14656×10^{-54}	4.91018×10^{-6}	2.85422×10^{-27}
Ag	4.73	3.68836×10^{-61}	3.15165×10^{-6}	1.21464×10^{-30}	9.22090×10^{-62}	7.87911×10^{-7}	3.03659×10^{-31}
Pt	6.35	4.59212×10^{-95}	1.28249×10^{-9}	1.35530×10^{-47}	1.14803×10^{-95}	3.20624×10^{-10}	3.38826×10^{-48}
Pb	4.14	4.19617×10^{-50}	0.000040	4.09691×10^{-25}	1.04904×10^{-50}	0.000010	1.02423×10^{-25}
Fe	4.50	9.20656×10^{-57}	8.67486×10^{-6}	1.91902×10^{-28}	2.30164×10^{-57}	2.16872×10^{-6}	4.79754×10^{-29}

the residual nucleus for various distances between their centers. The horizontal line E_α is the disintegration energy. There are three regions of interest. In the spherical region $r < r_1$ we are inside the nucleus and speak of a potential well with a depth of $-U_0$, where U_0 is taken as a positive number. Classically, the α-particle can move in this region with a kinetic energy $E_\alpha + U_0$ but cannot escape from it. The region $r_1 < r < r_2$ forms a potential barrier because here the potential energy is more than the total available energy E_α. The region $r > r_2$ is a classically permitted region outside the barrier. From the classical point of view, an α-particle in the spherical potential well would reverse its motion every time it tried to pass beyond $r = r_1$ of tunneling through such a barrier. A consistent model for this process assumes that the α-particle is bounded to the nucleus by a spherical potential well $V_1(r)$ or a spherical effective potential well $U_1(r)$ and that the α-particle is repelled from the residual nucleus by the central Coulomb potential barrier $V_2(r)$ or the effective central Coulomb potential barrier $U_2(r)$. The original radioactive nucleus has the charge Ze and the α-particle has the charge $2e$. So the Coulomb potential barrier is as follows:

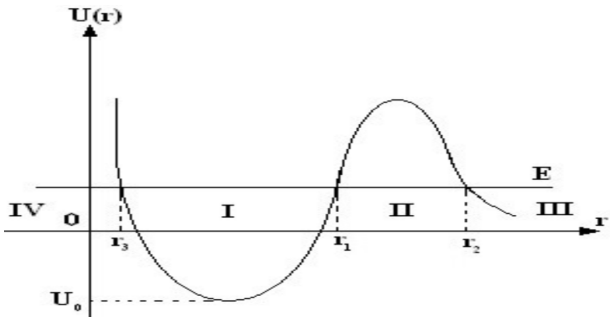

Figure 5. Effective potential function for the α-decay process of the nuclei.

$$V_2(r) = \frac{2(Z-2)e^2}{r} = \frac{c}{r}, [c = 2(Z-2)e^2].$$

Thus, the corresponding effective potential function is obtained as $U_2(r) = \frac{c}{r} + \frac{b}{r^2}, [b = \frac{\hbar^2}{2m} l(l+1)$ This effective potential is depicted in Figure 5. There are three domains in this effective potential.

According to the one-body theory, the disintegration constant λ of an alpha emitter is given by $\lambda = fT$, [Ttransmission coefficient]. Here, $f$is the frequency with which the α-particle presents itself at the barrier and $T$is the probability of transmission through the barrier. The quantity f is roughly of the order of $\frac{v}{2(r_1 - r_3)}$,

where v is the relative speed of the alpha particle inside the nucleus. $t_{1/2} = 0.693/\lambda$ is used for the calculations of nuclear half-life. The relative speed of the alpha particle can be found from its kinetic energy that is equal to the difference between the disintegration energy of the alpha particle, E_α, and the ground state energy of the nucleus, E_0, namely:

$$K.E. = \frac{1}{2}mv^2 = [E_\alpha - (-|E_0|)] = E_\alpha + |E_0|. \tag{70}$$

From Eq. (70), the following is obtained:

$$v = \sqrt{\frac{2(E_\alpha + |E_0|)}{m}} \text{ and } \frac{1}{v} = \sqrt{\frac{m}{2(E_\alpha + |E_0|)}}.$$

If the values of $1/v$ and $\lambda = fT$ are substituted into $t_{1/2} = 0.693/\lambda$, then the following formula is obtained:

$$t_{1/2} = \frac{0.693}{\lambda} = \frac{0.693}{fT} = 0.693\frac{2(r_1 - r_3)}{T}\frac{1}{v} = 0.693\frac{2(r_1 - r_3)}{T}\sqrt{\frac{m}{2(E_\alpha + |E_0|)}}.$$

To simplify the numerical calculations, this formula can be rewritten in following form:

$$t_{1/2} = 0.693\frac{2(r_1 - r_3)}{c}\sqrt{\frac{mc^2}{2(E_\alpha + |E_0|)}}\frac{1}{T}. \tag{71}$$

Here, $m = \frac{m_n m_\alpha}{m_n + m_\alpha}$ is the reduced mass. (m_n and m_α are the mass of the nucleus and α-particle).

7.4.2. Determination of the potential functions

In our numerical calculations, we have taken the harmonic oscillator type central potential as seen in Figure 6. The central potential parts $V_1(r)$,$V_2(r)$,$V_3(r)$ and the central effective potential parts $U_1(r)$,$U_2(r)$,$U_3(r)$ are as follows:

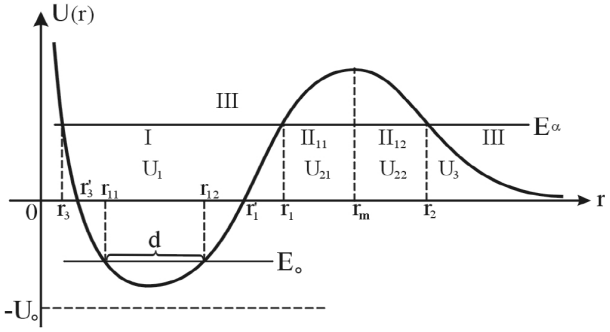

Figure 6. Central harmonic oscillator type effective potential.

$$V_1(r) = -U_0 + ar^2;\ U_1(r) = -U_0 + ar^2 + \frac{b}{r^2};\ [r_3 \le r \le r_1] \quad \text{[I region]}$$

$$V_{21}(r) = ar^2;\ U_{21}(r) = ar^2 + \frac{b}{r^2};\ [r_1 \le r \le r_m] \quad (\text{II}_1 \text{ region})$$

$$V_{22}(r) = \frac{c}{r};\ U_{22}(r) = \frac{c}{r} + \frac{b}{r^2};\ [r_m \le r \le r_2] \quad (\text{II}_2 \text{ region})$$

$$V_3(r) = \frac{c}{r};\ U_3(r) = \frac{c}{r} + \frac{b}{r^2};\ [r_2 \le r \le \infty] \quad \text{(III region)}$$

