



The Hirota–Satsuma coupled KdV hierarchy on noncommutative phase-space

Abdelali SAIDI^{1*}, Moulay Brahim SEDRA^{1,2}

¹LaboSimo, Department of Physics, Faculty of Sciences, Ibn Tofail University, Kenitra, Morocco

²FST, Moulay Ismail University, Errachidia, Morocco

Received: 01.03.2019

Accepted/Published Online: 10.05.2019

Final Version: 12.06.2019

Abstract: In the present contribution, the Hirota–Satsuma coupled KdV hierarchy on noncommutative phase-space is investigated using the noncommutative extension of Lax pair generating technique. In particular, the explicit representation of its associated Lax pair operators is constructed. It is shown that the obtained results for phase-space noncommutativity case reduce back to the standard commutative case when θ goes to $\frac{1}{2}$.

Key words: Noncommutative geometry, Hirota–Satsuma coupled KdV system, Moyal deformation

1. Introduction

The following system of equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} \frac{1}{2}u_{xxx} + 3uu_x - 6vv_x \\ -v_{xxx} - 3uv_x \end{pmatrix}, \quad (1.1)$$

is known as the Hirota–Satsuma coupled KdV system (HS-coupled KdV) proposed by Hirota and Satsuma [1], which describes interactions of two long waves with different dispersion relations. In [2], Hirota and Satsuma showed that the HS-coupled KdV system is the four-reduction of the KP hierarchy and its soliton solutions can be derived from those of the KP equation. The Lax representation of this system has been constructed in [3] by Dodd and Fordy. Different properties of the HS-coupled KdV system have been constructed such as Backlund transformation [4], Darboux transformation [5–7], bilinear form [1, 8], and Painleve property [9, 10]

More recently, the noncommutative geometry (NCG) [11, 12] has attracted attention from many researchers in physics [13–25]. On noncommutative (NC) spaces, the noncommutativity of the coordinates is defined as follows

$$[x^i, x^j] = i\theta^{ij}, \quad (1.2)$$

where x^i, x^j are noncommuting coordinates, and θ^{ij} are real constants called *the NC parameters*.

On the other hand, the multiplication of two arbitrary functions $f(x, p)$ and $g(x, p)$ on the two dimensional NC phase-space has been shown to satisfy the star product law defined as follows: [20–25]

$$f(x, p) \star g(x, p) = \sum_{s=0}^{\infty} \sum_{i=0}^s \frac{\theta^s}{s!} (-)^i c_s^i (\partial_x^i \partial_p^{s-i} f) (\partial_x^{s-i} \partial_p^i g), \quad (1.3)$$

*Correspondence: saidi220@gmail.com

with $c_s^i = \frac{s!}{i!(s-i)!}$.

In this context, on NC spaces we define the Moyal bracket given by the following expression [20–25]:

$$\{ f(x, p), g(x, p) \}_\theta = \frac{f \star g - g \star f}{2\theta}. \tag{1.4}$$

In the following section, we make a short review of the noncommutative Lax pair generating technique [26–28] which is relevant for this work. This technique will be useful to derive the noncommutative extension of the HS-coupled KdV hierarchy and obtain its associated Lax pair operators. A more detailed analysis in the commutative case can be found in [29–32].

2. Noncommutative Hirota–Satsuma coupled KdV hierarchy

An NC partial differential equation which has the Lax representation can be reformulated as follows:

$$\{L, T + \partial_t\}_\theta = 0. \tag{2.1}$$

This equation and the associated pair of operators (L, T) are called *the NC Lax differential equation* and *the Lax pair*, respectively, with $\partial_t \equiv \frac{\partial}{\partial t}$.

The main idea of the NC Lax pair generating technique is to find a T- operator satisfying (2.1) for a given L- operator. For this we have to consider the following ansatz on the T- operator, namely

$$T = p^n \star L + \tilde{T}, \tag{2.2}$$

where p^n are nothing but the momentum operators. With this last ansatz, the problem reduces to find the \tilde{T} - operator.

