Turkish Journal of Physics
http://journals.tubitak.gov.tr/physics/
Research Article
Turk J Phys
(2019) 43: 393 – 409
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doi:10.3906/fiz-1902-17

Continuous Virasoro algebra in open string field theory

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Received: 19.02.2019 • **Accepted/Published Online:** 19.06.2019 • **Final Version:** 02.08.2019

Abstract: It is known that the Takahashi–Tanimoto identity-based solution in open string field theory derives a kinetic operator which is a sum of twisted Virasoro generators. Applying the infinite circumstance description of conformal field theory, we derive continuous Virasoro algebra associated with the kinetic operator. Mode expansions of Virasoro generators and the modified BRST charge are given.

Key words: String field theory, conformal field theory, D-branes

1. Introduction

Understanding the nature of the tachyon vacuum has been an important issue in open string field theory (OSFT). It is well known that the wedge-based analytic solution [1–3] successfully explains the absence of open strings around the tachyon vacuum [4]. However, the physics at the tachyon vacuum has not yet been fully understood. Usually, an OSFT shifted by a classical solution defines a boundary conformal field theory (BCFT) which is different from the reference BCFT associated with the original OSFT. However, such BCFT for the tachyon vacuum is expected to be irregular since there are no more boundaries due to the absence of open strings. This leads to a question: what kind of physics described by the OSFT at the tachyon vacuum? More precisely, is there any (two dimensional) field theory associated with the tachyon vacuum? The analysis performed in [4] do not provide any information about this question since the cohomology simply vanishes. Number of attempts had been made to answer this question. They are explanation in terms of shrunken boundaries [5–11], deformation in ghost sector [12] and D-gD pair [13]. In spite of these efforts, it is fair to say that we do not yet have definite answer to the question raised above.

In this paper, we add one more attempt to the above list by applying the technique which has been developed in rather different context. That is so-called sine square deformation (SSD) which was originally introduced to reduce the boundary effect of the one dimensional open spin lattice [14]. The authors of [14] examined a specific boundary condition by deforming the open lattice Hamiltonian. They found that the ground state energy becomes almost identical to that of periodic lattice. The coincidence between open and periodic systems observed in SSD somewhat resembles tachyon condensation, where OSFT is expected to be deformed into closed string theory. Subsequently, the deformed Hamiltonian on the open spin lattice was interpreted as a Hamiltonian of bulk conformal field theory [15]:

$$H = \left(L_0 - \frac{1}{2}L_1 - \frac{1}{2}L_{-1}\right) + \left(\bar{L}_0 - \frac{1}{2}\bar{L}_1 - \frac{1}{2}\bar{L}_{-1}\right), \tag{1}$$

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where L_n is the Virasoro generator. The coincidence of ground state energy between open and periodic boundary conditions is explained by the fact that L_1 and L_{-1} vanish on the $SL(2,C)$ invariant vacuum. Furthermore, the spectrum of this Hamiltonian was explored in [16–18]. In [18], the authors developed the formalism of dipolar quantization in which bulk conformal fields are expanded by continuous mode numbers instead of discrete one. The Fourier modes of energy momentum tensor correspond to the continuously labeled Virasoro generators; they obey the commutation relation

$$[\mathcal{L}_\kappa, \mathcal{L}_\lambda] = (\kappa - \lambda)\mathcal{L}_{\kappa+\lambda}, \tag{2}$$

where κ and λ are *real numbers* rather than integers. And also, they called their formalism "infinite circumstance limit" of a CFT since the continuous label of Fourier modes indicates a system with infinite size. In fact, the authors of [18] presented a formula that embeds the infinite parameter to the complex plane and drew equal time contours derived from the formula.

It is not difficult to find resemblance between SSD and OSFT. It is known that the identity-based scalar solution of Takahashi and Tanimoto [19] yields the following kinetic operator upon gauge fixing [8]:

$$\mathcal{L}'_0 = \frac{1}{2}L'_0 - \frac{1}{4}L'_2 - \frac{1}{4}L'_{-2} + \frac{3}{2} \tag{3}$$

where $L'_n = L_n + nq_n + \delta_{n,0}$ is the twisted Virasoro generator [5]. In fact, this kinetic operator exactly coincides with the Z_2 symmetric deformation studied in SSD literature [18, 20]. The infinite circumstance formalism can be applied to this kinetic operator since L'_0, L'_2, L'_{-2} form $SL(2,R)$ algebra as L_0, L_1, L_{-1} of SSD does.

The aim of this paper is to derive the algebra associated with the modified BRST charge generated by the Takahashi–Tanimoto solution, by employing the formalism developed in [18]. This aim is accomplished by identifying eigenmodes of gauge fixed kinetic operator $\mathcal{L}'_0$, which turns out to have continuous mode numbers. Outline of this paper is as follows. In Section 2, we review the results of [8]. The kinetic operator (3) is obtained by gauge fixing the identity-based solution of [19]. Sections 3 and 4 are devoted to quantum analysis along the line with [18]. Section 3 provides basic tools for our investigation. Worldsheet geometry generated by $\mathcal{L}'_0$ (3) is described in detail. Then, continuous Fourier modes for conformal fields are introduced. Continuous Virasoro generators are derived from the modified BRST charge. Finally, it is shown that the continuous generators obey Virasoro algebra without anomaly. Section 4 is devoted to further investigation of the continuous modes. The mode expansion of modified BRST charge will be given. It turns out that all formulas obtained are "continuous version" of the discrete one. We conclude in Section 5 with some speculations. Please note that while completing the manuscript, we found a paper by Kishimoto et al. [21], which deals with same classical solution using SSD formalism. In the first version of our manuscript on ArXiv, the splitting of the theory into holomorphic and antiholomorphic sectors pointed out in [21] was not recognized; therefore, the description of the integration contour was inconsistent. The description of the contour is fixed in this article. We focus on the holomorphic sector.

