Turkish Journal of Physics
http://journals.tubitak.gov.tr/physics/
Research Article
Turk J Phys
(2019) 43: 59 – 66
© TÜBİTAK
doi:10.3906/fiz-1806-14

Modified Newton's gravitational law from deformed Poisson bracket

Ch'ng Han SIONG*
Methodist College Kuala Lumpur, Kuala Lumpur, Malaysia

Received: 13.06.2018 • **Accepted/Published Online:** 04.12.2018 • **Final Version:** 22.02.2019

Abstract: In this paper, we obtain the modified Newton's gravitational law from deformed Poisson bracket motivated by the generalized uncertainty principle. The modified Newton's gravitational law that we obtain contains a term that plays an important role only in the regime of first order in $\bar{\beta}$. In the ordinary situation or condition, the modified Newton's gravitational law reduces to the conventional Newton's gravitational law.

Key words: Generalized uncertainty principle, modified Newton's gravitational law

1. Introduction

The Planck length is predicted to be the smallest measureable length. This prediction is the consequence of the various approaches of quantum gravity, such as string theory and loop quantum gravity. These theories suggest the existence of the smallest measurable length, which is of the order of Planck length $L_P = \sqrt{\hbar G/c^3} \simeq 1.6 \times 10^{-35} m$. This existence of minimum measurable length has also led to the modification of the Heisenberg uncertainty principle [1–5]. The conventional Heisenberg uncertainty principle does not yield any minimal length. Hence, the conventional Heisenberg uncertainty principle is modified to the so-called generalized uncertainty principle. The generalized uncertainty principle is oriented on the deformation of commutation relation. In the classical limit, the quantum mechanical commutator is replaced by the Poisson bracket [6]. In this paper, the deformed Heisenberg algebra leads to the deformed Poisson bracket for corresponding classical variables. The deformed Poisson bracket yields the modified equations of motion.

The work in this paper is inspired by the generalized uncertainty principle. Some classical problems were already investigated within the framework of deformed Poisson brackets, such as in references [7–11]. In this paper, we study Newtonian gravitation from the classical limit of deformed Heisenberg algebra as proposed by Kempf et al. [1]. Their deformed Heisenberg algebra is given as $[x,p] = i\hbar\left(1+\beta p^2\right)$. Parameter β become important only when the scale is near the Planck scale. If $\beta \rightarrow 0$, then the modified Heisenberg algebra would reduce to the conventional Heisenberg algebra. Besides this, it is also well known that the correspondence between the quantum mechanical commutator and the Poisson bracket does exist. Consequently, the deformed Poisson bracket can be generated and the deformed equations of motion are formed. Newtonian mechanics are thus modified via the extra terms, which become important in extraordinary situations.

In this paper, we work on the classical limit of the deformed commutation relation. The classical limit of the deformed commutation relation is given by the deformed Poisson bracket. We derive a modified Newton's gravitational law from the deformed Poisson bracket. In order to differentiate parameter β as found in the

*Correspondence: chng_hs@yahoo.com

generalized uncertainty principle and in the classical limit, we denote β in the classical limit as $\bar{\beta}$. Parameter β in the work of the generalized uncertainty principle is taken to be relevant only when the scale is near the Planck scale, while parameter $\bar{\beta}$ in this study of classical limit is taken to be significant only when the length scale is in the regime of the so-called $\bar{\beta}$-scale. Besides this, we would also like to highlight that we consider the parameter $\bar{\beta}$ only up to first order in this study.

It has been known for a long time that an object can escape the gravitational field of another massive object if the object's velocity is equal to or more than a minimum velocity, which is called the escape velocity. This is done by establishing an equation from the principle of conservation of energy. The energy of an escaping object consists of kinetic and gravitational potential energy and is taken to be constant, where energy = kinetic energy + gravitational potential energy. The formula for escape velocity can be derived from this energy equation. However, a modified formula for escape velocity can be obtained from the deformed Poisson bracket. The energy equation or Hamiltonian equation is thus modified due to the modification of momentum. The extra term becomes important only when it is under certain circumstances. Extraordinary circumstances will be explained later.