Here, $R = R_0\left[(A-4)^{1/3}+4^{1/3}\right]$ is the total rayon of the nucleus and alpha particle. From the solution of the equation $E_\alpha = U_1(r) = -U_0+ar^2+\frac{b}{r^2}$, we can obtain $r_1 = \sqrt{\frac{(E_\alpha+U_0)+\sqrt{(E_\alpha+U_0)^2-4ab}}{2a}}; r_3 = \sqrt{\frac{(E_\alpha+U_0)-\sqrt{(E_\alpha+U_0)^2-4ab}}{2a}}$. From the solution of the equation $E_\alpha = U_{22}(r) = \frac{c}{r}+\frac{b}{r^2}$, we can obtain $r_2 = \frac{c+\sqrt{c^2+4bE_\alpha}}{2E_\alpha}$. From the solution of the equation $E_0 = U_1(r) = -U_0+ar^2+\frac{b}{r^2}$, we can obtain $r_{11} = \sqrt{\frac{(E_0+U_0)-\sqrt{(E_0+U_0)^2-4ab}}{2a}}$; $r_{12} = \sqrt{\frac{(E_0+U_0)+\sqrt{(E_0+U_0)^2-4ab}}{2a}}$; $d_0 = r_{12}-r_{11}; d = r_2-r_1$. From the solution of the equation $E_0 = -\frac{2\hbar^2}{m{d_0}^2}$, we obtain the ground state energy as follows:

$$E_0 = \sqrt{ab}-\frac{1}{2}\left[U_0+\sqrt{4ab-\frac{8a\hbar^2}{m}-4\sqrt{ab}U_0+{U_0}^2}\right].$$

With these grandeurs the coefficients of transmission are written as follow:

$$T_{wkb} = e^{-2P_{wkb}}; P_{wkb} = \sqrt{\frac{2m}{\hbar^2}}\left(\int_{r_1}^{r_m}\sqrt{U_{21}(r)-E_\alpha}dr+\int_{r_m}^{r_2}\sqrt{U_{22}(r)-E_\alpha}dr\right) \tag{72}$$

$$T_{new} = \frac{2}{\cosh[2Kd]+\cos[2\{Q_2(r_2)-Q_2(r_1)\}]} = \frac{2}{\cosh[2Kd]+\cos[2P_{new}]}, \tag{73}$$

$$P_{new} = \sqrt{\frac{2m}{\hbar^2}}\left(\int_{r_1}^{r_m}\sqrt{U_{21}(r)}dr+\int_{r_m}^{r_2}\sqrt{U_{22}(r)}dr\right), \tag{74}$$

$$t_{1/2}^{wkb} = 0.693\frac{2(r_1-r_3)}{c}\sqrt{\frac{mc^2}{2(E_\alpha-E_0)}}\frac{1}{T_{wkb}}, \tag{75}$$

$$t_{1/2}^{new} = 0.693\frac{2(r_1-r_3)}{c}\sqrt{\frac{mc^2}{2(E_\alpha-E_0)}}\frac{1}{T_{new}}. \tag{76}$$

The parameters a and r_m in potentials can be calculated. To calculate the parameter a, the following equation is used: $U_1(r) = -U_0+ar^2+\frac{b}{r^2} = 0$. The roots of this equation are found as follows:

$$r_1' = \sqrt{\frac{U_0+\sqrt{{U_0}^2-4ab}}{2a}}; r_3' = \sqrt{\frac{U_0-\sqrt{{U_0}^2-4ab}}{2a}}.$$

As it can be seen in Figure 6, the r_1' can be taken as sum of the radii of the nucleus and the alpha particle. That is, since the radius of the alpha particle is $R_\alpha = R_0 4^{1/3}$ and the radius of the nucleus $R_N = R_0(A-4)^{1/3}$, the total radius is $R_c = R_\alpha+R_N = R_0\left[4^{1/3}+(A-4)^{1/3}\right]$.

Thus it can be taken as follows: $\left(r_1{}'\right)^2 = {R_c}^2 = \left[U_0+\sqrt{{U_0}^2-4ab}\right]\Big/(2a)$. From this equation, the value of a is obtained as $a = \frac{U_0}{{R_c}^2}-\frac{b}{{R_c}^4} = \frac{U_0{R_c}^2-b}{{R_c}^4}$. To calculate the parameter r_m the equation $U_1(r) = U_{22}(r)$ is used and the resolution of this equation gives: $r_m = \frac{2\sqrt[3]{3}aU_0+\sqrt[3]{2}X}{6^{2/3}aX}$; $X = \left[9a^2c+\sqrt{3a^3\left(27ac^2-4{U_0}^3\right)}\right]^{2/3}$. Therefore, the potential depends only on R_0 and U_0 parameters, which simplifies the calculations.

7.4.3. Numerical calculations

To calculate the transmission coefficient T (or barrier penetration probability), Eqs. (72) and (73) have been used and to calculate the half-life values, Eqs. (75) and (76) have been used. Calculations were made for five nuclei and the results are visible in Tables 6 and 7. In these tables, the half-life values calculated from the new method and WKB method are compared with the experimental results. For the R_0 and U_0 parameters, first, R_0= $1.25fm$ and U_0= $40MeV$ values have been taken (Table 6) and it was seen that $t_{1/2}^{new}$ values had better results than $t_{1/2}^{wkb}$ values. Later, R_0 values in the interval $1.10-1.60fm$ by step 0.01 and U_0 values in the interval 30-50 MeV by step 1 have been changed and the most consistent values with the experimental results have been taken (Table 7). In all the tables, ${I_i}^{\pi_i}$ and ${I_f}^{\pi_f}$ are the initial and final state spins (parity) of the nucleus, respectively. l_α is the angular momentum of the alpha particle. From these tables, it can be seen that the new method is more appropriate than the classical method (WKB). Table 7 shows that the new method gives very good results with the experiment [25–32]. In these tables "exp" is the experimental value of the half-life. $y: year, d: day$.

Table 6. Comparison of the half-life values of the classical method with the new method for $R_0 = 1.25fm$ and U_0= $40MeV$ in the harmonic oscilator potential model.