Now, let us recall that the momentum Lax operator of the θ - deformed HS-coupled KdV hierarchy is defined as [3]

$$\begin{aligned} L &= (p^2 + u + v) \star (p^2 + u - v) \\ &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \end{aligned} \tag{2.3}$$

The T- operator can be written as follows:

$$T = p^{2n-3} \star L + \tilde{T}. \tag{2.4}$$

Straightforward algebraic computations based on the NC Lax pair generating technique lead to the explicit forms of the set of Moyal HS-coupled KdV hierarchy. The results are as follows:

- For $n = 0$, the NC Lax differential equation (2.1) leads the 1st-order NC HS-coupled KdV equations.

The NC Lax pair is given by

$$\begin{aligned} L &= (p^2 + u + v) \star (p^2 + u - v) \\ &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \end{aligned} \tag{2.5}$$

and

$$T = p^{-3} \star L + \tilde{T} \tag{2.6}$$

From the NC Lax differential equation (2.1), one finds

$$-\{L, \tilde{T}\}_\theta = (-2u_x - \frac{\dot{u}}{2\theta})p^2 + (4\theta v_{xx} + \dot{v}_x)p - 2\theta^2 u_{xxx} - 2uu_x + 2vv_x - \frac{1}{2}\theta \dot{u}_{xx} - \frac{u\dot{u}}{2\theta} + \frac{v\dot{v}}{2\theta}, \tag{2.7}$$

with the following system of 1st-order NC HS-coupled KdV equations

$$-\frac{\dot{u}}{2\theta} = 2u_x, \tag{2.8}$$

$$-\frac{\dot{v}}{2\theta} = 2v_x, \tag{2.9}$$

where $\dot{u} \equiv \frac{\partial u}{\partial t}$, $u_x \equiv \frac{\partial u}{\partial x}$, $u_{xx} \equiv \frac{\partial^2 u}{\partial x^2}$ and so on.

Hence, the NC Lax pair for the 1st-order NC HS-coupled KdV equations is explicitly given by

$$\begin{cases} L = p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \\ T = p \end{cases} \tag{2.10}$$

- For $n = 1$, the NC Lax differential equation (2.1) represents the 3rd-order NC HS-coupled KdV equations.

The NC Lax pair is given by

$$\begin{aligned} L &= (p^2 + u + v) \star (p^2 + u - v) \\ &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \end{aligned} \tag{2.11}$$

and

$$T = p^{-1} \star L + \tilde{T} \tag{2.12}$$

Then, we can take the \tilde{T} - operator as the following form

$$\tilde{T} = A.p + B, \tag{2.13}$$

where A and B are polynomials of $u, u_x, u_{xx} \dots etc.$ The NC Lax differential equation (2.1) takes the form

$$\begin{aligned} -\{L, \tilde{T}\}_\theta &= 2u_x p^4 - (8\theta u_{xx} + 4\theta v_{xx})p^3 + (6vv_x - 2uu_x - \frac{\dot{u}}{2\theta})p^2 - (8\theta^3 u_{4x} + 12\theta^3 v_{4x} \\ &\quad + 8\theta uu_{xx} + 8\theta uv_{xx} + 8\theta u_x v_x + \dot{v}_x)p - 2\theta^4 u_{5x} - 6\theta^2 uu_{3x} + 8\theta^2 v_x u_{xx} \\ &\quad + 22\theta^2 v_x v_{xx} - 10\theta^2 u_x u_{xx} + 2\theta^2 vv_{3x} - 4u_x u^2 + 4uvv_x - \frac{1}{2}\theta \dot{u}_{xx} - \frac{u\dot{u}}{2\theta} + \frac{v\dot{v}}{2\theta}. \end{aligned} \tag{2.14}$$

The left-hand side of Eq. (2.14) is explicitly given by

$$\begin{aligned}
 -\{L, \tilde{T}\}_\theta = & -4A_x p^4 - 4B_x p^3 + (2Au_x - 4uA_x - 4\theta^2 A_{xxx})p^2 + (4\theta A_x v_x - 4\theta A v_{xx} - 4\theta^2 B_{xxx} \\
 & - 4uB_x)p + 2\theta^2 Au_{xxx} + 2\theta^2 u_x A_{xx} + 2Auu_x - 2Avv_x + 4\theta B_x v_x
 \end{aligned} \quad (2.15)$$