2. Modified BRST charge

The identity-based solution of Takahashi and Tanimoto [19] is obtained by integrating primary fields multiplied by specific functions along "left" half of an open string:

$$\Psi_{TT} = \left[\int_{\gamma_L} \frac{dz}{2\pi i}\,(F(z)-1)\,j_B(z) - \int_{\gamma_L} \frac{dz}{2\pi i}\frac{(\partial F(z))^2}{F(z)}c(z)\right]|I\rangle\,, \tag{4}$$

where z represents the worldsheet coordinate of open string BCFT, which is taken to be entire complex plane in virtue of the doubling trick, and $j_B(z)$ and $c(z)$ are BRST current and conformal ghost respectively. $|I\rangle$ is the identity string field. The path γ_L is taken to be right half of the unit circle. The function $F(z)$ is explicitly chosen to be

$$\begin{aligned} F(z) &= 1 - \frac{1}{4}\left(z + \frac{1}{z}\right)^2 \\ &= \frac{1}{2} - \frac{1}{4}z^2 - \frac{1}{4}z^{-2}. \end{aligned} \tag{5}$$

An advantage of this solution is the use of left half integrated operators and identity string field, which identifies noncommutative star product between string fields with conventional operator algebra of BCFT. It can be shown that this solution satisfies the equation of motion of OSFT [19]. The OSFT action expanded around this solution is characterized by the modified BRST charge

$$Q' = \oint_\gamma \frac{dz}{2\pi i} F(z) j_B(z) - \oint_\gamma \frac{dz}{2\pi i} \frac{(\partial F(z))^2}{F(z)} c(z), \tag{6}$$

where γ is the unit circle enclosing the origin $z = 0$. Note that this circle represents an equal time contour in radial quantization therefore can be shrunk to arbitrary small radius. The contour integral can be evaluated by expanding $j_B(z)$ and $c(z)$ into Laurant series and picking up pole residues:

$$Q' = \frac{1}{2}Q_B - \frac{1}{4}(Q_2 + Q_{-2}) + 2c_0 + c_2 + c_{-2}, \tag{7}$$

where $j_B(z) = \sum_n Q_n z^{-m-1}$ and $c(z) = \sum_n c_n z^{-n+1}$. The gauge fixed kinetic operator in Siegel gauge is derived from the commutator between Q' and antighost zero mode b_0:

$$\begin{aligned} \mathcal{L}'_0 &= \{Q', b_0\} \\ &= \frac{1}{2}L'_0 - \frac{1}{4}\left(L'_2 + L'_{-2}\right) + \frac{3}{2}, \end{aligned} \tag{8}$$

where L'_n is the twisted Virasoro generator defined by

$$L'_n = L_n + nq_n + \delta_{n,0}, \tag{9}$$

and q_n is the nth mode of the ghost number current defined by

$$j_g(z) = c(z)b(z) = \sum_n q_n z^{-n+1}. \tag{10}$$

The total central charge of the matter and twisted ghost CFT is 24 rather than being zero. This value of central charge can be derived from Eq. (9) directly. Alternatively, it can be derived from the OPE between twisted energy momentum tensor in $\rho = \log z$ coordinate:

$$T'(\rho) = T(\rho) - \partial_\rho j_g(\rho), \tag{11}$$

or in z coordinate

$$T'(z) = T(z) - \frac{1}{z}\partial(z j_g(z)). \tag{12}$$

The twisted energy momentum tensor defined above is consistent with twisted ghost pair $c'(z)$ and $b'(z)$ rather than the conventional one. The twisted and untwisted ghost pairs are related by

$$c'(z) = z^{-1}c(z) = \sum_n c_n z^{-n}, \tag{13}$$

$$b'(z) = zb(z) = \sum_n b_n z^{-n-1}. \tag{14}$$

This correspondence between twisted and untwisted ghost CFTs will be used frequently.

The spectrum of the deformed theory corresponds to the cohomology of modified charge Q'. The cohomology was first derived by the authors of [22] in the gauge unfixed setting. Subsequently, the cohomology was also derived by authors of [8] in Siegel gauge. In the derivation of the gauge fixed cohomology, the identity

$$Q' = -\frac{1}{4}U'Q_B^{(2)}U'^{-1} \tag{15}$$

plays crucial role. The operator $U' = e^{1/2L'_{-2}}$ is a finite conformal transformation and $Q_B^{(2)}$ is the shifted charge obtained by applying the replacement

$$c_n \to c_{n+2}, \quad b_n \to b_{n-2} \tag{16}$$

to the original BRST charge. The cohomology of Q' is obtained by mapping the cohomology of $Q^{(2)}$, which is nothing but a shifted version of the original cohomology of Q_B. In this way, the cohomology of Q' is identified as

$$|\Psi\rangle_{TZ} = U'\left(|P\rangle \otimes b_{-2}|0\rangle + |P'\rangle \otimes |0\rangle\right), \tag{17}$$

where $|P\rangle$ and $|P'\rangle$ are DDF states in matter CFT and $|0\rangle$ is the conventional $SL(2,R)$ vacuum of the ghost CFT defined by $c_n|0\rangle = 0 \quad (n \geq 2)$ and $b_n|0\rangle = 0 \quad (n \geq -1)$. Surprisingly, the existence of nontrivial cohomology does not contradict with Sen's conjecture that identifies the classical solution in Eq. (4) as the tachyon vacuum. This is simply because the cohomology in Eq. (17) does not contribute to any pertubative amplitudes due to mismatch of ghost number [19].

3. Continuous Virasoro algebra

3.1. Geometrical analysis

In order to reformulate the system described by the kinetic operator in Eq. (8), we will identify the nature of time evolution generated by it. The twist involved in Eq. (8) is irrelevant for this purpose. Thus, we only need to consider the untwisted generator

$$\mathcal{L}_0 = \frac{1}{2}L_0 - \frac{1}{4}\left(L_2 + L_{-2}\right). \tag{18}$$

Following [18], we introduce a classical representation of $\mathcal{L}_0$:

$$l_0 = -g(z)\frac{\partial}{\partial z}, \tag{19}$$

where the function $g(z)$ is chosen to be

$$g(z) = zF(z) = \frac{1}{2}z - \frac{1}{4}z^3 - \frac{1}{4}z^{-1}. \tag{20}$$

Next, we will find an eigenfunction of l_0 which satisfies

$$g(z)\partial_z f_\kappa(z) = \kappa f_\kappa(z). \tag{21}$$

A solution of the above equation is easily found to be

$$f_\kappa(z) = e^{\kappa \int^z \frac{dz'}{g(z')}} = e^{\frac{2\kappa}{z^2-1}}, \tag{22}$$

where z is the radial coordinate of conventional CFT. Note that the eigenfunction is regular for any real κ. Therefore, l_0 exhibits continuous spectrum. Furthermore, the eigenfunctions $f_\kappa(z)$ can be used to define continuously indexed generators