$\mathbf{{}_{z}^{A}X}$	$\mathbf{E_\alpha(MeV)}$	$\mathbf{{I_i}^{\pi_i}}$	$\mathbf{{I_f}^{\pi_f}}$	$\mathbf{l_\alpha}$	$\mathbf{t_{1/2}^{exp}}$	$\mathbf{t_{1/2}^{wkb}}$ (75)	$\mathbf{t_{1/2}^{new}}$ (76)	$\mathbf{t_{1/2}^{wkb}/t_{1/2}^{exp}}$	$\mathbf{t_{1/2}^{new}/t_{1/2}^{exp}}$
${}_{84}{}^{208}Po$	5.215	0^+	0^+	0	2.898 y	0.0463 y	5.6950 y	0.16	1.97
${}_{84}{}^{210}Po$	5.407	0^+	0^+	0	138.28 d	1.2767 d	267.865 d	0.009	1.94
${}_{97}{}^{247}Bk$	5.889	$\frac{3}{2}^-$	$\frac{5}{2}^-$	2	1380 y	16.6389 y	2521.68 y	0.012	1.83
${}_{84}{}^{209}Po$	4.979	$\frac{1}{2}^-$	$\frac{5}{2}^-$	2	102 y	3.6377 y	96.3285 y	0.036	0.94
${}_{83}{}^{237}Np$	4.959	$\frac{5}{2}^+$	$\frac{3}{2}^-$	1	2.14 $\times 10^6$ y	76826.3 y	1.44 $\times 10^6$ y	0.036	0.67

Table 7. Comparison of the experimental half-life values and the results calculated using the new methods with the harmonic oscillator type well potential, calculated values a and r_m parameters for the parameters r_0 and U_0 which are very well matched with the experimental results.

$\mathbf{{}_{z}^{A}X}$	$\mathbf{E_\alpha(MeV)}$	$\mathbf{{I_i}^{\pi_i}}$	$\mathbf{{I_f}^{\pi_f}}$	$\mathbf{l_\alpha}$	$\mathbf{r_0(fm)}$	$\mathbf{U_0(MeV)}$	$\mathbf{t_{1/2}^{exp}}$	$\mathbf{t_{1/2}^{new}}$	$\mathbf{t_{1/2}^{new}/t_{1/2}^{exp}}$
${}_{84}{}^{208}Po$	5.215	0^+	0^+	0	1.30	45	2.898 y	2.899 y	1.0004
${}_{84}{}^{210}Po$	5.407	0^+	0^+	0	1.30	47	138.40 d	138.41 d	1.0001
${}_{97}{}^{247}Bk$	4.979	$\frac{1}{2}^-$	$\frac{5}{2}^-$	2	1.24	35	102 y	102 y	1.0010
${}_{84}{}^{209}Po$	4.959	$\frac{5}{2}^+$	$\frac{3}{2}^-$	1	1.23	49	2.14 $\times$ 10^6 y	2.15 $\times$ 10^6 y	1.0028
${}_{83}{}^{237}Np$	5.889	$\frac{3}{2}^-$	$\frac{5}{2}^-$	2	1.29	43	1380 y	1375 y	0.9964

Here, the general transmission coefficient formula for a potential barrier with an arbitrary form has been easily calculated without making any approximation. In this calculation, a new method that we developed for the solution of the radial SE has been used. The transmission coefficient obtained from the new method is given by the formula in Eq. (73). In this formula, it could be difficult to calculate the integral $\int \sqrt{U_2(r)}dr$. If these calculations cannot be made analytically, they should then be performed by numerical methods.

In the application of the general transmission coefficient formula to the α-decay, three dimensional harmonic oscillator potential well has been used. The results are given in the tables along with the experimental values. The tables also contain the "ratio" column for comparison. It can be seen in Table 7 that the ratios for the most of the nuclei are close to 1 (one). The deviations from 1 are within the experimental error. Hence, it is said that the results obtained using the new method are more realistic. In the WKB method, the wave function

is sinusoidal inside and outside the potential barrier. However, it is not a sinusoidal but an exponential function while entering into the potential barrier. So, the entering wave into the potential barrier is not sinusoidal and after the potential barrier, it becomes sinusoidal again. But in the new method, the wave function includes a sinusoidal multiplier inside the potential barrier, but the sinusoidal multipliers are different inside and outside the potential barrier. Thus, the wave function has different phases inside and outside the potential barrier, but it advances everywhere as sinusoidal functions. It can also be said that it is more accurate and realistic. Besides, the WKB method gives an approximate wave function. In the new method, the wave function is exact because there is no approximation. That is why the theoretical calculated half-life values match better with the experimental values. From these, we conclude that the transmission coefficient given in Eq. (73) is more correct and realistic. By using the new transmission coefficient and half-life formulas, the half-life values of nuclei can easily be calculated. The general transmission coefficient formula can be used for the other tunneling phenomenon such as the cold emission from the metals.

8. Scattering theory

8.1. Calculation of the scattering amplitudes

Let us consider a spherical wave progressing at the direction of $0z$ axis from left to right, and arriving to a central potential field, sitting at the origin of the $0xyz$ coordinate system. When we consider scattering, we shall assume that the interaction between the scattering particle and the scatter can be represented by an effective central potential energy function $U(r)$, where r is the relative radial variable. The effective potential $U(r)$ can include attractive and repulsive parts. Such a central potential is schematically represented in Figure 7. The total energy of the incoming particle beam is E and the incoming particle beam can be represented by the spherical wave. This progressive spherical wave progress from right to left and arrives to the point $r = r_1$ in Figure 7. We divide the potential region into four zones and examine the motion of the particle beam in these four zones.

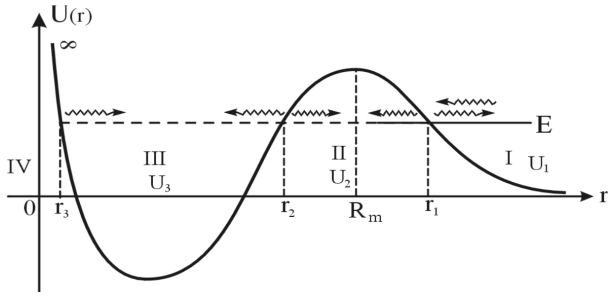

Figure 7. General schematic representation of scattering by central potential.