Identifying Eqs. (2.14) and (2.15) leads to the following constrains equations:

$$A = -\frac{1}{2}u \quad ; \quad B = 2\theta u_x + \theta v_x \quad (2.16)$$

Hence, Eq. (2.13) becomes

$$\tilde{T} = -\frac{1}{2}up + 2\theta u_x + \theta v_x \quad (2.17)$$

Furthermore, one obtains the following system of 3rd-order NC HS-coupled KdV equations:

$$-\frac{\dot{u}}{2\theta} = 2\theta^2 u_{xxx} + 3uu_x - 6vv_x \quad (2.18)$$

$$-\frac{\dot{v}}{2\theta} = -4\theta^2 v_{xxx} - 3vv_x \quad (2.19)$$

This coincides with the standard HS-coupled KdV equations given in Eq. (1.1) once the limit $\theta \rightarrow \frac{1}{2}$ and the scaling transformation $-\frac{1}{2\theta} \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t}$ are performed. Consequently, the NC Lax pair associated to the θ -deformed 3rd-order HS-coupled KdV equations is explicitly given by

$$\begin{cases} L = p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \\ T = p^3 + \frac{3}{2}up - 3\theta v_x \end{cases} \quad (2.20)$$

Let us note that the same analysis used for $n = 0$ and $n = 1$ is actually extended to build the 5th-order, 7th-order and the 9th-order NC HS-coupled KdV equations and their associated Lax pair operators.

- Now, for $n = 2$, the NC Lax differential equation given in Eq. (2.1) yields the 5th-order NC HS-coupled KdV equations.

The NC Lax pair is given by

$$\begin{aligned}
 L = & (p^2 + u + v) \star (p^2 + u - v) \\
 = & p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2
 \end{aligned} \quad (2.21)$$

and

$$T = p \star L + \tilde{T} \quad (2.22)$$

From Eq. (2.1), the \tilde{T} - operator is explicitly given by

$$\tilde{T} = \frac{1}{2}up^3 - \theta(2u_x + v_x)p^2 + (4\theta^2 v_{xx} + 3\theta^2 u_{xx} + \frac{7}{8}u^2 - \frac{1}{4}v^2)p - \frac{5}{2}\theta uv_x - 2\theta uu_x - 2\theta^3 u_{xxx} + 2\theta vv_x - 5\theta^3 v_{xxx} \quad (2.23)$$

After calculation, we find the following system of 5th-order NC HS-coupled KdV equations:

$$-\frac{\dot{u}}{2\theta} = 2\theta^4 u_{5x} - 10\theta^2 v v_{xxx} + 5\theta^2 u u_{xxx} + 10\theta^2 u_x u_{xx} - 10\theta^2 v_x v_{xx} + \frac{15}{4} u_x u^2 - 5u v v_x - \frac{5}{2} u_x v^2 \quad (2.24)$$

$$-\frac{\dot{v}}{2\theta} = -8\theta^4 v_{5x} - 10\theta^2 u v_{xxx} - 10\theta^2 u_x v_{xx} - 5\theta^2 v_x u_{xx} - \frac{5}{4} v_x u^2 - \frac{5}{2} v_x v^2 \quad (2.25)$$

Hence, the NC Lax pair for the 5th-order NC HS-coupled KdV equations is explicitly given by

$$\begin{cases} L = p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \\ T = p^5 + \frac{5}{2} up^3 - 5\theta v_x p^2 + (5\theta^2 u_{xx} + \frac{15}{8} u^2 - \frac{5}{4} v^2) p - 5\theta^3 v_{xxx} - \frac{5}{2} \theta u v_x. \end{cases} \quad (2.26)$$

- Similarly, for $n = 3$, the NC Lax differential equation given in Eq. (2.1) becomes the 7th-order NC HS-coupled KdV equations.