$$l_\kappa = -g(z)f_\kappa(z)\frac{\partial}{\partial z}. \tag{23}$$

It is easily confirmed that they form continuous Witt algebra

$$[l_\kappa, l_\lambda] = (\kappa - \lambda)l_{\kappa+\lambda}. \tag{24}$$

Let us now describe time evolution generated by l_0. Note that Eq. (19) is the generator of time translation which acts on a conformal field. We also note that the worldsheet time t should be paired with another parameter s along a string to define complex coordinate $\rho = t + is$. We require

$$\frac{\partial}{\partial \rho} = g(z)\frac{\partial}{\partial z}. \tag{25}$$

This defines a relation between complex coordinates z and ρ. The z dependence of ρ can be obtained by rewriting above equation to

$$\frac{d\rho}{dz} = \frac{1}{g(z)}, \tag{26}$$

and integrating this with respect to z. Thus, we obtain

$$\rho = \frac{2}{z^2-1}. \tag{27}$$

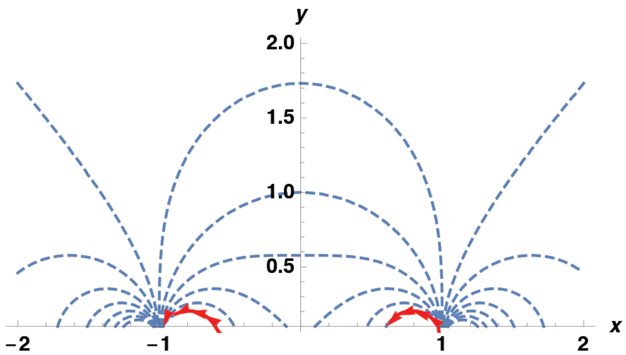
Let us investigate the equal time contours of Eq. (27). Decomposing right hand side of Eq. (27) into real and imaginary parts with $z = x + iy$ and comparing them to left hand side, we obtain

$$t = \frac{2(x^2-y^2-1)}{(x^2+y^2)^2 - 2(x^2-y^2)+1}, \tag{28}$$

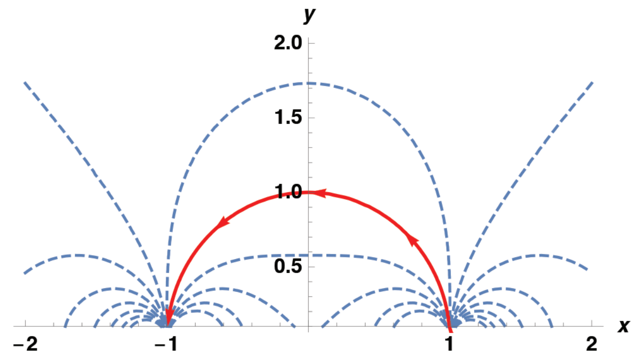
$$s = -\frac{4xy}{(x^2+y^2)^2 - 2(x^2-y^2)+1}. \tag{29}$$

We identify the worldsheet of a string as a whole ρ plane, i.e. $-\infty < t < \infty$ and $-\infty < s < \infty$. From Eq. (27), we see that only half of the z plane is covered by the trajectories of a string. How it is covered depends on a choice of branch cut on the z plane. We would like to choose the upper half z as an image of whole ρ plane as this choice is compatible with the result of [8].

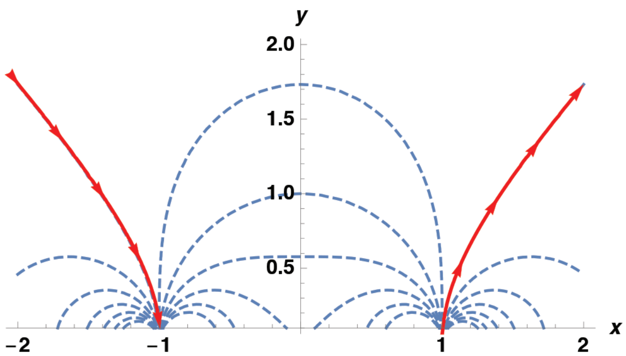
The contours for various t are plotted in Figure 1. First we note that $z = 1$ and $z = -1$ correspond to $s = -\infty$ and $s = \infty$ respectively. They are remnants of open string boundaries which are kept fixed against



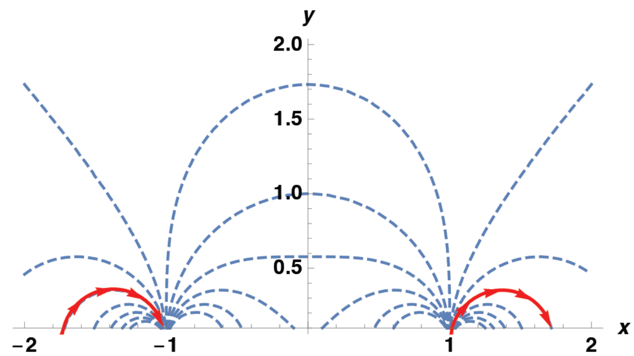
(a) $t=-3$



(b) $t=-1$



(c) $t=0$



(d) $t=1$

Figure 1. Equal time contours for $t=-3,-1,0$ and 1. Arrows in each solid line indicates increase of t.

time evolution generated $\mathcal{L}_0$. Thus, "missing boundaries" provides an evidence that open string vanishes at the tachyon vacuum.

The global structure of contours depend on the value of t. For $t\leq -2$, a contour splits into two open curves within the unit circle. The contour for $t=-1$ is just upper half of the unit circle. For $-1<t<0$ a contour does not split. The contour at $t=0$ is upper half of the hyperbola $x^2-y^2=1$. For $t>0$, a contour is placed outside the unit circle and splits into two open curves again.

3.2. Mode expansion

Next we introduce the mode expansion of conformal fields according to [18]. Consider a primary field $\phi(z)$ with weight h. The Fourier mode of this field is now continuously labeled

$$\phi_\kappa=\int_{\gamma_+}\frac{dz}{2\pi i}g(z)^{h-1}f_\kappa(z)\phi(z), \tag{30}$$

where the path γ_+ is one of the constant t contours in Figure 1 in the upper half plane. It starts from $z=1$ and ends at $z=-1$[1]. We also have the inverse relation

$$\phi(z)=g(z)^{-h}\int_{-\infty}^{\infty}d\kappa\phi_\kappa f_{-\kappa}(z). \tag{31}$$

[1] Here we note that the integral (30) encounters essential singularities at the endpoints $z=\pm1$ because of the $f_\kappa(z)$ insertion. It can be shown that this singularity can be regularized by deforming the contour [21].