Zone I is the region before the effective potential from where free particle comes; zone $II, III,$ and IV are the effective potential regions where the particle beam is affected. These regions may include attractive and repulsive potential segments. Zone IV is the region where the particle beam is not able to penetrate because the potential is infinite. So, the wave function is zero in this region. These are presented with four regions along with the central potentials in Figure 7. We assume that $r = r_1$ and $r = r_2$ at the interface between zone I and II, and zone II and III, respectively. The effective potential segments in the zones are represented as $U_1(r)$, $U_2(r)$, and $U_3(r)$ according to the zone numbers. The effective potential $U_4(r)$ can be assumed infinite since the particle does not enter this region. The central potential can be taken as zero at much far from zone I so that the particle is free in that region and the effective potential is composed of only the centrifugal

term due to the incoming particle angular momentum or spin. The Coulomb interaction potential should also be added to $U_1(r)$ if it is available. The total energy of the incoming particle and the centrifugal term are always positive and the latter is less than the former. According to the functions given in Eqs. (52a)–(52f), the following functions are determined for the zones that are taken into account:

In zone $I: E>0, U_1(r)>0$ and $E>U_1(r); k=im_1\sqrt{E}=iK; (K=m_1\sqrt{E})$;

$$G_1(r)=im_1\int\sqrt{U_1(r)}dr=iQ_1(r);\quad \left[m_1=\sqrt{\frac{2m}{\hbar^2}}, Q_1(r)=m_1\int\sqrt{U_1(r)}dr\right].$$

In zone $II: E>0, U_2(r)>0$ and $E<U_2(r), k=m_1\sqrt{E}=K$

$$G_2(r)=m_1\int\sqrt{U_2(r)}dr=Q_2(r); \left[k=m_1\sqrt{E}, m_1=\sqrt{\frac{2m}{\hbar^2}}\right].$$

In zone III, the conditions $E<0$ (This zone corresponds to the bound state. So, the negative energy should be taken.), $U_3(r)<0$ and $E>U_3(r)$ leads to the expressions below:

$$k=im_1\sqrt{E}=im_1\sqrt{-|E|}=\pm K, \left[K=m_1\sqrt{|E|}\right],$$

$$G_3(r)=im_1\int\sqrt{U_3(r)}dr=im_1\int\sqrt{-|U_3(r)|}dr=\pm Q_3(r).$$

The wave function in zone I should vanish at large distance. Under these circumstances, the radial wave functions in the three zones can be put in the forms below with regard to the general functions given in Eqs. (52a)–(52f):

$$F_1(r)=A_1e^{iKr-Q_1(r)}+e^{-iKr-Q_0(r)};\quad [B_1=1, Q_1(r)>0, Q_0(r)>0], \tag{77a}$$

$$F_2(r)=A_2e^{Kr\pm iQ_2(r)}+B_2e^{-Kr\mp iQ_2(r)}, \tag{77b}$$

$$F_3(r)=A_3e^{-Kr\pm iQ_3(r)}+B_3e^{Kr\mp iQ_3(r)}. \tag{77c}$$

The wave function in zone IV vanishes since the effective potential in this region is infinite. Here, $Q_0(r)$ is the function resulting from the angular momentum of the incoming particle. This function $Q_0(r)$ can also be taken as zero because it does not have any contribution to the calculation of the scattering cross-section as it will be seen soon.

The potential in zone III can also be complex in some cases (usually called the optical potential). If $U_3(r)$ is the optical potential, it can be written as follows:

$$U_3(r)=|U_3(r)|e^{i\emptyset}=U_{31}(r)+iU_{32}(r)=\sqrt{{U_{31}}^2(r)+{U_{32}}^2(r)}e^{i\emptyset}. \tag{78a}$$

Here, $\tan(\emptyset)=\frac{U_{32}(r)}{U_{31}(r)}, \emptyset=arctan\left[\frac{U_{32}(r)}{U_{31}(r)}\right]$; $k=im_1\sqrt{E}=im_1\sqrt{-|E|}=K, \left[k=m_1\sqrt{|E|}\right]$;
$G_3(r)=im_1\int\sqrt{U_3(r)}dr=im_1\int\sqrt{-|U_3(r)|}dr=\pm Q_3(r)$

$$Q_3(r)=m_1\int\sqrt{|U_3(r)|}dr=m_1\int\sqrt{\sqrt{{U_{31}}^2(r)+{U_{32}}^2(r)}}dr=m_1\int\sqrt[4]{{U_{31}}^2(r)+{U_{32}}^2(r)}dr. \tag{78b}$$

In Eqs. (77a)–(78b), the functions $Q_p(r)$ can also be written briefly as follows:

$$Q_p(r) = m_1 \int \sqrt{|U_p(r)|}dr, [p = 0, 1, 2, 3].$$

The terms containing A_1 and B_1 in the functions in Eqs. (77a)–(77c) give the outgoing and incoming waves, respectively. We assume that the amplitude of the incoming wave at the boundary of zone I and II is constant. The second (77b) and the third (77c) functions represent the states of the wave in the effective region of the potential. Applying the continuity conditions on $F_p(r_j)$ and $F_p^{'}(r_j)$, $[p, j = 1, 2, 3]$ functions, the coefficients A_i and B_i in Eqs. (77a)–(77c) can be determined. These conditions at the boundary points of the three zones can be written in the following form:

$$F_1(r_1) = F_2(r_1); \; F_1^{'}(r_1) = F_2^{'}(r_1); \; F_2(r_2) = F_3(r_2); \; F_2^{'}(r_2) = F_3^{'}(r_2) \tag{79}$$

$$F_3(r_3) + F_3^{'}(r_3)/K = 0, \; Q_p^{'}(r_j) = K, \quad (p, j = 1, 2, 3).$$

The coefficients A_1,A_2,B_2,A_3,B_3 in the functions given in Eqs. (77a)–(77c) can be found by solving five linear equations, which can be obtained by using the conditions given in Eq. (79) for each of the functions given in Eqs. (77a)–(77c).

The essential coefficient for the scattering cross-section is A_1, as described below. Therefore, there is no need to give other coefficients here. The A_1 coefficient, which is obtained from four equations, is computed by taking into account the lower and upper signs in the exponential expressions in Eq. (79) as follows:

8.1.1. The A_1 coefficient found using the lower signs:

$$A_1 = \frac{1}{2}(1+i)\, e^{-2iKr_1 - Q_0(r_1) + Q_1(r_1)} \left\{ -1 + (2-i)\, e^{[2K(r_3 - r_1) + 2i[Q_2(r_1) - Q_2(r_2) + Q_3(r_2) - Q_3(r_3)]} \right\} \tag{80a}$$

8.1.2. The A_1 coefficient found using the upper signs:

$$A_1 = \frac{(1+i)\, e^{2(1-i)Kr_1 - Q_0(r_1) + Q_1(r_1) + 2i[Q_2(r_1) + Q_3(r_2)]}}{(2+i)\, e^{[2Kr_3 + 2i[Q_2(r_2) + Q_3(r_3)]} - e^{[2Kr_1 + 2i[Q_2(r_1) + Q_3(r_2)]}}. \tag{80b}$$

The terms with A_1 in the functions given in Eqs. (77a)–(77c) representing the outgoing wave from the center of potential, includes both the scattered by the potential and the incoming wave. Therefore, we must subtract away the latter to find the amplitude of only the scattered wave. Thus, we obtain the scattering amplitude and radial wave function representing the scattered wave as:

$$C_s(r_1) = A_1 e^{-Q_1(r_1)} - e^{-Q_0(r_1)}, (scattering amplitude) \tag{81a}$$

$$R_s(r) = \frac{F_s(r)}{r} = C_s(r_1)\frac{e^{iKr}}{r} = \left[A_1 e^{-Q_1(r_1)} - e^{-Q_0(r_1)}\right]\frac{e^{iKr}}{r}. \tag{81b}$$

Eqs. (81a) and (81b) represent the elastic scattering wave by the potential.

