

## Application of the fractional oscillator model to describe damped vibrations

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Received: 20.11.2018

Accepted/Published Online: 01.04.2019

Final Version: 12.06.2019

**Abstract:** We consider a model of damped vibrations based on fractional differentiation. The model given is completely consistent with the classical model of vibration with viscous damping. We find the relation between the order of fractional differentiation in the equation of motion and Q-factor of an oscillator. The proposed approach seems more appropriate for the physical nature of the described system. The experiment with a vibrating piezoelectric plate, performed as part of the study, showed good agreement with the model and confirmed that the fractional oscillator model can be used to describe strongly damped vibrations.

**Key words:** Fractional differentiation, linear oscillator, damped vibrations, damping, dynamic friction force, dynamic storage function

**PACS:** 05.45.Df; 45.50.Dd

### 1. Introduction

Damping in any physical oscillatory system always leads to energy dissipation. In particular, damping in mechanical systems is caused by various friction processes (viscous and dry friction, air resistance, etc.) and scattering of elastic waves. Attenuation in electrical systems is the result of the conversion of electromagnetic energy into heat (the Joule–Lenz law, electrical or magnetic hysteresis in dielectrics or magnets) or the emission of electromagnetic waves, which in various problems is determined by the Poynting vector. Oscillations in mechanical and electric systems are often described by the same differential equations. This electromechanical analogy is often applied in modeling of complicated oscillation processes.

As a rule, damping force is proportional to the velocity of the process. In practice, this means that friction force may be equivalently replaced by a viscous damping force [1]. The dissipation of energy in the case of an equivalent damping is the same as the one caused by a real friction.

In this paper, we discuss the applicability of the fractional calculus tools for describing free vibration with damping. The fractional calculus reveals many effective applications in the modern theory of dissipative processes modeling [2,3]. An oscillator described by a motion equation with fractional derivatives (or integrals) is usually called a fractional oscillator. The concept of a fractional oscillator and the physical interpretation of a fractional integral were presumably first given in [4]. At present, many works devoted to fractional oscillators have been published. We refer to papers [5–15], in which different approaches to the concept and the current

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state of research on the topic can be found. Nevertheless, a detailed comparative analysis of fractional and classical models with the use of experimental data has not been conducted yet. In addition, there have been no explicit expressions offered so far for the relation between the order of fractional differentiation and measurable physical parameters. We consider these issues here. We compare the classical and fractional approaches and give a physical interpretation of parameters of the fractional model. In addition, our work contains the results of an experimental investigation of the vibration of a piezoelectric plate. We consider a free vibration and a vibration in polystyrene foam. The experiment showed good agreement with the model.

A detailed review of recent works in the field of fractional calculus application to dynamic problems of solid mechanics has been reported in [13]. In particular, the authors of this survey conclude that the use of the concept of "fractional" inertia in oscillation models is unpromising for an engineer since there are no explicit methods for calibrating the parameters of fractional models. Our paper fills, inter alia, this gap.

## 2. Classical model

First, as an example of a linear differential equation of vibration with damping, consider the spring pendulum equation given in the dimensionless form

$$\frac{d^2u}{d\xi^2} + 2D\frac{du}{d\xi} + u = 0, \quad (1)$$

$$u(\xi) = \frac{x(\xi)}{x_0}, \quad \xi = \omega_0 t,$$

where  $x(\xi)$  is the pendulum displacement,  $t$  stands for the time,  $x_0$  is the initial displacement,  $D$  is the damping ratio, and  $\omega_0$  is the natural frequency.

We add to Eq. (1) the initial conditions

$$u(0) = 1, \quad u'(0) = 0. \quad (2)$$

For the solution of the Cauchy problem, Eqs. (1) and (2) can be written in the form

$$u(\xi) = \exp(-D\xi) \left[ \cos(\sqrt{1-D^2}\xi) + \frac{D}{\sqrt{1-D^2}} \sin(\sqrt{1-D^2}\xi) \right]. \quad (3)$$

Let us point out some defects of the model in Eqs. (1)–(3). As follows from Eq. (3), the oscillation never stops in a finite time. This does not match actual physical processes. Moreover, the dissipative force in Eq. (1) cannot be derived from the principle of least action, because this principle is applicable to conservative systems only. Despite these defects, the model in Eqs. (1)–(3) is fruitfully applied to describe nature experiments.