These relations correspond to Fourier transformation and its inverse rather than discrete Fourier series. Our concern is bosonic string theory, whose fundamental fields are $\partial X^{\mu}(z)$, $c(z)$ and $b(z)$. Their Fourier modes can be pulled out by applying Eq. (30) to each of them. Thus, we have

$$\mathcal{A}^{\mu}_{\kappa} = i\sqrt{\frac{2}{\alpha'}}\int_{\gamma_+}\frac{dz}{2\pi i}f_{\kappa}(z)\partial X_{\mu}(z), \tag{32}$$

$$\mathcal{C}_{\kappa} = \int_{\gamma^+}\frac{dz}{2\pi i}g(z)^{-2}f_{\kappa}(z)c(z), \tag{33}$$

$$\mathcal{B}_{\kappa} = \int_{\gamma_+}\frac{dz}{2\pi i}g(z)f_{\kappa}(z)b(z). \tag{34}$$

Furthermore, we introduce the twisted versions of Eqs. (33) and (34). Using the fact that $c'(z) = zc(z)$ and $b'(z) = z^{-1}b(z)$ behave as weight 0 and 1 fields respectively, we can write Fourier modes for them as

$$\mathcal{B}'_{\kappa} = \int_{\gamma_+}\frac{dz}{2\pi i}f_{\kappa}(z)zb(z), \tag{35}$$

$$\mathcal{C}'_{\kappa} = \int_{\gamma_+}\frac{dz}{2\pi i}g(z)^{-1}f_{\kappa}(z)z^{-1}c(z). \tag{36}$$

Inverse formulas for the Fourier modes can be obtained by applying Eq. (31) to each fields.

3.3. Virasoro generator

Now we are ready to introduce continuous Virasoro generators, which is our main interest. The result of [8] implies that the twisted BCFT is suitable for our purpose. Therefore, we would like to deal with

$$\mathcal{L}'_{\kappa} = \{Q', \mathcal{B}'_{\kappa}\}, \tag{37}$$

where Q' is the modified BRST charge defined in Eq. (6). We will show $\mathcal{L}'_{\kappa}$ form continuous Virasoro algebra as expected. First, we derive the explicit form of the twisted generator. This is done in similar fashion to that of conventional CFT [18], by integrating operator product expansion around a single pole. As a warming up,

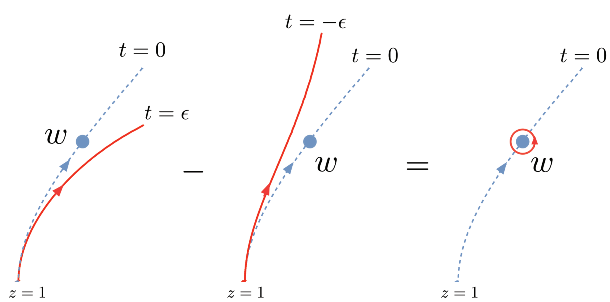


Figure 2. A contour integral associated with the evaluation of the commutator $\{Q', \mathcal{B}_0\}$. The curve for $\mathcal{B}'$ integral (dotted line) is fixed to $t=0$ contour. The curve for Q' is placed slightly back and forward in time. Evaluating the commutator amount to pick pole residues around w.

let us begin with $\mathcal{L}'_0$. First, we rewrite Eq. (6) in terms of $g(z)$:

$$Q' = \oint_{\gamma}\frac{dz}{2\pi i}z^{-1}g(z)j_B(z) - \oint_{\gamma}\frac{dz}{2\pi i}\frac{z\left(\partial(z^{-1}g(z))\right)^2}{g(z)}c(z), \tag{38}$$

where the integration path $\gamma = \gamma_+ + \gamma_-$, where γ_+ is one of the equal time contours already explained in Section 3.1 and γ_- is its mirror image in the lower half plane. Furthermore, we would like to choose γ_+ to be close to $t = 0$ contour. Then, line integrals included Q' and $\mathcal{B}_0$ can be performed as Figure 2. The evaluation of commutator can be carried out in similar fashion to the conventional CFT as

$$\begin{aligned}\mathcal{L}'_0 &= \{Q', \mathcal{B}'_0\} \\ &= \oint_w dz \int_{\gamma_+} dw (z^{-1} g(z)) w \mathbf{T}\left(j_B(z) b(w)\right) \\ &- \oint_w dz \int_{\gamma_+} dw \frac{z\left(\partial(z^{-1}g(z))\right)^2}{g(z)} w \mathbf{T}\left(c(z) b(w)\right),\end{aligned} \tag{39}$$

where $\mathbf{T}$ denotes time ordering. Each time-ordered product in the last line is replaced with operator product expansion that takes form of a Laurent expansion around $z = w$:

$$\mathbf{T}\left(j_B(z) b(w)\right) = \frac{3}{(z-w)^3} + \frac{j_g(w)}{(z-w)^2} + \frac{T(w)}{(z-w)}, \tag{40}$$

$$\mathbf{T}\left(c(z) b(w)\right) = \frac{1}{z-w}. \tag{41}$$

Then, by picking up pole residues, we obtain the result which can be summarized in a compact notation

$$\mathcal{L}'_0 = \left[gT + hj_g + \frac{3}{2}\frac{gk}{g} - \frac{h^2}{g}\right], \tag{42}$$

where the square bracket simply denotes a contour integral of a product of functions or fields,

$$[ab] \equiv \int_{\gamma_+} \frac{dz}{2\pi i} a(z) b(z), \tag{43}$$

and functions $h(z)$ and $k(z)$ are defined by

$$h(z) = z\frac{d}{dz}\left(\frac{g(z)}{z}\right), \quad k(z) = z\frac{d^2}{dz^2}\left(\frac{g(z)}{z}\right). \tag{44}$$

The generator with nonzero κ can be derived in similar manner. In this case, it is soon realized that the result can be obtained just by inserting f_κ in Eq. (42). Thus, we have

$$\mathcal{L}'_\kappa = \{Q', \mathcal{B}'_\kappa\} \tag{45}$$

$$= [f_\kappa g T] + [f_\kappa h j_g] + \left[\frac{f_\kappa}{g}\left\{\frac{3}{2} gk - h^2\right\}\right] \tag{46}$$