8.2. Calculation of particle currents

8.2.1. Calculation of the scattered particle current

Using Eq. (81b) and the equation $J_s(r) = \frac{\hbar}{2mi}\left[{R_s}^*(r)\frac{dR_s(r)}{dr} - R_s(r)\frac{d{R_s}^*(r)}{dr}\right]$, the current of scattered particles per unit area at $r = r_1$ point can be found as:

$$J_s(r_1) = \frac{1}{{r_1}^2}\frac{\hbar K}{m}|C_s(r_1)|^2. \tag{82}$$

8.2.2. Calculation of the incoming particle current

The radial wave function of the incoming beam, passing through zone I and toward zone II, can be written as $R_g(r) = \frac{F_g(r)}{r} = \frac{1}{r}e^{-iKr-Q_0(r_1)} = C_g(r_1)\frac{e^{-iKr}}{r}, \left[C_g(r_1) = e^{-Q_0(r_1)}\right]$. Here, $C_g(r_1) = e^{-Q_0(r_1)}$ represents the amplitude of the incoming wave. The incident current per unite area at $r = r_1$ point can be obtained in the way that is applied to the current equation:

$$J_g(r_1) = \frac{1}{{r_1}^2}\frac{\hbar K}{m}|C_g(r_1)|^2 = \frac{1}{{r_1}^2}\frac{\hbar K}{m}e^{-2Q_0(r_1)}. \tag{83}$$

8.3. Calculations of scattering cross-sections

8.3.1. Calculation of elastic scattering differential cross-section

The probability per unit differential surface of a sphere of radius r_1 that an incident particle is scattered into the differential surface area on the sphere of radius $r_1, dS = {r_1}^2 d\Omega, [d\Omega = \sin(\theta)\, d\theta d\phi]$ is expressed as the ratio of the scattered current to the incident current, that is:

$$\frac{d\sigma_s}{dS} = \frac{d\sigma_s}{{r_1}^2 d\Omega} = \frac{J_s(r_1)}{J_g(r_1)} \rightarrow \frac{d\sigma_s}{d\Omega} = \frac{J_s(r_1)}{J_g(r_1)}{r_1}^2. \tag{84}$$

The differential elastic cross-section can be expressed in a simple form by putting Eqs. (82) and (83) into Eq. (84):

$$\frac{d\sigma_s}{d\Omega} = \frac{C_s(r_1)\,{C_s}^*(r_1)}{C_g(r_1)\,{C_g}^*(r_1)}{r_1}^2 = \frac{|C_s(r_1)|^2}{|C_g(r_1)|^2}{r_1}^2. \tag{85}$$

Since the scattering is azimuthally symmetrical, the angle ϕ can be integrated out so that the expression given in Eqs. (84) and (85) can be written as follows:

$$\frac{d\sigma_s}{d\theta} = 2\pi\frac{J_s(r_1)}{J_g(r_1)}{r_1}^2\sin(\theta) = 2\pi\frac{|C_s(r_1)|^2}{|C_g(r_1)|^2}{r_1}^2\sin(\theta). \tag{86}$$

The expressions given in Eq. (86) show the elastic scattering differential cross-sections in the angle $d\theta$ which is usually measured experimentally.

8.3.2. Calculation of differential inelastic or reaction (no-elastic) cross-section

Differential reaction (capture of particle, emission of particle, inelastic collision…) cross-section per the solid angle can be found through the difference between the incoming current and the outgoing current divided by

the former. By analogy with Eq. (86), the differential reaction cross-section can be expressed as follows:

$$\frac{d\sigma_r}{d\theta}=2\pi\frac{[J_g(r_1)-J_s(r_1)]}{J_g(r_1)}{r_1}^2\sin(\theta)=2\pi\frac{\left[|C_g(r_1)|^2-|C_s(r_1)|^2\right]}{|C_g(r_1)|^2}{r_1}^2\sin(\theta). \tag{87}$$

8.3.3. Calculation of total cross-sections

The total elastic scattering cross-section is the total probability to be elastic scattered in any direction and it can be determined through the integral of differential cross-section given in Eq. (85):

$$\sigma_s = \int d\sigma_s = \int \frac{d\sigma_s}{d\Omega}d\Omega = \iint \frac{J_s(r_1)}{J_g(r_1)}{r_1}^2\sin(\theta)\,d\theta d\phi,$$

$$\sigma_s=4\pi{r_1}^2\frac{J_s(r_1)}{J_g(r_1)}=4\pi{r_1}^2\frac{|C_s(r_1)|^2}{|C_g(r_1)|^2}. \tag{88}$$

By analogy with Eq. (88), the total reaction cross-section can be expressed as follows:

$$\sigma_r=4\pi{r_1}^2\frac{[J_g(r_1)-J_s(r_1)]}{J_g(r_1)}=4\pi{r_1}^2-\sigma_s. \tag{89}$$

In Eq. (89), it is seen that if $J_s(r_1)=J_g(r_1)$, then $\sigma_r=0$, full elastic scattering; if $J_s(r_1)>J_g(r_1)$, then $\sigma_r<0$, it is taken out of the particle from the target (emission of particle from target) and if $J_s(r_1)<J_g(r_1)$, then $\sigma_r>0$, it is captured (absorbed) the particle by the target.

The total scattering cross-section, including all process [elastic plus reaction (all of no-elastic events)]:

$$\sigma_t = \sigma_s+\sigma_r=4\pi{r_1}^2\frac{J_s(r_1)}{J_g(r_1)}+4\pi{r_1}^2\frac{[J_g(r_1)-J_s(r_1)]}{J_g(r_1)}=4\pi{r_1}^2. \tag{90}$$

Then the cross-sections σ_s,σ_r,σ_t can be expressed through the A_1 coefficients given in Eqs. (80a) and (80b).