The NC Lax pair is given by

$$\begin{aligned} L &= (p^2 + u + v) \star (p^2 + u - v) \\ &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \end{aligned} \quad (2.27)$$

and

$$T = p^3 \star L + \tilde{T}. \quad (2.28)$$

The \tilde{T} - operator is explicitly given by

$$\begin{aligned} \tilde{T} &= \frac{3}{2} up^5 - 3\theta(v_x + 2u_x)p^4 + (\frac{19}{2}\theta^2 u_{xx} + 12\theta^2 v_{xx} + \frac{27}{8}u^2 - \frac{3}{4}v^2)p^3 + (6\theta v v_x - 8\theta^3 u_{xxx} - 16\theta^3 v_{xxx} \\ &\quad - \frac{21}{2}\theta u v_x - 6\theta u u_x)p^2 + (4\theta^4 v_{4x} + \frac{9}{2}\theta^4 u_{4x} + \frac{5}{2}\theta^2 v_x^2 - 8\theta^2 v v_{xx} + \frac{23}{2}\theta^2 u u_{xx} - \frac{13}{8}\theta^2 u_x^2 \\ &\quad + \frac{35}{16}u^3 - \frac{21}{8}u v^2)p + 7\theta^5 v_{5x} - 2\theta^5 u_{5x} + 2\theta^3 v v_{xxx} + 6\theta^3 v_x v_{xx} + \frac{21}{4}\theta v_x v^2 - 2\theta^3 u u_{xxx} \\ &\quad - 6\theta^3 u_x u_{xx} + 21\theta^3 u_x v_{xx} - \frac{21}{8}\theta v_x u^2 + \frac{7}{2}\theta^3 u v_{xxx}. \end{aligned} \quad (2.29)$$

Finally, we get the following system of 7th-order NC HS-coupled KdV equations:

$$\begin{aligned} -\frac{\dot{u}}{2\theta} &= 2\theta^6 u_{7x} + 14\theta^4 v v_{5x} + 7\theta^4 u u_{5x} + 21\theta^4 u_x u_{4x} - 14\theta^4 v_x v_{4x} + 35\theta^4 u_{xx} u_{xxx} - \frac{21}{4} u u_x v^2 - \frac{21}{4} v v_x u^2 \\ &\quad - \frac{7}{2}\theta^2 u_{xxx} v^2 - 56\theta^4 v_{xx} v_{xxx} + \frac{35}{4}\theta^2 u_{xxx} u^2 + 14\theta^2 v u_x v_{xx} - 21\theta^2 u v_x v_{xx} + 35\theta^2 u u_x u_{xx} \\ &\quad + 7\theta^2 u v v_{xxx} - 7\theta^2 u_x v_x^2 + \frac{21}{2} v_x v^3 + \frac{35}{4}\theta^2 u_x^3 + \frac{35}{8} u_x u^3 \end{aligned} \quad (2.30)$$

$$\begin{aligned} -\frac{\dot{v}}{2\theta} &= 16\theta^6 v_{7x} + 28\theta^4 u v_{5x} + 56\theta^4 u_x v_{4x} + 7\theta^4 v_x u_{4x} + 42\theta^4 v_{xx} u_{xxx} + 70\theta^4 u_{xx} v_{xxx} + \frac{21}{2}\theta^2 v_{xxx} u^2 \\ &\quad + 21\theta^2 u u_x v_{xx} + \frac{7}{2}\theta^2 u v_x u_{xx} - \frac{7}{4}\theta^2 v_x u_x^2 + 14\theta^2 v_x^3 + 35\theta^2 v v_x v_{xx} + 7\theta^2 v_{xxx} v^2 \\ &\quad - \frac{7}{8} v_x u^3 + \frac{21}{4} u v_x v^2. \end{aligned} \quad (2.31)$$

Consequently, the NC Lax pair for the 7th-order NC HS-coupled KdV equations is explicitly given by

$$\left\{ \begin{aligned} L &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \\ T &= p^7 + \frac{7}{2}up^5 - 7\theta v_x p^4 + \left(\frac{35}{2}\theta^2 u_{xx} + \frac{35}{8}u^2 - \frac{7}{4}v^2\right)p^3 - (28\theta^3 v_{xxx} + \frac{21}{2}\theta uv_x)p^2 \\ &\quad + \left(\frac{21}{2}\theta^4 u_{4x} - \frac{7}{2}\theta^2 v_x^2 + \frac{35}{8}\theta^2 u_x^2 + \frac{35}{2}\theta^2 uu_{xx} - 14\theta^2 vv_{xx} - \frac{21}{8}uv^2 + \frac{35}{16}u^3\right)p \\ &\quad + \frac{7}{2}\theta^3 uv_{xxx} + 21\theta^3 u_x v_{xx} + 7\theta^5 v_{5x} + \frac{21}{4}\theta v_x v^2 - \frac{21}{8}\theta v_x u^2 \end{aligned} \right. \quad (2.32)$$