Note that last term in the final expression in Eq. (46) is a constant. Explicit evaluation of this constant requires convenient choice of $t = 0$ as was done in [18]. With this choice, we can convert the contour integral to the one along a straight line as

$$[f_k a] = \int_{\gamma_+} \frac{dz}{2\pi} \frac{f_\kappa(z)}{g(z)} g(z) a(z) \tag{47}$$

$$= \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{i\kappa s} g\left(\sqrt{1 + \frac{2}{is}}\right) a\left(\sqrt{1 + \frac{2}{is}}\right), \tag{48}$$

where we have used $\rho = is = 2/(1-z^2)$. In this way, a contour integral is evaluated by inverse Fourier transform of $g(\sqrt{1+2/(is)})a(\sqrt{1+2/(is)})$. This quantity can be unambiguously evaluated if $a(z)$ involve even powers of z only. Fortunately, this is the case for the last term of Eq. (46):

$$g(z)k(z) = \frac{(z^4+3)(z^2-1)^2}{8z^4} = \frac{-2(s^2-is-1)}{s^2(s-2i)^2}, \tag{49}$$

$$h(z)^2 = \frac{(z^4-1)^2}{4z^4} = \frac{-4(s-i)^2}{s^2(s-2i)^2}. \tag{50}$$

Fourier transform of these functions can be evaluated analytically, although the results turn out to be distributions rather than ordinary functions:

$$\left[\frac{f_\kappa}{g} gk\right] = \frac{3}{2}e^{-2\kappa}\kappa\theta(\kappa) + \frac{1}{4}\kappa\epsilon(\kappa), \tag{51}$$

$$\left[\frac{f_\kappa}{g} h^2\right] = e^{-2\kappa}(\kappa-1)\theta(\kappa) + \frac{1}{2}(\kappa+1)\epsilon(\kappa), \tag{52}$$

where $\theta(\kappa)$ and $\epsilon(\kappa)$ are Heaviside step function and sign function respectively. Plugging these back to Eq. (46), we arrive at the final expression:

$$\mathcal{L}'_\kappa = [f_\kappa gT] + [f_\kappa h j_g] + a_\kappa, \tag{53}$$

where

$$a_\kappa = \frac{1}{4}(5\kappa+4)e^{-2\kappa}\theta(\kappa) - \frac{1}{8}(\kappa+4)\epsilon(\kappa). \tag{54}$$

3.4. Virasoro algebra

We would like to derive the commutator between continuous Virasoro generator $\mathcal{L}'_\kappa$. We divide Eq. (46) into untwisted part and the remaining:

$$\mathcal{L}'_\kappa = \mathcal{L}_\kappa + \delta\mathcal{L}_\kappa + a_\kappa, \tag{55}$$

where

$$\mathcal{L}_\kappa = [f_\kappa gT], \tag{56}$$

$$\delta\mathcal{L}_\kappa = [f_\kappa h j_g], \tag{57}$$

$$a_\kappa = \left[\frac{3}{2}\frac{f_\kappa}{g}gk - \frac{f_\kappa}{g}h^2\right]. \tag{58}$$

Then, the commutator is expanded as

$$[\mathcal{L}'_\kappa, \mathcal{L}'_\lambda] = [\mathcal{L}_\kappa, \mathcal{L}_\lambda] + [\mathcal{L}_\kappa, \delta\mathcal{L}_\lambda] + [\delta\mathcal{L}_\kappa, \mathcal{L}_\lambda] + [\delta\mathcal{L}_\kappa, \delta\mathcal{L}_\lambda]. \tag{59}$$

According to [18], the untwisted generators satisfy Virasoro algebra:

$$[\mathcal{L}_\lambda, \mathcal{L}_\lambda] = (\kappa-\lambda)\mathcal{L}_{\kappa+\lambda}. \tag{60}$$

There is no central term since we consider matter plus ghost CFT with vanishing total central charge. Next we would like to evaluate first order term in δ:

$$[\mathcal{L}_\kappa, \delta\mathcal{L}_\lambda] + [\delta\mathcal{L}_\kappa, \mathcal{L}_\lambda] = [\mathcal{L}_\kappa, \delta\mathcal{L}_\lambda] - [\mathcal{L}_\kappa, \delta\mathcal{L}_\lambda]. \tag{61}$$

The first term in the right hand side of the above equation can be evaluated as:

$$\begin{aligned}
[\mathcal{L}_\kappa, \delta\mathcal{L}_\lambda] &= \oint_w \frac{dz}{2\pi i}\int_{\gamma_+}\frac{dw}{2\pi i} g(z)f_\kappa(z)h(w)f_\lambda(w)\mathbf{T}(T(z)j_g(w)) \\
&= \oint_w \frac{dz}{2\pi i}\int_{\gamma_+}\frac{dw}{2\pi i} g(z)f_\kappa(z)h(w)f_\lambda(w)\left(\frac{-3}{(z-w)^3}+\frac{j_g(w)}{(z-w)^2}+\frac{\partial j_g(w)}{z-w}\right) \\
&= -\frac{3}{2}\left[(gf_\kappa)''(hf_\lambda)\right] + \left[(gf_\kappa)'hf_\lambda j_g\right] + \left[ghf_\kappa f_\lambda \partial j_g\right] \\
&= -\frac{3}{2}\left[(gf_\kappa)''(hf_\lambda)\right] + \left[(gf_\kappa)'hf_\lambda j_g\right] \\
&= -\frac{3}{2}\left[(gf_\kappa)''(hf_\lambda)\right] + \left[ghf_\kappa' f_\lambda j_g\right] \\
&= -\frac{3}{2}\left[(gf_\kappa)''hf_\lambda\right] + \kappa\left[hf_\kappa f_\lambda j_g\right] \\
&= -\frac{3}{2}\left[\kappa\frac{g'h}{g}f_{\kappa+\lambda} + \kappa^2\frac{h}{g}f_{\kappa+\lambda}\right] + \kappa\left[hf_{\kappa+\lambda}j_g\right].
\end{aligned} \tag{62}$$

Then, we have

$$\begin{aligned}
[\mathcal{L}_\kappa, \delta\mathcal{L}_\lambda] + [\delta\mathcal{L}_\kappa, \mathcal{L}_\lambda] &= (\kappa-\lambda)\left[hf_{\kappa+\lambda}j_g\right] \\
&\quad + \left[\frac{f_{\kappa+\lambda}}{g}\left\{-\frac{3}{2}(\kappa-\lambda)g'h - \frac{3}{2}(\kappa^2-\lambda^2)h\right\}\right].
\end{aligned} \tag{63}$$