## 3. Fractional model

Let us derive a fractional oscillator equation. We write an equation of motion in the form

$$p(x, t) = \int_0^t G(t-t')F(x, t')dt', \quad (4)$$

where  $p(x, t)$  is the momentum,  $F(x, t)$  is the force, and  $G(t)$  is the storage function. The storage function  $G(t)$  determines the momentum change in response to the acting force. For a system without dissipation, the storage

function is the Heaviside step function. This case corresponds to a conservative system with ideal memory. That is, the momentum does not change after an impulse force action. A system with dissipation gradually “forgets” the initial force action. The Heaviside step function smears out and, in simple case, transforms into the two-parameter power function

$$G(t) = \frac{1}{\Gamma(\alpha)} \frac{b}{(\omega_0 t)^{1-\alpha}}, \quad 0 < \alpha \leq 1, \quad (5)$$

where  $b$  is a positive number and  $\alpha$  is a parameter that determines the intensity of the energy dissipation. There is no dissipation if  $\alpha = 1$ . We substitute Eq. (5) into Eq. (4) and invert the fractional integral. From the formula for elastic force  $F(x, t) = -kx$  ( $k$  is the spring stiffness), we get the fractional oscillator equation

$$\frac{d^{1+\alpha}u}{d\xi^{1+\alpha}} + bu = 0, \quad (6)$$

where  $\frac{d^{1+\alpha}}{d\xi^{1+\alpha}}$  stands for the Caputo fractional derivative of order  $1 + \alpha$ :

$$\frac{d^{1+\alpha}u(\xi)}{d\xi^{1+\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^\xi \frac{d^2u(s)}{ds^2} \frac{ds}{(\xi-s)^\alpha}.$$

For the solution of the Cauchy problem Eq. (2) for Eq. (6) can be written in the form

$$u(\xi) = E_{1+\alpha}(-b\xi^{1+\alpha}). \quad (7)$$

Here  $E_\rho(z)$  is the Mittag-Leffler function, which is defined by

$$E_\rho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + 1)}, \quad \rho \geq 0.$$

It is known [16] that function  $E_\rho(z)$  has at most a finite number of real zeroes for  $\rho \in (0, 2)$ , which depends on  $\rho$ . This means that the fractional oscillation has a limited duration, in contrast to the classical model [15].

#### 4. Interpretation of parameters

Thus, we have two models of vibration with damping for a string pendulum. The former takes the dissipation into account by appending an additional term, namely the viscous friction force, to the motion equation (1). The latter uses the storage function (Eq. (5)) in order to consider the dissipation that gives the fractional differential equation (6). To compare solutions in Eq. (3) and Eq. (7), and to determine the numerical values of the parameters  $D$ ,  $b$ , and  $\alpha$ , providing the highest closeness of these solutions, a computing experiment was performed. The calculations were performed using Mathcad 15.0. The Mittag-Leffler function  $E_{1+\alpha}(-b\xi^{1+\alpha})$  in Eq. (7) was numerically calculated as an approximate solution to the Volterra-type integral equation

$$u(\xi) + \frac{b}{\Gamma(1+\alpha)} \int_0^\xi (\xi-\eta)^\alpha u(\eta) d\eta = 1.$$

This approach allowed us to calculate the value  $E_{1+\alpha}(-b\xi^{1+\alpha})$  with good accuracy for large  $\xi$  (in contrast to an approach based on summation of the power series, for example). The optimization was undertaken by the method of least squares with respect to all three parameters. The computation showed that the parameter  $b$  is close to unity for the wide range of  $D$  and  $\alpha$ , and, without loss of acceptable accuracy, we can set  $b = 1$  in Eq. (6). Taking into account the asymptotic formula for the Mittag-Leffler function (see [16], Theorem 1.5.2), the optimization with respect to the parameters  $D$  and  $\alpha$ , with  $b = 1$ , yields the following approximate relation between the parameters of the models:

$$\alpha = \frac{\pi}{\operatorname{arccotg}\left(-\frac{1}{2Q}\right)} - 1, \quad Q = \frac{\pi}{\delta}, \quad (8)$$