- For $n = 4$, the NC Lax differential equation given in Eq. (2.1) is the 9th-order NC HS-coupled KdV equations.

The NC Lax pair is given by

$$\begin{aligned} L &= (p^2 + u + v) \star (p^2 + u - v) \\ &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \end{aligned} \quad (2.33)$$

and

$$T = p^5 \star L + \tilde{T}. \quad (2.34)$$

The 9th-order NC HS-coupled KdV equations are explicitly given by

$$\begin{aligned} -\frac{\dot{u}}{2\theta} &= 2\theta^8 u_{9x} + 45\theta^2 vv_x^3 - \frac{135}{16}v^2 u_x u^2 + \frac{45}{4}uv_x v^3 - \frac{45}{8}vv_x u^3 + 9\theta^6 uu_{7x} - 18\theta^6 v_x v_{6x} \\ &\quad + \frac{105}{8}\theta^2 u_{xxx} u^3 + 84\theta^6 u_{xx} u_{5x} + 126\theta^6 u_{xxx} u_{4x} + \frac{63}{4}\theta^4 u_{5x} u^2 - 102\theta^6 v_{xx} v_{5x} \\ &\quad + \frac{483}{4}\theta^4 u_{xxx} u_x^2 - 30\theta^4 u_{xxx} v_x^2 + 36\theta^6 u_x u_{6x} - \frac{9}{2}\theta^4 u_{5x} v^2 + \frac{651}{4}\theta^4 u_x u_{xx}^2 \\ &\quad - 150\theta^6 v_{xxx} v_{4x} - 93\theta^4 u_x v_{xx}^2 + 30\theta^6 vv_{7x} + \frac{45}{2}\theta^2 v_{xxx} v^3 + \frac{45}{16}u_x v^4 \\ &\quad + \frac{315}{64}u_x u^4 - \frac{15}{4}\theta^2 vv_{3x} u^2 - \frac{45}{4}\theta^2 uu_{xxx} v^2 - \frac{135}{4}\theta^2 v_x v_{xx} u^2 + \frac{315}{2}\theta^4 uu_{xx} u_{xxx} \\ &\quad - 120\theta^4 uv_{xx} v_{xxx} + \frac{225}{2}\theta^2 v_x v_{xx} v^2 - 45\theta^4 uv_x v_{4x} + \frac{189}{2}\theta^4 uu_x u_{4x} - 84\theta^4 u_x v_x v_{xxx} \\ &\quad + 33\theta^4 uvv_{5x} - 3\theta^4 vv_x u_{4x} + \frac{315}{4}\theta^2 u_x u_{xx} u^2 + 105\theta^4 vv_{xxx} u_{xx} + 45\theta^4 vv_{xx} u_{xxx} \\ &\quad - 105\theta^4 v_x v_{xx} u_{xx} + 99\theta^4 vv_x v_{4x} - \frac{45}{2}\theta^2 u_x u_{xx} v^2 - \frac{75}{2}\theta^2 uu_x v_x^2 - \frac{75}{4}\theta^2 vv_x u_x^2 \\ &\quad - 30\theta^2 uu_{xx} vv_x - 15\theta^2 uu_x vv_{xx} + \frac{315}{8}\theta^2 uu_x^3 \end{aligned} \quad (2.35)$$