The presence of minimal length due to the deformed Heisenberg algebra brings a new development to classical mechanics as well as quantum theory. Some problems were investigated. These include works on the equivalence principle reconciling with the generalized uncertainty principle [7], a composite system in deformed space with minimal length [6], some classical dynamics based on the minimal length uncertainty principle [8,9], and a universe investigated in the framework of the generalized uncertainty principle [12,13].

In Section 2, we introduce the mathematical formalism that is needed. In Section 3, we review the conventional Newton's gravitational law. In Section 4–6, we derive the modified Newton's gravitational law based on a deformed Poisson bracket. In Section 7, we present the discussion and conclusion.

2. Mathematical formalism

Let us start by considering the conventional Heisenberg algebra:

$$[\hat{x}, \hat{p}] = i\hbar$$

or

$$\frac{1}{i\hbar}[\hat{x}, \hat{p}] = 1. \tag{1}$$

Paul Dirac [6] proposed that there is a correspondence between the quantum mechanical commutator and Poisson bracket, given as follows:

$$\frac{1}{i\hbar}[\hat{x}, \hat{p}] = (x, p)\,. \tag{2}$$

Eq. (2) reduces to Eq. (1) since $(x,p) = \frac{\partial x}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial x}{\partial p}\frac{\partial p}{\partial x} = 1$. Recently, many works have been done on the generalized uncertainty principle, such as in references [14–19]. One of the deformed Heisenberg algebras is the one proposed by Kempf et al. [1], described as follows:

$$\frac{1}{i\hbar}[\hat{x}, \hat{p}] = \left(1 + \beta\hat{p}^2\right)(x, p)\,. \tag{3}$$

In general, we may write Eq. (3) above as:

$$\frac{1}{i\hbar}[\hat{x}, \hat{p}] = f(\hat{p})\,(x, p)\,, \tag{4}$$

where $f(\hat{p})$ is the general function of $\hat{p}$. It is also known that for any two observables A, B, we have

$$\frac{1}{i\hbar}\left[\hat{A},\hat{B}\right]=\{A,B\}\,. \tag{5}$$

Comparing Eq. (4) with Eq. (5), we conclude:

$$\{A,B\}=f\left(p\right)\cdot\left(\frac{\partial A}{\partial x}\frac{\partial B}{\partial p}-\frac{\partial A}{\partial p}\frac{\partial B}{\partial x}\right). \tag{6}$$

Now let us introduce a Hamiltonian of the following form:

$$H=\frac{p^2}{2m}+V, \tag{7}$$

where potential energy $V=V\left(x\right)$. We have to keep in mind that momentum p in Eq. (7) is not the conventional momentum; it also contain the correction terms in $\bar{\beta}$-scale. In other words, the momentum in Eq. (7) can deal with the $\bar{\beta}$-scale. Besides this, we also take the assumption that potential energy V in Eq. (7) is also valid in the regime of the $\bar{\beta}$-scale. The modified Hamilton equations are then given as:

$$v=\frac{dx}{dt}=\{x,H\}=f\left(p\right)\cdot\frac{p}{m} \tag{8}$$

and

$$F=\frac{dp}{dt}=\{p,H\}=-f\left(p\right)\cdot\frac{dV}{dx}, \tag{9}$$

where v and F are velocity and force, respectively. Eqs. (8) and (9) can reduce to conventional Hamilton equations for the limit $f\left(p\right)\rightarrow 1$. Up to the first order in $\bar{\beta}$, p can be written approximately as [7,10]:

$$p=p_0+\bar{\beta}p_1 \tag{10}$$

where $p_0=mv$ is the conventional formula for momentum. In other words, $p_0=mv$ is the formula of momentum under conventional Newtonian mechanics. If $\bar{\beta}\rightarrow 0$, then $p=p_0$. However, if $\bar{\beta}\neq 0$, then momentum p is different from the conventional momentum $p_0=mv$. Thus, momentum p is taken to be the momentum that is valid in the regime of first order in $\bar{\beta}$. The momentum p is approximated only up to first order in $\bar{\beta}$. As indicated in the Taylor series expansion, the momentum $p$is slightly divergent from p_0. In other words, the magnitude of momentum p is close to p_0. We should highlight that the subscript 0 is used to represent the formula under conventional Newtonian mechanics. For example, the conventional Newton's second law is $F=\frac{dp_0}{dt}$ and conventional momentum$p_0=mv$. We would also like to stress here that since $x=x_0$, the potential energy V is taken to be valid in all scales, including the $\bar{\beta}$-scale.

3. Conventional Newtonian gravitational law

Let us consider an object of mass m escaping away from a gravitational field of a bigger object of mass M. By employing the spherical coordinates system (r,θ,ϕ) and assuming the object moves along the radial coordinate, the Hamiltonian of the object of mass m is of the following form:

$$H=\frac{p_0^2}{2m}-\frac{GMm}{r}, \tag{11}$$

where potential energy $V = V(r) = -\frac{GMm}{r}$. Substituting Eq. (11) into Eqs. (8) and (9) and taking $f(p) \to 1$ or $\bar{\beta} \to 0$, we obtain:

$$v = \frac{dr}{dt} = \{r, H\} = \frac{p_0}{m}, \tag{12}$$

$$F = \frac{dp_0}{dt} = \{p_0, H\} = -\frac{dV}{dr}. \tag{13}$$

The force F in Eq. (13) is the conventional Newton's gravitational law since $f(p) \to 1$ and it is given as:

$$F = -\frac{GMm}{r^2}. \tag{14}$$

4. Modified Newtonian gravitational law

We now take $f(p) = 1 + \bar{\beta}p^2$. Hence, Eqs. (8) and (9) become:

$$v = \left(1 + \bar{\beta}p^2\right) \cdot \frac{p}{m} \tag{15}$$

and

$$F = -\left(1 + \bar{\beta}p^2\right) \cdot \frac{dV}{dr}. \tag{16}$$

We substitute Eq. (10) into $\bar{\beta}p^2$ in Eq. (15) and, up to the first order of $\bar{\beta}$, we obtain an equation as follows:

$$v = \frac{p + \bar{\beta}p_0^2 p}{m}. \tag{17}$$

Again substituting Eq. (10) into $\bar{\beta}p_0^2 p$ in Eq. (17), we get

$$p = p_0 - \bar{\beta}p_0^3, \tag{18}$$

where $p_0 = mv$. We now substitute Eq. (10) into $\bar{\beta}p^2$ in Eq. (16), and up to the first order of $\bar{\beta}$, we obtain

$$F = -\left(1 + \bar{\beta}p_0^2\right) \cdot \frac{GMm}{r^2}. \tag{19}$$

In the limit $\bar{\beta} \to 0$, the conventional Newtonian gravitational law is recovered. In general, the gravitational force is dependent on conventional momentum $p_0 = mv$. In conventional classical mechanics, the total energy of an object moving in a gravitational field is given by $E_0 = \frac{p_0^2}{2m} - \frac{GMm}{r}$, where $p_0 = mv$. Therefore, Eq. (19) can also be written as:

$$F = -\frac{GMm}{r^2} - \bar{\beta}\left(2E_0 m + \frac{2GMm^2}{r}\right)\frac{GMm}{r^2}. \tag{20}$$

The gravitational force is thus also said to be dependent on the total energy of an object in a gravitational field.