Finally, δ^2 term is evaluated as

$$\begin{aligned}
[\delta\mathcal{L}_\kappa, \delta\mathcal{L}_\lambda] &= \oint_w \frac{dz}{2\pi i}\int_{\gamma_+}\frac{dw}{2\pi i} h(z)f_\kappa(z)h(w)f_\lambda(w)\mathbf{T}(j_g(z)j_g(w)) \\
&= \oint_w \frac{dz}{2\pi i}\int_{\gamma_+}\frac{dw}{2\pi i} h(z)f_\kappa(z)h(w)f_\lambda(w)\frac{1}{(z-w)^2} \\
&= \left[(hf_\kappa)'hf_\lambda\right] \\
&= \left[h'f_\kappa hf_\lambda + hf_\kappa' hf_\lambda\right] \\
&= \left[\frac{f_{\kappa+\lambda}}{g}(ghh' + \kappa h^2)\right].
\end{aligned} \tag{64}$$

Our result can be summarized as:

$$[\mathcal{L}'_\kappa, \mathcal{L}'_\lambda] = (\kappa-\lambda)\left(\mathcal{L}_{\kappa+\lambda} + \delta\mathcal{L}_{\kappa+\lambda}\right) + u(\kappa,\lambda), \tag{65}$$

where

$$u(\kappa,\lambda) = \left[\frac{f_{\kappa+\lambda}}{g}\left\{-\frac{3}{2}(\kappa-\lambda)g'h - \frac{3}{2}(\kappa^2-\lambda^2)h + ghh' + \kappa h^2\right\}\right]. \tag{66}$$

The constant $u(\kappa,\lambda)$ can be explicitly evaluated in terms of Fourier transformation. The result turns out to be

$$\begin{aligned}u(\kappa,\lambda) &= (\kappa-\lambda)\left\{\frac{1}{4}(5\kappa+5\lambda+4)e^{-2(\kappa+\lambda)}\theta(\kappa+\lambda) - \frac{1}{8}(\kappa+\lambda+4)\epsilon(\kappa+\lambda)\right\} \\ &= (\kappa-\lambda)a_{\kappa+\lambda}, \end{aligned} \tag{67}$$

where $a_{\kappa+\lambda}$ is already defined in Eq. (54). Putting back this result to Eq. (65), we obtain

$$\begin{aligned}[\mathcal{L}'_\kappa, \mathcal{L}'_\lambda] &= (\kappa-\lambda)\left(\mathcal{L}_{\kappa+\lambda} + \delta\mathcal{L}_{\kappa+\lambda}\right) + (\kappa-\lambda)a_{\kappa+\lambda} \\ &= (\kappa-\lambda)\mathcal{L}'_{\kappa+\lambda}. \end{aligned} \tag{68}$$

Thus, we have shown that the generator $\mathcal{L}'_\kappa$ satisfy Virasoro algebra without anomaly although it is defined in terms of twisted generators.

4. Mode expansion

4.1. Commutation relations

Here we would like to derive the algebra formed by Fourier modes of various conformal fields. As an example, let us evaluate the commutator between $\mathcal{B}_\kappa$ and $\mathcal{C}_\kappa$. The commutator can be evaluated in a similar way to the derivation of continuous Virasoro algebra, where OPE and contour integral is used:

$$\begin{aligned}\{\mathcal{B}_\kappa, \mathcal{C}_\lambda\} &= \oint_w \frac{dz}{2\pi i}\int_{\gamma_+}\frac{dw}{2\pi i} f_\kappa(z) f_\lambda(w) g(z)\mathbf{T}\left(b(z)c(w)\right) \\ &= \int_{\gamma_+}\frac{dz}{2\pi i}\frac{f_{\kappa+\lambda}(z)}{g(z)} \\ &= \int_{-\infty}^{\infty}\frac{ds}{2\pi}e^{i(\kappa+\lambda)s} \\ &= \delta(\kappa+\lambda), \end{aligned} \tag{69}$$

where we transformed variable z into s by choosing $t=0$ contour. The commutation relation for the twisted pairs $\mathcal{B}'_\kappa$ and $\mathcal{C}'_\kappa$ yield exactly the same result since the extra weight factors (z^{-1} for $b'(z)$ and z for $c(z)$) do not change the commutator. Thus, we have

$$\{\mathcal{B}'_\kappa, \mathcal{C}'_\lambda\} = \delta(\kappa+\lambda). \tag{70}$$

The commutator for $\mathcal{A}^\mu_\kappa$ is evaluated similarly as

$$[\mathcal{A}^\mu_\kappa, \mathcal{A}^\nu_\lambda] = \kappa\eta^{\mu\nu}\delta(\kappa+\lambda). \tag{71}$$

Note that the commutation relations derived here can be understood as "continuous version" of the discrete one

$$\{b'_m, c'_n\} = \delta_{m+n}, \quad [\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n}, \tag{72}$$

where m and n are integers.

We can also derive commutators between $\mathcal{L}'_\kappa$ and other modes. This can be done in much the same way as we did for Virasoro algebra of commutation relations between Fourier modes. In this case, relevant OPEs are those between $T(z)$ or $j_g(z)$ with other fundamental fields. Explicitly, they are

$$T(z)\partial X^\mu(w) \sim \frac{\partial X^\mu(w)}{(z-w)^2} + \frac{\partial^2 X^\mu(w)}{(z-w)} + \cdots, \tag{73}$$

$$T(z)c(w) \sim \frac{-c(w)}{(z-w)^2} + \frac{\partial c(w)}{(z-w)} + \cdots, \tag{74}$$

$$T(z)b(w) \sim \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{(z-w)} + \cdots, \tag{75}$$

$$j_g(z)c(w) \sim \frac{c(w)}{z-w} + \cdots, \tag{76}$$

$$j_g(z)b(w) \sim \frac{-b(w)}{z-w} + \cdots. \tag{77}$$

These OPEs can be translated to commutators

$$[\mathcal{L}'{}_\kappa, \mathcal{A}^\mu_\lambda] = -\lambda \mathcal{A}^\mu{}_{\kappa+\lambda}, \tag{78}$$

$$[\mathcal{L}'{}_\kappa, \mathcal{B}'_\lambda] = (\kappa-\lambda)\mathcal{B}'_{\kappa+\lambda}, \tag{79}$$

$$[\mathcal{L}'{}_\kappa, \mathcal{C}'_\lambda] = (-2\kappa-\lambda)\mathcal{C}'_{\kappa+\lambda}. \tag{80}$$

The correspondence between discrete and continuous algebras is worth to mention. The commutators for $\mathcal{B}'_\lambda$ and $\mathcal{C}'_\lambda$ turns out to be "continuous version" of the untwisted commutators rather than twisted ones:

$$[L_m, b_n] = (m-n)b_{m+n}, \quad [L_m, c_n] = (-2m-n)c_{m+n}. \tag{81}$$

This is surprising but consistent with the fact that $\mathcal{L}'_\kappa$ obeys Virasoro algebra without anomaly.