$$\begin{aligned}
 -\frac{\dot{v}}{2\theta} = & 32\theta^8 v_{9x} + \frac{195}{2}\theta^2 uvv_x v_{xx} + 105\theta^4 uu_{xxx} v_{xx} + 240\theta^4 vv_{xx} v_{xxx} + 120\theta^4 u_x u_{xx} v_{xx} \\
 & + 180\theta^4 uu_x v_{4x} + \frac{33}{2}\theta^4 uu_{4x} v_x + 12\theta^4 v_x u_x u_{xxx} + \frac{45}{2}\theta^2 uv_{xxx} v^2 + \frac{45}{2}\theta^2 u_x v_{xx} v^2 \\
 & + \frac{45}{4}\theta^2 u_x v_{xx} u^2 + \frac{45}{4}\theta^2 u_{xx} v_x v^2 - \frac{15}{8}\theta^2 u_{xx} v_x u^2 + 195\theta^4 uu_{xx} v_{xxx} + 129\theta^4 vv_x v_{4x} \\
 & - \frac{45}{64}v_x u^4 + \frac{45}{16}v_x v^4 + 114\theta^6 v_{xx} u_{5x} + 480\theta^6 u_{xxx} v_{4x} + 216\theta^6 u_x v_{6x} + 15\theta^6 v_x u_{6x} \\
 & + 72\theta^6 uv_{7x} + \frac{21}{4}\theta^4 v_x u_{xx}^2 + 18\theta^4 v_{5x} v^2 + \frac{195}{2}\theta^4 v_{xxx} u_x^2 + 318\theta^6 v_{xxx} u_{4x} + 246\theta^4 v_{xxx} v_x^2 \\
 & + 357\theta^4 v_x v_{xx}^2 + 420\theta^6 u_{xx} v_{5x} - \frac{45}{8}\theta^2 uvv_x u_x^2 + \frac{75}{2}\theta^2 vv_x v_x^2 + 30\theta^2 uv_x^3 + \frac{45}{16}v_x u^2 v^2 \\
 & + 45\theta^4 v_{5x} u^2 + \frac{15}{4}\theta^2 v_{xxx} u^3
 \end{aligned} \tag{2.36}$$

Finally, the NC Lax pair for the 9th-order NC HS-coupled KdV equations is explicitly given by

$$\left\{ \begin{aligned}
 L &= p^4 + 2up^2 - 4\theta v_x p + 2\theta^2 u_{xx} + u^2 - v^2 \\
 T &= p^9 + \frac{9}{2}up^7 - 9\theta v_x p^6 + (42\theta^2 u_{xx} + \frac{63}{8}u^2 - \frac{9}{4}v^2)p^5 - (\frac{45}{2}\theta uv_x + 75\theta^3 v_{xxx})p^4 + (\frac{105}{16}u^3 \\
 &+ \frac{105}{8}\theta^2 u_x^2 + \frac{315}{4}\theta^2 uu_{xx} + 63\theta^4 u_{4x} - \frac{45}{8}uv^2 - 15\theta^2 v_x^2 - \frac{75}{2}\theta^2 vv_{xx})p^3 + (\frac{45}{4}\theta v_x v^2 \\
 &- 51\theta^5 v_{5x} - \frac{105}{2}\theta^3 v_x u_{xx} + 45\theta^3 u_x v_{xx} - 60\theta^3 uv_{xxx} - \frac{135}{8}\theta v_x u^2)p^2 + (\frac{315}{16}\theta^2 uu_x^2 \\
 &+ \frac{315}{128}u^4 + \frac{45}{32}v^4 - \frac{75}{4}\theta^2 uv_x^2 - 30\theta^2 uvv_{xx} - \frac{15}{4}\theta^2 u_x vv_x - \frac{45}{4}\theta^2 u_{xx} v^2 + \frac{315}{8}\theta^2 u_{xx} u^2 \\
 &+ \frac{189}{4}\theta^4 u_x u_{xxx} + \frac{189}{4}\theta^4 uu_{4x} - 42\theta^4 v_x v_{xxx} - \frac{51}{2}\theta^4 vv_{4x} - \frac{135}{32}u^2 v^2 + \frac{651}{8}\theta^4 u_x^2 \\
 &+ 18\theta^6 u_{6x} - \frac{183}{2}\theta^4 v_{xx}^2)p - 15\theta^3 uu_{xx} v_x + \frac{45}{2}\theta^3 uu_x v_{xx} + 45\theta^3 vv_x v_{xx} + \frac{45}{8}\theta uv_x v^2 \\
 &- \frac{45}{8}\theta^3 v_x u_x^2 + \frac{105}{2}\theta^5 u_{xx} v_{xxx} - \frac{3}{2}\theta^5 v_x u_{4x} + \frac{33}{2}\theta^5 uv_{5x} + 75\theta^5 u_x v_{4x} + 60\theta^5 v_{xx} u_{xxx} \\
 &- \frac{15}{8}\theta^3 v_{xxx} u^2 + \frac{45}{4}\theta^3 v_{xxx} v^2 - \frac{45}{16}\theta v_x u^3 + 15\theta^7 v_{7x} + \frac{45}{2}\theta^3 v_x^3
 \end{aligned} \right. \tag{2.37}$$