$$\omega = \omega_0 \sin\left(\frac{\pi}{1+\alpha}\right), \quad (9)$$

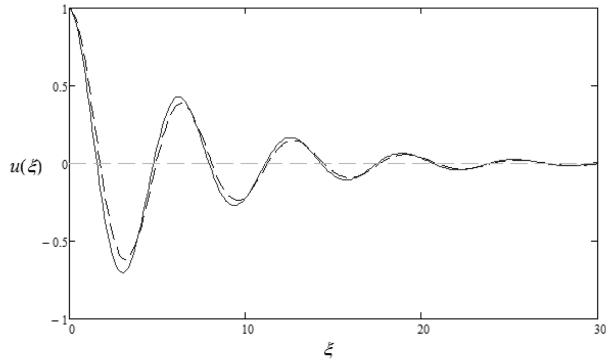
where  $Q = 1/(2D)$  is the Q-factor of the oscillatory system,  $\delta$  is the logarithmic decrement, and  $\omega$  is the damped natural frequency. It follows from the result of the computing experiment that the relations between Eq. (8) and Eq. (9) are held with quite good accuracy for  $Q > 3$ . As an example, Figure 1 shows the graphs of rapidly damped oscillations corresponding to Eq. (3) and Eq. (7) for  $b = 1$ ,  $D = 0.2$ , and  $\alpha = 0.745$ . The increase in the oscillator damping enlarges the quantitative difference between Eq. (3) and Eq. (7), but the qualitative correspondence remains. When the Eqs. (8) and (9) are satisfied, the quantitative difference between Eq. (3) and Eq. (7) becomes negligible for high-Q oscillatory systems. Expanding Eq. (8) into series in terms of powers of  $1/Q$ , we get [17]

$$\alpha \approx 1 - \frac{2}{\pi Q}. \quad (10)$$

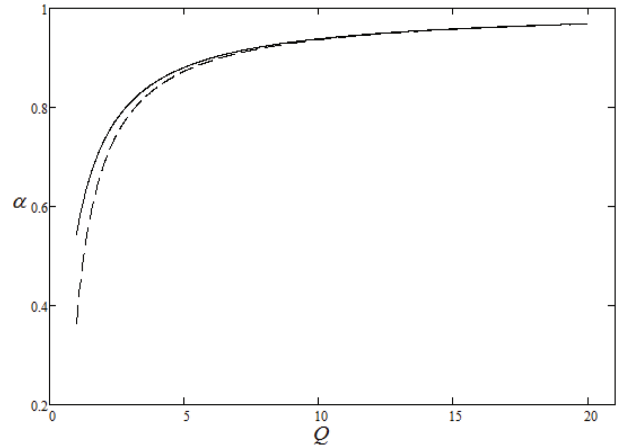
This formula gives a simple relation between the order of fractional differentiation and the Q-factor. Figure 2 shows the graphs of the function  $\alpha(Q)$  constructed using Eq. (8) and Eq. (10). From this figure, it can be seen that the approximate formula (Eq. (10)) can be used for  $Q \geq 10$ . It follows from the above that an experimental verification of the fractional oscillator model must be carried out for strongly damped oscillatory systems.

## 5. Experiment

To test the model under consideration, we conducted an experiment with damped vibration of a piezoelectric plate. The experiment was conducted under standard laboratory conditions (at temperature of 25 °C and atmospheric pressure of 100 kPa). Electric signals were generated and measured by an USB PC oscilloscope and a signal generator Velleman PCSGU250. We investigated the vibrations of the free plate and the vibrations of the plate in polypropylene foam. The plate used consists of a brass base with a diameter of 27 mm and a thickness of 0.1 mm, and a metal-coated piezoelectric element with a diameter of 18.5 mm and a thickness of 0.35 mm. A square wave signal of 2.3 kHz was fed to the plate. The waveform allowed us to observe both inverse and direct piezoelectric effects during the experiment. Due to the inverse piezoelectric effect, each level shift of the signal provoked a sharp displacement of the plate. This initiated the damped vibration of the plate, and the direct piezoelectric effect gave rise to an alternating electrical signal that reflected the natural vibrations of the plate. Figure 3 shows examples of the waveforms.

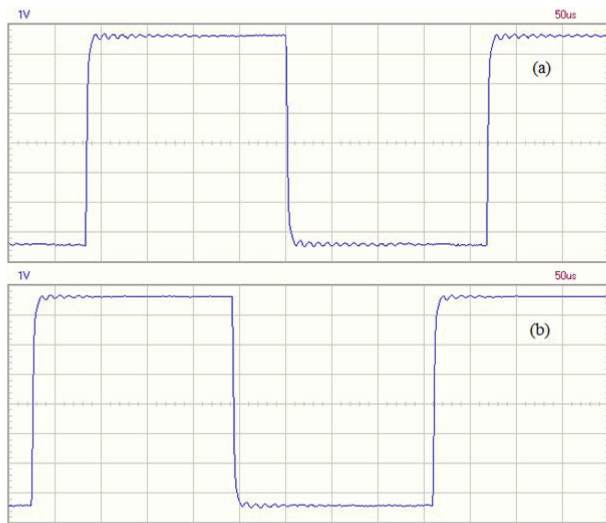


**Figure 1.** The damped vibration of a string pendulum: the dotted line corresponds to formula in Eq. (3); the solid line corresponds to formula in Eq. (7).