4.2. Mode expansion of Virasoro generators

We would like to derive the Fourier mode expansion of the Virasoro generator

$$\mathcal{L}'_\kappa = \mathcal{L}^m_\kappa + \mathcal{L}'{}^g_\kappa + a_\kappa, \tag{82}$$

where

$$\begin{aligned}\mathcal{L}^m_\kappa &= \int_{\gamma_+} \frac{dz}{2\pi i} f_\kappa g T^m \\ &= -\frac{1}{\alpha'}\left[f_\kappa g : \partial X^\mu \partial X_\mu :\right]\end{aligned} \tag{83}$$

$$\begin{aligned}\mathcal{L}'{}^g_\kappa &= -\int_{\gamma_+} \frac{dz}{2\pi i} f_\kappa g : \partial b c + 2b\partial c : - \int_{\gamma_+} \frac{dz}{2\pi i} f_\kappa h : bc : \\ &= -\left[f_\kappa g : \partial b c + 2b\partial c :\right] - \left[f_\kappa h : bc :\right].\end{aligned} \tag{84}$$

Here the normal ordering is defined through the time ordering prescription we have already worked out. Fourier mode expansion of the Virasoro generator is obtained by replacing each field in the Virasoro generator with the inverse Fourier expansion according to Eq. (31):

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}g(z)^{-1}\int d\kappa\,\mathcal{A}^\mu_\kappa f_{-\kappa}(z), \tag{85}$$

$$c'(z) = \int d\kappa\,\mathcal{C}'_\kappa f_{-\kappa}(z), \tag{86}$$

$$b'(z) = g(z)^{-1}\int d\kappa\,\mathcal{B}'_\kappa f_{-\kappa}(z). \tag{87}$$

Evaluation of matter part proceeds straightforwardly. Two $g^{-1}(z)$ from ∂X^μ and another $g(z)$ in the generator multiplies to total weight $g^{-1}(z)$. In addition, two $f_\kappa(z)$ from ∂X^μs and another one in the generator give rise to a delta function

$$\int_{\gamma_+}\frac{dz}{2\pi i}\frac{f_{\kappa-\kappa_1-\kappa_2}(z)}{g(z)} = \delta(\kappa-\kappa_1-\kappa_2). \tag{88}$$

Then this delta function is integrated with the oscillator $\mathcal{A}^\mu_\kappa$ and yields

$$\mathcal{L}^m_\kappa = \frac{1}{2}\int d\kappa'\,:\mathcal{A}^\mu_{\kappa-\kappa'}\mathcal{A}_{\mu,\kappa'}:, \tag{89}$$

which is merely a continuous version of the discrete expression.

The evaluation of ghost part is rather involved. We first replace the ghost pairs with the twisted ones in terms of the relation

$$c(z) = zc'(z), \quad b(z) = z^{-1}b'(z). \tag{90}$$

This replacement reads

$$\begin{aligned} -\left[f_\kappa g : \partial bc + 2b\partial c:\right] - \left[f_\kappa h : bc:\right] &= -\left[f_\kappa g : \partial b'c' + 2b'\partial c':\right] - \left[f_\kappa h : b'c':\right] \\ &\quad - \left[f_\kappa z^{-1} : b'c':\right]. \end{aligned} \tag{91}$$

Then, by replacing b' and c' with the Fourier expansion in Eqs. (86) and (87), and evaluating each term carefully leads to rather simple result

$$\mathcal{L}'^g_\kappa = \int d\kappa'(2\kappa-\kappa') : \mathcal{B}'_{\kappa'}\mathcal{C}'_{\kappa-\kappa'}:. \tag{92}$$

Note that this is again a continuous version of the untwisted ghost Virasoro generator

$$L^g_m = \sum_k (2m-k)b_k c_{m-k}. \tag{93}$$

This result is again convinced from the fact that the continuous generator satisfies untwisted algebra. In summary, the total Virasoro generator is the continuous version of the untwisted one up to the constant a_κ:

$$\mathcal{L}'_\kappa = \frac{1}{2}\int d\kappa'\,:\mathcal{A}^\mu_{\kappa-\kappa'}\mathcal{A}_{\mu,\kappa'}: + \int d\kappa'(2\kappa-\kappa') : \mathcal{B}'_{\kappa'}\mathcal{C}'_{\kappa-\kappa'}: + a_\kappa. \tag{94}$$

4.3. Mode expansion of modified BRST charge

Having obtained mode expansion of fundamental fields, we next derive the mode expansion of the modified BRST charge, which will be used to investigate physical state. This can be done straightforwardly by inserting the expanded fields into the original expression of the modified BRST charge. First, we rewrite the original expression of Eq. (6) in terms of twisted ghosts. Explicit expression of the BRST current in terms and matter and ghost CFT is

$$j_B(z) = cT^m - cb\partial c + \frac{3}{2}\partial^2 c \tag{95}$$

$$= zc'T^m - zc'b'\partial c' + \frac{3}{2}\partial^2(zc'), \tag{96}$$

where we have used the relation $c(z) = zc'(z)$ and $b(z) = z^{-1}b'(z)$. Using this expression, we can write the modified BRST charge as

$$\begin{aligned} Q' &= \oint_\gamma \frac{dz}{2\pi i} zg(z)j_B(z) - \oint_\gamma \frac{dz}{2\pi i}\frac{z^2\partial\left(z^{-1}g(z)\right)}{g(z)}c(z) \\ &= [gc'T^m] - [g : c'b'\partial c' :] + \left[+\frac{3}{2}\partial^2(zc') - \frac{z^2\partial\left(z^{-1}g(z)\right)}{g(z)}zc'\right] + \cdots \\ &= [gc'T^m] - [g : c'b'\partial c' :] + \left[\left(\frac{3}{2}k - \frac{h^2}{g}\right)c'\right] + \cdots, \end{aligned} \tag{97}$$

where $k(z)$ and $h(z)$ are those defined in Eq. (44). The dots denote contributions from antiholomorphic sector which is irrelevant to the following discussion.