3. Conclusion

In commutative case, it is well known that the existence of Lax pair operators (L, T) for nonlinear differential equations is a strong indication of integrability, that is, the existence of a complete set of conservation laws, of multisoliton solutions and so on. In this context, many efforts have been devoted to extend the Lax pair operators for several $(1 + 1)$ and $(2 + 1)$ -dimensional integrable systems on noncommutative spaces [24–28]. The noncommutative extension of Lax pair generating technique is one of the most powerful methods to do so.

In this paper, we have presented a systematic study of HS-coupled KdV hierarchy on noncommutative phase-space by using the noncommutative Lax pair generating technique. We have obtained a set of higher-order noncommutative HS-coupled KdV equations which are also integrable equations. This integrability is due to

the existence of a noncommutative Lax pair operators (L, T) which is explicitly derived. Also, we have shown that the obtained results reduce to the commutative case once the limit $\theta \rightarrow \frac{1}{2}$ and the scaling transformation $-\frac{1}{2\theta} \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t}$ are considered.

References

- [1] Hirota R, Satsuma J. Soliton solutions of a coupled Korteweg-de Vries equation. *Physics Letters A* 1981; 85 (8-9): 407-408. doi: 10.1016/0375-9601(81)90423-0
- [2] Hirota R, Satsuma J. A coupled KdV equation is one case of the four-reduction of the KP hierarchy. *Journal of the Physical Society of Japan* 1982; 51 (10): 3390-3397. doi: 10.1143/JPSJ.51.3390
- [3] Dodd R, Fordy A. On the integrability of a system of coupled KdV equations. *Physics Letters A* 1982; 89 (4): 168-170. doi: 10.1016/0375-9601(82)90199-2
- [4] Levi D. A hierarchy of coupled Korteweg-de Vries equations. *Physics Letters A* 1983; 95 (1): 7-10. doi: 10.1016/0375-9601(83)90766-1
- [5] Leble SB, Ustinov NV. Darboux transforms, deep reductions and solitons. *Journal of Physics A: Mathematical and General* 1993; 26 (19): 5007-5016. doi: 10.1088/0305-4470/26/19/029
- [6] Hu HC, Liu QP. New darboux transformation for Hirota–Satsuma coupled KdV system. *Chaos, Solitons and Fractals* 2003; 17 (5): 921-928. doi: 10.1016/S0960-0779(02)00309-0
- [7] Hu HC, Liu Y. New positon, negaton and complexiton solutions for the Hirota–Satsuma coupled KdV system. *Physics Letters A* 2008; 372 (36): 5795-5798. doi: 10.1016/j.physleta.2008.07.030
- [8] Tam HW, Ma WX, Hu XB, Wang DL. The Hirota-Satsuma coupled KdV equation and a coupled Ito system revisited. *Journal of the Physical Society of Japan* 2000; 69 (1): 45-52. doi: 10.1143/JPSJ.69.45
- [9] Weiss J. The sine-Gordon equations: Complete and partial integrability. *Journal of Mathematical Physics* 1984; 25 (7): 2226. doi: 10.1063/1.526415
- [10] Weiss J. Modified equations, rational solutions, and the Painlevé property for the Kadomtsev–Petviashvili and Hirota–Satsuma equations. *Journal of Mathematical Physics* 1985; 26 (9): 2174. doi: 10.1063/1.526841
- [11] Connes A. *Noncommutative geometry*. San Diego, CA, USA: Academic Press, 1994.
- [12] Connes A, Douglas MR, Schwarz A. Noncommutative geometry and Matrix theory. *Journal of High Energy Physics* 1998; 268 (2): 03. doi: 10.1088/1126-6708/1998/02/003
- [13] Madore J. *An Introduction to Noncommutative Geometry and its Physical Applications*. Cambridge, UK: Cambridge University Press, 1999.
- [14] Landi G. *Introduction to Noncommutative Spaces and their Geometries*. Berlin, Germany: Springer-Verlag, 1997.
- [15] Green MB, Schwarz JH, Witten E. *Superstring Theory*. Cambridge, UK: Cambridge University Press, 1987.
- [16] Polchinski J. *String Theory*. Cambridge, UK: Cambridge University Press, 1998.
- [17] Seiberg N, Witten E. String theory and noncommutative geometry. *Journal of High Energy Physics* 1999; 1999 (09): 032. doi: 10.1088/1126-6708/1999/09/032
- [18] Douglas MR, Hull C. D-branes and the noncommutative torus. *Journal of High Energy Physics* 1998; 1998 (02): 008. doi: 10.1088/1126-6708/1998/02/008
- [19] Taylor W. D-brane field theory on compact spaces. *Physics Letters B* 1997; 394 (3-4): 283-287. doi: 10.1016/S0370-2693(97)00033-6
- [20] Das A, Popowicz Z. Properties of Moyal–Lax representation. *Physics Letters B* 2001; 510 (1-4): 264-270. doi: 10.1016/S0370-2693(01)00561-5