5. Modified Newtonian gravitational law when $E_0 = 0$

We now consider a special case where $E_0 = 0$. The object can escape from the gravitational field and hence

$$v = \sqrt{\frac{2GM}{r}}. \tag{21}$$

We now differentiate velocity v in Eq. (21) with respect to time t to obtain the gravitational acceleration g_0 as follows:

$$g_0 = \frac{dv}{dt} = -\frac{GM}{r^2}. \tag{22}$$

Eq. (22) is indeed the standard Newton's gravitational acceleration. It is appropriate to set $E_0 = 0$. The velocity v in Eq. (21) gives the velocity of the escaping object as a function of radial distance in the entire gravitational field. The differentiation of v with respect to time t is the Newton's gravitational acceleration $g_0 = -\frac{GM}{r^2}$. If $E_0 = 0$, the object can just barely escape the gravitational field. If $E_0 > 0$, the kinetic energy is dominant and the object will keep moving and will not stop even beyond the gravitational field. If $E_0 < 0$, the potential energy is dominant and the object will be attracted by the gravitational force and fall down. Since force F in Eq. (20) is intended to be able to provide the gravitational force in the entire gravitational field, it is appropriate to set $E_0 = 0$. In the case where $E_0 = 0$, Eq. (20) reduces to the following:

$$F = -\frac{GMm}{r^2} - \bar{\beta}\left(\frac{2G^2M^2m^3}{r^3}\right). \tag{23}$$

Eq. (23) is the modified Newton's gravitational law. We could obtain the gravitational potential energy by integrating gravitational force over radial distance. By integrating Eq. (23) over the radial distance from r to ∞, the modified gravitational potential energy is given as follows:

$$\bar{V} = -\frac{GMm}{r} - \bar{\beta}\left(\frac{2G^2M^2m^3}{r^2}\right). \tag{24}$$

Both $\bar{V}$ and $V(r) = -\frac{GMm}{r}$ are potential energy, but what is the difference between them? To answer this question, we may want to recall Eq. (19), $F = -\left(1 + \bar{\beta}p_0^2\right) \cdot \frac{GMm}{r^2}$. We can actually rewrite this equation as follows:

$$F = -\frac{d\bar{V}}{dr} = -\left(1 + \bar{\beta}p_0^2\right) \cdot \frac{dV}{dr}, \tag{25}$$

where $V = -\frac{GMm}{r}$ and $\bar{V} = -\frac{GMm}{r} - \bar{\beta}\left(\frac{2G^2M^2m^3}{r^2}\right)$. We can see from Eq. (25) that the potential momentum-independent energy is given by $V = -\frac{GMm}{r}$, while the potential momentum-dependent energy is given by $\bar{V} = -\frac{GMm}{r} - \bar{\beta}\left(\frac{2G^2M^2m^3}{r^2}\right)$.

Now we would like to find the gravitational acceleration g (we have to highlight here that $g \neq g_0$, where $g_0 = -\frac{GM}{r^2}$). In a simple way, we can establish an equation as follows:

$$mg = -\frac{GMm}{r^2} - \bar{\beta}\left(\frac{2G^2M^2m^3}{r^3}\right). \tag{26}$$

Subsequently, g is given as follows:

$$g = -\frac{GM}{r^2} - \bar{\beta} m^2 \left(\frac{2G^2 M^2}{r^3} \right). \tag{27}$$

One might think that the equivalence principle is violated since now the gravitational acceleration g in Eq. (27) is dependent on the mass of the object, m. However this is not true. From references [6,7], we can see that parameter $\bar{\beta}$ for a macroscopic object can be written as

$$\bar{\beta} = \sum_i {\mu_i}^3 \bar{\beta}_i, \tag{28}$$

where $\mu_i = \frac{m_i}{\sum_i m_i}$. The symbol $\bar{\beta}_i$ denotes the parameters of individual particles that form the macroscopic object or system, while m_i is the mass of the individual particle. If the macroscopic object or system consists of the same N particles, then $m_1 = m_2 = m_3 = ... = m_N$ and $\bar{\beta}_1 = \bar{\beta}_2 = \bar{\beta}_3 = ... = \bar{\beta}_N$. The total mass of the macroscopic object or system is given as $m = m_1 + m_2 + m_3 + ... + m_N = N m_N$. The $\bar{\beta}$ in Eq. (28) can then be rewritten as follows:

$$\bar{\beta} = \frac{\bar{\beta}_N}{N^2}. \tag{29}$$

Hence, $\bar{\beta} m^2 = \frac{\bar{\beta}_N m^2}{N^2} = \bar{\beta}_N m_N^2$. Therefore, g in Eq. (27) is now not dependent on mass m but dependent on the term $\bar{\beta}_N m_N^2$. The term $\bar{\beta}_N m_N^2$ is the same for all objects of different mass. This is due to the assumption that all objects consist of the same particles. Hence, the equivalence principle is preserved. The authors in [7] also showed that, by supposing $\sqrt{\bar{\beta}_1} m_1 = \sqrt{\bar{\beta}_2} m_2 = ... = \sqrt{\bar{\beta}_N} m_N = \gamma$, the effective parameter $\bar{\beta}$ for a macroscopic body is given by $\bar{\beta} = \frac{\gamma^2}{m^2}$, where γ is considered as a fundamental constant and taking the same value for all objects. Therefore, the parameter $\bar{\beta}$ is proportional to $\frac{1}{m^2}$. Since γ^2 is a fundamental constant, Eqs. (23), (24), and (27) can be rewritten in the following way:

$$F = -\frac{GMm}{r^2} - \frac{KM^2 m}{r^3}, \tag{30}$$

$$\bar{V} = -\frac{GMm}{r} - \frac{KM^2 m}{r^2}, \tag{31}$$

and

$$g = -\frac{GM}{r^2} - \frac{KM^2}{r^3}, \tag{32}$$

where constant $K = 2G^2\gamma^2$.

6. Modified Newtonian gravitational law when $E_0 = -\frac{GMm}{2r}$

In this section, we consider a case where an object is orbiting around a central mass M. From the conventional Newton's law of gravitation, the energy of an object orbiting around a central mass M is given by:

$$E_0 = -\frac{GMm}{2r}. \tag{33}$$

Hence, Eq. (20) reduces to:

$$F = -\frac{GMm}{r^2} - \bar{\beta}m^2\left(\frac{G^2M^2m}{r^3}\right). \tag{34}$$

The gravitational potential energy and gravitational acceleration are thus given by:

$$\bar{V} = -\frac{GMm}{r} - \bar{\beta}m^2\left(\frac{G^2M^2m}{r^2}\right) \tag{35}$$

and

$$g = -\frac{GM}{r^2} - \bar{\beta}m^2\left(\frac{G^2M^2}{r^3}\right). \tag{36}$$

Eqs. (34), (35), and (36) can be rewritten as follows:

$$F = -\frac{GMm}{r^2} - \frac{LM^2m}{r^3}, \tag{37}$$

$$\bar{V} = -\frac{GMm}{r} - \frac{LM^2m}{r^2}, \tag{38}$$

and

$$g = -\frac{GM}{r^2} - \frac{LM^2}{r^3}, \tag{39}$$

where $L = G^2\bar{\beta}m^2 = G^2\gamma^2$. Now we can determine the magnitude of rotation velocity v from the following formula: $g = \frac{v^2}{r}$. Therefore:

$$v^2 = \frac{GM}{r} + \frac{LM^2}{r^2}. \tag{40}$$

In conventional Newtonian mechanics, Eq. (40) reduces to $v^2 = \frac{GM}{r}$ where v decreases with respect to r. However, we now show that the square of rotation velocity, v^2, is corrected by the second term of the right-hand side.