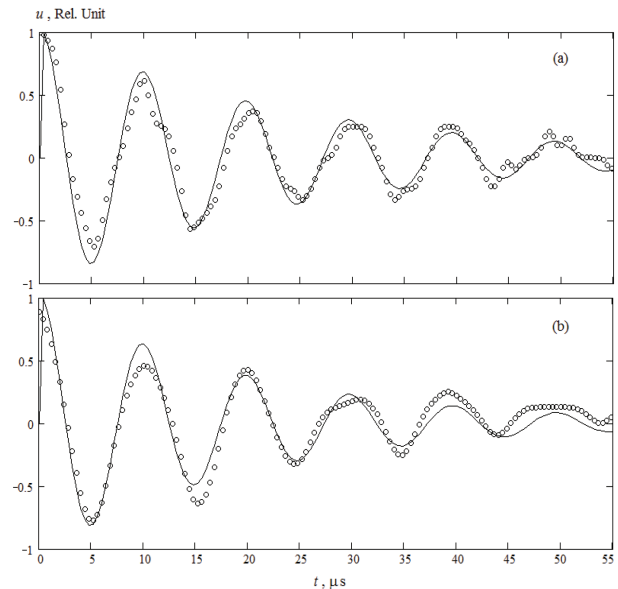


**Figure 2.** The dependence of the order of the fractional derivative in the oscillation equation in Eq. (6) on the Q-factor: the dotted line corresponds to formula in Eq. (8); the solid line corresponds to formula in Eq. (10).

Figure 4 shows the highlighted segments of the waveforms that correspond to the vibrations of the piezoelectric plate. The circles refer to the data of the experiment, and the solid curve refers to the values calculated by formula (7). As can be clearly seen, the fractional model agrees well with the experimental data. The damped natural frequency  $\omega$  and the fractional derivative of order  $\alpha$  for the free plate take the values  $0.632 \text{ rad}/\mu\text{s}$  and  $0.998$ , respectively. In the case of the plate in polystyrene foam, those are  $\omega = 0.629 \text{ rad}/\mu\text{s}$  and  $\alpha = 0.98$ . These values allow us to obtain the Q-factors for both cases:  $Q = 318.31$  and  $Q = 31.831$ , as well as the undamped natural frequency of the piezoelectric plate:  $f_0 = \omega_0/(2\pi) \approx 0.1 \text{ MHz}$ .



**Figure 3.** The waveforms: (a) corresponds to the free plate; (b) corresponds to the plate in the polystyrene foam.



**Figure 4.** The highlighted waveforms of the piezoelectric plate: (a) refers to the case of the free plate; (b) refers to the case of the plate placed in the polystyrene foam.

Thus, the experiment showed that the parameter  $\alpha$  in the model based on Eq. (6) characterizes energy dissipation and vibration damping. In practice, this can serve as a basis for a new method for studying oscillatory electromechanical processes using the fractional oscillator model and a Q-meter instrument.

## 6. Conclusions

Fractional differentiation can have a constructive role in modeling various oscillatory processes with damping. The fractional derivative in the equation of motion (Eq. (6)) takes into account energy dissipation, and formula in Eq. (8) gives a relation between the order of fractional differentiation and the Q-factor of the oscillatory system.

Moreover, this approach can be extended to describe dissipative processes in various oscillatory physical systems, such as vibration of solids or propagation of elastic waves in viscoelastic media. The use of Eq. (4) as the equation of motion with the storage function in Eq. (5) leads to the fractional diffusion-wave equation

$$\frac{\partial^{1+\alpha} u}{\partial t^{1+\alpha}} - \lambda \Delta u = f(\mathbf{r}, t). \quad (11)$$

Here  $\Delta$  denotes the Laplace operator and  $f(\mathbf{r}, t)$  is a source function. Eq. (11) can be used to simulate the propagation of sound waves in strongly scattering media. This equation also seems to be quite an appropriate alternative to the telegraph equation in electrical engineering. Investigations of various initial-boundary value problems for Eq. (11) broaden the opportunities for analytical study of the dissipation effect in problems of mechanics and electrodynamics.

## Acknowledgments

The reported study was funded by RFBR (project 18-51-45005) and DST (project INT/RUS/RFBR/308).

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