σ_s elastic scattering cross-section found using the coefficient (80a) is as follows:

$$X_1=3e^{4Kr_1}+5e^{4Kr_3}+2e^{4Kr_1}\left[\cos(2Kr_1)+\sin(2Kr_1)\right],$$

$$X_2=-2e^{2K(r_1+r_3)}[3\cos(2(Kr_1-Y))+2\cos(2Y)+\sin(2(Kr_1-Y))+\sin(2Y),$$

$$\frac{\sigma_s}{4\pi{r_1}^2} = \frac{1}{2}e^{-4Kr_1}\{X_1+X_2\}. \tag{91a}$$

σ_s elastic scattering cross-section found using the coefficient (80b) is as follows:

$$P_1=3e^{4Kr_1}+5e^{4Kr_3}+2e^{4Kr_1}\left[\cos(2Kr_1)+\sin(2Kr_1)\right],$$

$$P_2=-2e^{2K(r_1+r_3)}\left[3cos(2Kr_1-2Y)+2\cos(2Y)+\sin(2Kr_1-2Y)+\sin(2Y)\right],$$

$$P_3 = e^{4Kr_1}+5e^{4Kr_3}-2e^{2K(r_1+r_3)}\left[2\cos(2Y)+\sin(2Y)\right]$$

$$\frac{\sigma_s}{4\pi{r_1}^2} = \frac{P_1+P_2}{P_3}. \tag{91b}$$

In both cases, the reaction and total scattering cross-section σ_r and σ_t are as follows:

$$\frac{\sigma_r}{4\pi {r_1}^2} = 1 - \frac{\sigma_s}{4\pi {r_1}^2} \text{ and } \frac{\sigma_t}{4\pi {r_1}^2} = \frac{\sigma_s}{4\pi {r_1}^2} + \frac{\sigma_r}{4\pi {r_1}^2} = 1 \text{ or } \sigma_t = 4\pi {r_1}^2. \tag{91c}$$

In these expressions, Y is given by the following equation:

$$Y = Q_2(r_1) - Q_2(r_2) + Q_3(r_2) - Q_3(r_3) = m_1 \int_{r_2}^{r_1} \sqrt{|U_2(r)|}dr + m_1 \int_{r_3}^{r_2} \sqrt{|U_3(r)|}dr. \tag{91d}$$

The integrals in Eq. (91d) can be solved numerically if the functions $Q_2(r)$ and $Q_3(r)$ cannot be calculated analytically. It can be seen from these formulas given in Eq. (91a)–(91d) that the scattering cross-sections (σ_s and σ_r) depend on the total energy $E\,[withK(E)]\,,Y$ integral and the effective radius r_1,r_2,r_3 of the scatter potential, separately, but total scattering cross-section $\sigma_t = \sigma_s + \sigma_r$ only depends on the parameter r_1 so the energy E. Here, r_1 can be considered as impact or collision parameter, classically.

8.4. Examples of the calculation of scattering cross-section

8.4.1. Model potentials, wave functions, and their ingredients

To calculate a scattering cross-section, a model potential should be considered. Here as an example, we consider the Wood–Saxon shape potential plus the spin-orbit, centrifugal, and Coulomb potentials. The potential zones are defined in Figure 7 and shown in Figure 8 for these model potentials. Wood–Saxon potential depends on three parameters (V_0,a_c,R_c). The calculations have been thus performed with these potentials. The scattering event affects only the relative motion. In the center of the mass reference frame (CM), the scattering cross-section of the incoming (incident or projectile) particle $\sigma(\Omega)$ depends on the energy $E_r = M_t E_L/(M_p + M_t)$, where M_p and M_t are respectively mass of incident (projectile) and target particles; E_L, Laboratory; and E_r, relative energies. The boundary values of the potential zones taken in the calculations: the maximum potential energy occurs at the distance $r_2 = R_m = R_0({A_p}^{1/3} + {A_t}^{1/3})$, the addition of the projectile (incident particle) radius to the target radius, provided that they are spherical, where A_p and A_t are the mass number of the projectile and the target, respectively. The zones and $r_k, (k = 1, 2, 3)$ values are shown in Figure 8. The effective potential energy is as follows:

$$U(r) = V_{ws}(r) + V_{LSJ}(r) + V_S(r) + V_c(r), \tag{92}$$

$V_{ws}(r) = -\frac{V_0}{1+Exp[(r-R_c)/a_c]}$, Wood–Saxon potential, $V_{LSJ}(r) = -\frac{\hbar^2}{2{M_i}^2 c^2}\frac{1}{r}\frac{dV(r)}{dr} < \vec{L}.\vec{S} >$, spin-orbit potential,

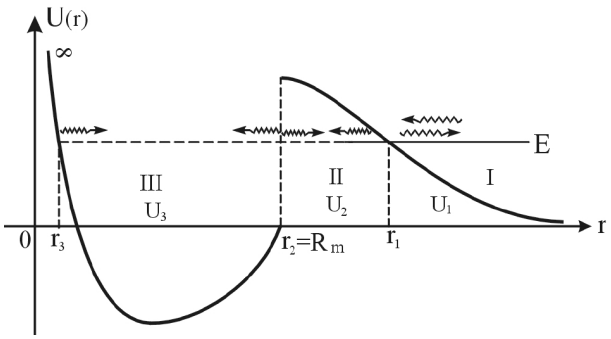

Figure 8. Effective Wood–Saxon potential and Coulomb potential zones used in the calculations.

$$<\vec{L}.\vec{S}> = \frac{1}{2}\left[J\left(J+1\right)-L\left(L+1\right)-S(S+1)\right]$$

$V_S\left(r\right) = \frac{b}{r^2}; [b = \frac{\hbar^2 L(L+1)}{2M_i}]$, centrifugal potential,

$V_c\left(r\right) = \frac{C_c}{r}, \left[C_c = \left(Z_p e\right)\left(Z_t e\right) = Z_p Z_t e^2\right]$ potential energy of Coulomb.

Here, $[L, S, J]$ are the relative orbital, spin, and total angular momentum quantum numbers between the target and projectile, respectively. $M_i is$ the reduced mass of the target and projectile. $R_c = R_0 {A_t}^{1/3}$, (R_0 *is the parameter*).

If $r_{sc} = \left[C_c+\sqrt{{C_c}^2+4bE_r}\right]\Big/[2E_r]$ is the positive root of the equation $E_r = \frac{b}{r^2}+\frac{C_c}{r} = V_s\left(r\right)+V_c\left(r\right)$ and $r_2 = R_m$ then we get $r_1 = r_2+r_{sc} = r_2+\left[C_c+\sqrt{{C_c}^2+4bE_r}\right]\Big/[2E_r].Z_p$ and Z_t are the charge numbers of the projectile and the target, respectively. The r_3 value is determined by equalizing $V_s\left(r\right) to E_r$. Consequently, r_3,r_2,r_1 values are obtained as follows:

$$r_3 = \sqrt{b/E_r}; \quad r_2 = R_m = R_0\left({A_p}^{\frac{1}{3}}+{A_t}^{\frac{1}{3}}\right); \quad r_1 = r_2+\frac{C_c+\sqrt{{C_c}^2+4bE_r}}{2E_r}.$$