The Fourier mode expansion is obtained by inserting expanded fields into this expression,

$$T^m(z) = g(z)^{-2}\int_{-\infty}^{\infty} d\kappa \mathcal{L}^m_\kappa f_{-\kappa}(z), \tag{98}$$

$$c'(z) = \int_{-\infty}^{\infty} d\kappa\, \mathcal{C}'_\kappa f_{-\kappa}(z), \tag{99}$$

$$b'(z) = g(z)^{-1}\int_{-\infty}^{\infty} d\kappa\, \mathcal{B}'_\kappa f_{-\kappa}(z). \tag{100}$$

After some algebra, we reach the following expression:

$$Q' = \int_{-\infty}^{\infty} d\kappa\, \mathcal{C}'{}_\kappa \mathcal{L}^m_{-\kappa} + \frac{1}{2}\int_{-\infty}^{\infty} d\kappa\, d\lambda(\lambda-\kappa) : \mathcal{C}'_\kappa \mathcal{B}'_{-\kappa-\lambda}\mathcal{C}'_\lambda : + \int d\kappa\, \mathcal{C}'{}_\kappa a_\kappa. \tag{101}$$

Furthermore, this result can be compared to the expansion with Virasoro generator in Eq. (94). Similar to the discrete case, the BRST charge can be expressed in terms of ghost Virasoro as

$$Q' = \int d\kappa \left(\mathcal{C}'_\kappa \mathcal{L}^m_{-\kappa} + \frac{1}{2} : \mathcal{C}'_\kappa \mathcal{L}'^g{}_{-\kappa} :\right) + \int d\kappa\, \mathcal{C}'{}_\kappa a_\kappa. \tag{102}$$

This expression can be confirmed explicitly by inserting Eq. (94) into Eq. (101). Again, this can be compared to the expression of continuous version

$$Q_B = \sum_n \left(c_n L^m_{-n} + \frac{1}{2} : c_n L^g_{-n} :\right). \tag{103}$$

In closing this section, we would like to summarize our results. Fourier mode expansion of $\mathcal{L}'_\kappa$ and Q' are obtained from those of L_n and Q_B by the following procedure:

1. Replace α_n^μ, c_n, b_n with their continuum counterpart $\mathcal{A}_\kappa^\mu$, $\mathcal{C}'_\kappa$, $\mathcal{B}'_\kappa$.
2. Replace sum with the κ integral.
3. Include a constant a_κ to the Virasoro generator.

Having obtained the oscillator expansion which looks similar to the discrete counterpart, one may be tempted to derive the cohomology of Q' by applying the Kato–Ogawa derivation [23] to the continuous modes. In order to carry out such analysis, a vacuum for the continuous modes is indispensable. However, the relation between such vacuum and conventional $SL\ (2,\mathbb{R})$ vacuum seems to be quite nontrivial and is not at hand yet. Therefore, we leave the derivation of cohomology derivation as a future task.

5. Summary and discussion

We analyzed the identity-based solution of Takahashi and Tanimoto by adopting the infinite circumstance formalism. We obtained the continuous Virasoro algebra of matter plus ghost system. The oscillator expression of conformal fields is introduced, and it turns out that it can be obtained by extending an integer mode number to continuous variable.

We would like to explain the strong resemblance between discrete and continuous theories by introducing another coordinate system[2]. It is

$$Z = f_0(z) = e^{\frac{2}{z^2-1}} = e^\rho. \tag{104}$$

Applying the standard transformation law of primary field, we obtain

$$\phi_\kappa = \int_C dZ Z^{\kappa+h-1}\tilde{\phi}(Z), \tag{105}$$

where $\tilde{\phi}(Z)$ is a transformed field. Note that this expression looks similar to that of conventional CFT

$$\phi_n = \oint dz z^{n+h-1}\phi(z), \tag{106}$$

except for the continuous mode number. However, it should also be noted that the correspondence between Z and ρ is not one-to-one. The contour C winds infinitely many times around $Z=0$ because of the relation

$$\rho = \log Z. \tag{107}$$

Therefore, the worldsheet in Z coordinate is a Riemann surface composed of infinitely many sheets. This interpretation explains the resemblance between discrete and continuous algebras, since the commutation relation is evaluated by a product between two operators nearby, which only reflects local property within same sheet.

The noncompact Riemann surface of $\log Z$ could lead to further speculation about the nature of the tachyon vacuum. Intuitively, the noncompact worldsheet can be interpreted as a collection of open string

[2] This coordinate system is also discussed in [21].

worldsheets. Therefore, the tachyon vacuum encodes infinitely many open strings in certain manner. More concretely, let us consider a subset of continuous Virasoro generators labeled by a positive integer q and arbitrary integer n:

$$\mathcal{L}_{\frac{n}{q}}. \tag{108}$$

By keeping q fixed, we introduce rescaled generators

$$\mathfrak{l}^q_n = q\mathcal{L}_{\frac{n}{q}}. \tag{109}$$

It is obvious that these generators form subalgebra since $[\mathfrak{l}^q_m, \mathfrak{l}^q_n] = (m-n)\mathfrak{l}^q_{m+n}$. Therefore, infinitely many discrete algebras are embedded in the continuous algebra.

Another interpretation is possible by considering nonscaled generators. The spectrum of nonscaled generators can be described by a real number λ and an integer n as

$$[\mathcal{L}_0, \mathcal{L}_{-\lambda n}] = \lambda n \mathcal{L}_{-\lambda n}. \tag{110}$$

The spectrum becomes denser for small λ. This is reminiscent of the spectrum found in [24] , reported as the landscape of boundary string field theory.

Our analysis revealed unexpected richness of the Hilbert space of OSFT. In particular, the noncompact worldsheet introduced in this section will be important to understand the nature of tachyon vacuum. We expect further progress in this direction. It will also be interesting to extend our analysis to the wedge-based analytic solutions.

Acknowledgments

We thank the organizer and participants of “Workshop on sine square deformation and related topics” at RIKEN. In particular, we thank T. Tada, T. Takahashi, and S. Ryu for their valuable discussions. This work is partially supported by RIKEN as for the workshop, and also supported by our institution, Yokote Seiryo Gakuin High School.

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