7. Discussion and conclusions

We consider two cases: in the first case, an object is escaping from a gravitational field, whereas in the second case, an object is orbiting around a central mass. We have obtained the momentum-dependent gravitational potential energy, momentum-dependent gravitational force, and momentum-dependent gravitational acceleration. They are Eqs. (30) and (37), Eqs. (31) and (38), and Eqs. (32) and (39). The K-terms and L-terms become important if the magnitude of M^2/r^3 is large. The values of K and L are yet to be determined from experiments. If we assume that the K-terms or L-terms become important only if $\frac{KM^2}{r^3} \geq x$ or $\frac{LM^2}{r^3} \geq x$, then $\frac{M^2}{r^3} \geq \frac{x}{K}$ or $\frac{M^2}{r^3} \geq \frac{x}{L}$. We believe that under normal conditions, such as for an ordinary star or planet, the K-terms or L-terms are expected to be negligible and hence the conventional Newton's gravitational law must be used. The gravitational accelerations of Eqs. (27) and (36) seem to be dependent on m; however, this is not true. In the future, it would be worthwhile to investigate the theoretical value of fundamental constant $\gamma^2 = \bar{\beta}m^2$.

In order to apply Eqs. (30), (31), (32), (37), (38), (39), and (40), the object must be moving in a gravitational field so that the object has momentum; otherwise, the conventional Newton's law of gravitation must be used. In the future, we would like to study the possibility of explaining flat galaxies' rotation curves based on the momentum-dependent formulas. It is learned from conventional Newtonian mechanics that the rotation velocity of an object in a galaxy decreases with distance from the center. However, observations indicate that the rotation velocity is independent of the distance from the center of a galaxy (flat rotation curves). To solve this problem, the second term on the right-hand side of Eq. (40) is treated as a correction term for compensating the decrease in rotation velocity. Hence, Eq. (40) may be used to explain the flat galaxies' rotation curves. Nevertheless, its validity in explaining the flat galaxies' rotation curves depends on the empirical value of constant L.

References

[1] Kempf, A.; Mangano, G.; Mann, R. B. *Phys. Rev. D* **1995**, *52*, 1108-1118.

[2] Ali, A. F.; Das, S.; Vagenas, E. C. *Phys. Lett. B* **2009**, *678*, 497-499.

[3] Das, S.; Vagenas, E. C.; Ali, A. F. *Phys. Lett. B* **2010**, *690*, 407-412.

[4] Pedram, P. *Phys. Lett. B* **2012**, *714*, 317-323.

[5] Pedram, P. *Phys. Lett. B* **2012**, *718*, 638-645.

[6] Dirac, P. A. M. *The Principles of Quantum Mechanics*; Clarendon: Oxford, UK, 1958.

[7] Quesne, C.; Tkachuk, V. M. *Phys. Rev. A* **2010**, *81*, 012106.

[8] Tkachuk, V. M. *Phys. Rev. A* **2012**, *86*, 062112.

[9] Benczik, S.; Chang, L. N.; Minic, D.; Okamura, N.; Rayyan, S., Takeuchi, T. *Phys. Rev. D* **2002**, *66*, 026003.

[10] Chung, W. S. *Int. J. Theor. Phys.* **2016**, *55*, 825-836.

[11] Silagadze, Z. *Phys. Lett. A* **2009**, *373*, 2643-2645.

[12] Ali, A. F.; Majumder, B. *Class. Quantum Grav.* **2014**, *31*, 215007.

[13] Battisti, M. V. *Phys. Rev. D* **2009**, *79*, 083506.

[14] Pedram, P. *Int. J. Mod. Phys. D* **2010**, *12*, 2003-2009.

[15] Miao, Y. G.; Zhao, Y. J. *Int. J. Mod. Phys. D* **2014**, *23*, 1450062.

[16] Deb, S.; Das, S.; Vagenas, E. C. *Phys. Lett. B* **2016**, *755*, 17-23.

[17] Shababi, H.; Chung, W. S. *Phys. Lett B* **2017**, *770*, 445-450.

[18] Ali, A. F.; Tawfik, A. *Adv. High Energy Phys.* **2013**, *2013*, 126528.

[19] Scardigli, F.; Lambiase, G.; Vagenas, E. C. *Phys. Lett. B.* **2017**, *767*, 242-246.