The wave functions for zones 1–3 used in the calculation can be expressed in the following ways, respectively:

$$Q_0\left(r\right) = m_\hbar \int \sqrt{\frac{\hbar^2 L_0(L_0+1)}{2M_i r^2}}dr = \sqrt{L_0(L_0+1)}\ln\left(r\right),$$

$$Q_1\left(r\right) = m_\hbar \int \sqrt{\frac{C_c}{r}+\frac{b}{r^2}}dr = 2m_\hbar\left\{\sqrt{b+C_c r}-\sqrt{b}arctanh\left[\sqrt{\frac{b+C_c r}{b}}\right]\right\},$$

$$Q_2\left(r\right) = m_\hbar \int \sqrt{\frac{C_c}{r}+\frac{b}{r^2}}dr = 2m_\hbar\left\{\sqrt{b+C_c r}-\sqrt{b}arctanh\left[\sqrt{\frac{b+C_c r}{b}}\right]\right\},$$

$$Q_3\left(r\right) = m_\hbar \int \sqrt{|U_3\left(r\right)|}dr = m_\hbar \int \sqrt{|V_{ws}\left(r\right)+V_{LSJ}\left(r\right)+V_S\left(r\right)|}dr. \tag{93}$$

Here, $m_\hbar = \sqrt{2M_i}/\hbar$ and L_0 is the angular momentum of the incident particle.

It is seen from Eq. (91c) that even though σ_s and σ_r have changed with the parameters V_0 and a_c;$\sigma_t = \sigma_s+\sigma_r$ has not changed for a certain value of R_0. In other words, the total cross-section does not depend on the parameters V_0 and a_c and, therefore, does not also depend on the potential. So, R_0 parameter can be obtained from the solution of the equation $4\pi {r_1}^2 = {\sigma_t}^{exp}$ as follows (${\sigma_t}^{exp}$ is experimental total cross-section):

$$R_0 = \frac{-\sqrt{\pi}10\left[C_c+\sqrt{{C_c}^2+4bE_r}\right]+E_r\sqrt{10{\sigma_t}^{exp}}}{20E_r\sqrt{\pi}\left[\sqrt[3]{A_p}+\sqrt[3]{A_t}\right]} \tag{94}$$

To calculate σ_s and σ_r separately, the parameter V_0 in the range 20–60 by step 0.0001 and a_c in the range 0.40–0.60 by step 0.01 have been changed in the potential until ${\sigma_r}^{cal} > 0$ and round $[{\sigma_r}^{exp}]$ = round $[\sigma_r^{cal}]$ and floor $[{\sigma_r}^{exp}]$ = floor $[{\sigma_r}^{cal}]\}$, during the calculations. So, rough V_0 and a_c values have been found. Then, with

these approximate values, $\left\{{\sigma_s}^{exp} = {\sigma_s}^{cal} \text{ and } {\sigma_r}^{exp} = {\sigma_r}^{cal}\right\}$ system of equations has been solved and the exact values of parameters V_0 and a_c have been found. With these V_0 and a_c parameters, the exact σ_s and σ_r have been recalculated. We have taken in the calculations: e^2= $1.439976 MeV fm$;M_u= $931.502 MeV/c^2$; $\hbar c = 197.329 MeV fm$.

8.4.2. Thermal neutron cross-sections

For example, thermal neutron (E_L= $0.025eV$) scattering cross-sections for some targets have been calculated and compared with the measured values. In calculation, because their angular momentums are zero even nuclei have been taken as targets. So, the relative angular momentums are those of the projectile, they are $L = 0, S$ =1/2, J =1/2 for neutron. For the calculations of σ_s,σ_r, and σ_t, the formulas given in Eq. (84) which has been obtained by the lower sign functions. The results are compared with the measured total cross-sections in Table 8. The experimental values have been taken from [25–32]. It is seen that agreements are fairly good as seen in Table 8. In this table, the first column: target; the second column: (R_0,V_0, $and a_c$) parameters; the third column: (R_c,R_m = r_2,r_1); the fourth column: cross-sections; the fifth column: values of calculation; and the sixth column: values of the experiment.

Table 8. $[n(0,1) + Xn(Z,N)]$ Thermal neutron cross-section comparisons with those measured.

Target $Xn(Z,N)$	R_0 V_0 a_c	R_c R_m r_1	Cross-sections (mb)	Calculations (mb)	Experiment (mb)
H (1,2)	2.29845	2.89586	$\sigma_s \rightarrow$	3390	3390 ± 12
	21.3275	5.19431	$\sigma_r \rightarrow$	0.519	0.519 ± 0.007
	0.40	5.19431	$\sigma_t \rightarrow$	3390.52	3390.52
C (6,12)	1.86896	4.27885	$\sigma_s \rightarrow$	4746	4746 ± 2
	35.2934	6.14781	$\sigma_r \rightarrow$	3.53	3.53 ± 0.07
	0.40	6.14781	$\sigma_t \rightarrow$	4749.53	4749.53
O (8,16)	1.5543	3.91659	$\sigma_s \rightarrow$	3761	3761 ± 6
	41.2421	5.47089	$\sigma_r \rightarrow$	0.190	0.190 ± 0.019
	0.40	5.47089	$\sigma_t \rightarrow$	3761.19	3761.19
Si (14,28)	1.02922	3.12533	$\sigma_s \rightarrow$	1992	1992 ± 6
	21.3192	4.15456	$\sigma_r \rightarrow$	177	177 ± 5
	0.40	4.15456	$\sigma_t \rightarrow$	2169	2169
Ca (20,40)	1.1803	4.03655	$\sigma_s \rightarrow$	3010	3010 ± 8
	39.0960	5.21685	$\sigma_r \rightarrow$	410	410 ± 20
	0.40	5.21685	$\sigma_t \rightarrow$	3420	3420

8.4.3. $_2{}^3$He Cross-sections on some targets in the intermediate energy

The cross-sections σ_t at three different He-3 energies have been calculated for 5 different targets [^9_4Be, $^{12}_6C$, $^{16}_8O$, $^{28}_{14}Si$, and $^{40}_{20}Ca$]. The calculated cross-sections have been compared with those measured taken from [32,38–40]. The comparisons are made in the way that is described above and given in Table 9. There is no need for relative angular momentums here.

Table 9. $[He[2,3] + Xn[Z,N]$ Total cross-section comparison with those measured.