- [21] Boulahoual A, Sedra MB. Non-standard extended noncommutativity of coordinates. *Chinese Journal of Physics* 2004; 42 (5): 591-597.
- [22] Boulahoual A, Sedra MB. The Moyal momentum algebra. *African Journal of Mathematical Physics* 2005; 510: 111-113.
- [23] Boulahoual A, Sedra MB. The Moyal momentum algebra applied to θ -deformed 2d conformal models and KdV-hierarchies. *Chinese Journal of Physics* 2005; 43 (3): 408-444.
- [24] Sedra MB. Moyal noncommutative integrability and the Burgers–KdV mapping. *Nuclear Physics B* 2006; 740 (3): 243-270. doi: 10.1016/j.nuclphysb.2006.01.005
- [25] Dafounansou O, El boukili A, Sedra MB. Some aspects of Moyal deformed integrable systems. *Chinese Journal of Physics* 2006; 44 (4): 274-289.
- [26] Hamanaka M, Toda K. Towards noncommutative integrable systems. *Physics Letters A* 2003; 316 (1-2): 77-83. doi: 10.1016/S0375-9601(03)01138-1
- [27] Hamanaka M, Toda K. Noncommutative Burgers equation. *Journal of Physics A: Mathematical and General* 2003; 36 (48): 11981-11998. doi: 10.1088/0305-4470/36/48/006
- [28] Toda K. Extensions of soliton equations to non-commutative $(2 + 1)$ dimensions. In: *Proceedings of workshop on Integrable Theories, Solitons and Duality, Sao Paulo, Brazil; 2002.* pp. 038.
- [29] Toda K, Yu SJ. The investigation into the Schwarz–Korteweg–de Vries equation and the Schwarz derivative in $(2 + 1)$ dimensions. *Journal of Mathematical Physics* 2000; 41 (7): 4747. doi: 10.1063/1.533374
- [30] Toda K, Yu SJ. The investigation into new equations in $(2 + 1)$ dimensions. *Journal of Nonlinear Mathematical Physics* 2001; 08 (1): 272-277. doi: 10.2991/jnmp.2001.8.s.47
- [31] Toda K, Yu SJ. Note on an extension of the CDF equation to $(2 + 1)$ dimensions. *Reports on Mathematical Physics* 2001; 48 (1-2): 255-262. doi: 10.1016/S0034-4877(01)80086-0
- [32] Toda K, Yu SJ. A study of the construction of equations in $(2 + 1)$ dimensions. *Inverse Problems* 2001; 17 (4): 1053. doi: 10.1088/0266-5611/17/4/331