$Xn[Z,N]$ (target)	Energy (projectile) E_L (MeV)	$R_0\,(fm)$	$R_c(fm)$	$R_m = r_2\ (fm)$	$r_1(fm)$ (impact parameter)	Calculation $\sigma_t(mb)$	Experiment $\sigma_t(mb)$
Be[4,9]	96.4	0.673279	1.40048	2.37051	2.53101	805	805 ± 30
	137.8	0.623868	1.29770	2.19747	2.30905	670	670 ± 30
	167.3	0.607056	1.26273	2.13825	2.23016	625	625 ± 30
C[6,12]	96.4	0.620244	1.42000	2.31455	2.53885	810	810 ± 40
	137.8	0.594922	1.36203	2.22006	2.37697	710	710 ± 30
	167.3	0.572480	1.31065	2.13631	2.26556	645	645 ± 35
O[8,16]	96.4	0.631332	1.59086	2.50140	2.78546	975	975 ± 35
	137.8	0.606261	1.52768	2.40206	2.60079	850	850 ± 50
	167.3	0.595506	1.50058	2.35945	2.52313	800	800 ± 25
Si[14,28]	96.4	0.600731	1.82417	2.69058	3.15392	1250	1250 ± 65
	137.8	0.603057	1.83124	2.70099	3.02513	1150	1150 ± 70
	167.3	0.590377	1.79273	2.64420	2.91119	1065	1065 ± 40
Ca[20,40]	96.4	0.544438	1.86195	2.64717	3.28976	1360	1360 ± 90
	137.8	0.563942	1.92866	2.74200	3.19154	1280	1280 ± 85
	167.3	0.565988	1.93565	2.75195	3.12222	1225	1225 ± 75

The calculation of cross-sections through solution of the radial SE by the partial wave expansion is very difficult. In many cases, some approximations are needed for these kinds of solutions. In the present study, firstly, differential elastic scattering, inelastic (or reaction) scattering, and total cross-sections have been calculated without using any approximation. These calculations have been performed using a simple method, improved for the solution of the radial SE, for an incident particle being in a central field of any form. We have obtained the general formulas of the scattering amplitudes and elastic, inelastic (no-elastic) and total scattering cross-sections. Secondly, we have made some applications. In these applications, the potentials have been assumed to have Saxon–Woods shape plus spin-orbit interaction, centrifugal and Coulomb potentials. With these potentials, first, for the thermal neutrons; the elastic, inelastic (neutron radiative capture), and total scattering cross-sections of different targets have been calculated. Then, total scattering cross-sections for ${}_2{}^3He$ particles of three different energies on 5 targets have been calculated. The calculated results have been compared with the experimental results. The results calculated have given satisfactory agreement with the available experimental results. More cross-section calculations can be found in [33].

The calculations have also shown that the total cross-sections depend on the mean potential range. Thus, it is also proved that the total cross-sections can be calculated easily using even very complex potentials. The same calculations have also been performed using optical potentials, but the results have not been included here due to getting the same scattering distance and cross-sections. The use of two parameters is seen to be enough in the agreement of the calculated results with the measured results, whereas this agreement is ensured using more parameters in the partial wave expansion method.

9. Conclusion and some explanations

We have found a simple procedure for the general solution of the time-independent SE in one dimension without using any infinite series or other approximations. The wave functions, which are always periodic in the bound

states, are given in Eqs. (52a)–(57). In our procedure of solution, there are two difficulties: one is to solve the equation $E = U(x)$ to find the energy; and the other is to integrate $\sqrt{U(x)}$ namely, to calculate $\int U(x)\,dx$ to find the exact normalized wave function. If these calculations cannot be done analytically, then it should be done by numerical methods. To find the energy values, there is no need to calculate this integral, it is sufficient to find the classical turning points by solving the equation $E = U(x)$. Thus, there is no need to know the wave functions to find the energy values of the states; it is enough to know only the potential energy function. The SE has been solved for a particle in many potential wells and their energies and normalized wave functions were calculated as examples.

Using a simple procedure, the solution of the radial SE for spherically symmetric potentials without using any infinite series has been achieved in this study. The wave functions which are always periodicals are given in Eqs. (52a)–(57). By using this procedure, the radial SE has been solved for a particle found in many spherically symmetric central potential wells and two different solutions have been found. One of them is symmetric function and the other is antisymmetric function. From these expressions, it is observed that these functions are periodic and they are similar to each other in form for all potential wells. This simple solution was applied to scattering and tunneling theories and it yielded good results.

The solution that we propose is a general solution. The points of view supporting the method we presented here is more realistic which are as follows:

As it is known, the SE is a second order differential equation with variable coefficients. The solutions of such equations are based generally on series method and special functions (Hermit, Bessel...) in quantum physics. In the expanding of power series, the consecutive relations between the coefficients of the series are found. Some approaches are taken to make the series convergent and by using them, the energy values and the wave functions are determined. In all of these solutions, one or more approximations are used. Some other methods such as perturbation, WKB, and variation are also applied for solving the SE and approximate solutions are obtained. However, the solution that is proposed here is neither based on series methods nor special functions, and no approximation is used. We think that these solutions, which do not have any approach, are more realistic.

It is seen that the functions found with the known methods and with the present method are different in form. However, when we do numerical calculations, the results are not very different from each other and are consistent with their well-known values. For example, for the one dimensional harmonic oscillator, our result and the result obtained by the known methods are very close to each other.

The SE, the fundamental equation of the quantum mechanics, is also known as the Schrödinger wave equation. In physics, the harmonic waves are represented with periodic functions e.g., sin and cos. However, most of the known solutions of the SE are in the form of polynomials. Especially the solution of the harmonic oscillator should be periodic function. However, the known solution is polynomial. As it is seen in the functions given in Eqs. (5.1.3) and (5.1.4), our solution gives sin and cos functions, which makes it more realistic.

It is said that the quantum mechanics includes classical mechanics. Hence, the results of the classical mechanics should be obtained from the results found in quantum mechanics. As it is seen from the functions given in Eq. (5.1.4), it is very difficult to obtain sin and cos functions from polynomials. However, the classical solutions could be easily obtained if $\Psi(x)$ is taken very small in the functions given in (5.1.3).

In quantum mechanics textbooks, it is said that all infinitely high potential wells are similar to each other. But when we look at their solutions, we see that all of the solutions are different in form; whereas our solutions are similar to each other in form for all such potential wells.

In quantum mechanics, if there is no exact solution, sometimes, the variation principle is used to find the

ground state energy. The calculations of variation are made with a trial function, and it is seen that the results are not very dependent on the trial function. However, the wave functions are not necessary to find the ground or excited state energy in our solutions. It is sufficient to know the classical turning points of the potential function.

Special conditions are not required in order to find the energy values and wave functions. The continuity of the wave functions and their derivatives at the classical turning points are enough.

The application of our procedure is very easy. Most of the problems that could not be solved analytically with the known methods can easily be solved using our procedure. A complete solution of the SE used in all branches of physics has been made. A problem that has been worked on by many theoretical physicists has been solved. I think that this solution is very useful and helpful for those physicists interested in quantum mechanics and applications.

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