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Second-order gauge-invariant perturbation theory and conserved charges in cosmological Einstein gravity

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Abstract: Recently a new approach in constructing the conserved charges in cosmological Einstein gravity was given. In this new formulation, instead of using the explicit form of the field equations, a covariantly conserved rank-four tensor was used. In the resulting charge expression, instead of the first derivative of the metric perturbation, the linearized Riemann tensor appears along with the derivative of the background Killing vector fields. Here we give a detailed analysis of the first-order and the second-order perturbation theory in a gauge-invariant form in cosmological Einstein gravity. The linearized Einstein tensor is gauge-invariant at the first order but it is not so at the second order, which complicates the discussion. This method depends on the assumption that the first-order metric perturbation can be decomposed into gauge-variant and gauge-invariant parts and the gauge-variant parts do not contribute to physical quantities.

Key words: Second-order perturbation theory, gauge-invariant perturbation theory, conserved charges, Taub charges, constraint equations

1. Introduction

In general relativity finding the exact solution to the field equations is generally too hard and therefore one needs to use perturbation theory by starting from the exact solution to the background field equations, which has symmetries. This technique yields a lot of information about the physical problem at hand. In the absence of a source, any generic gravity field equations in local coordinates read as follows:

$$\mathscr{E}_{\mu\nu}(g(\lambda)) = 0, \tag{1}$$

where λ parametrizes the solution set. We have the exact solution plus the perturbations defined as:

$$g(\lambda = 0) := \bar{g}, \qquad h_{\mu\nu} := \frac{dg_{\mu\nu}}{d\lambda}\Big|_{\lambda=0}, \qquad k_{\mu\nu} := \frac{1}{2}\frac{d^2 g_{\mu\nu}}{d\lambda^2}\Big|_{\lambda=0}, \tag{2}$$

where $\bar{g}$ is the background solution that we carry out the perturbations around, h denotes the first-order expansion of the metric tensor, and k denotes the second-order expansion. When we consider the expansion of the field equations (1) about the background spacetime metric $\bar{g}$, we obtain perturbations of the field equations up to $\mathcal{O}(\lambda^3)$ as:

$$\bar{\mathscr{E}}_{\mu\nu}(\bar{g}) + \lambda(\mathscr{E}_{\mu\nu})^{(1)}(h) + \lambda^2\Big((\mathscr{E}_{\mu\nu})^{(2)}(h,h) + (\mathscr{E}_{\mu\nu})^{(1)}(k)\Big) = 0. \tag{3}$$

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Here by assumption $\bar{\mathscr{E}}_{\mu\nu}(\bar{g}) = 0$ and $(\mathscr{E}_{\mu\nu})^{(1)}(h)$ denotes the first-order linearized field equations, while the combination $(\mathscr{E}_{\mu\nu})^{(2)}(h,h) + (\mathscr{E}_{\mu\nu})^{(1)}(k)$ shows the expansion of the field equations at the second order. Of course, not all background solutions can be used to obtain an exact solution, since once $\bar{g}$ solves the background field equations, the solution to the first-order perturbation of the field equations, h, must satisfy the given relation (2). Similarly, the second-order metric perturbation must satisfy the given definition with the second-order field equations:

$$(\mathscr{E}_{\mu\nu})^{(2)}(h,h) + (\mathscr{E}_{\mu\nu})^{(1)}(k) = 0. \tag{4}$$

It means that even if we find the linearized solutions, h, to the linear order expansion of the field equations $(\mathscr{E}_{\mu\nu})^{(1)}(h) = 0$, due to the the second-order expansion of the field equations there exists a constraint on it. In order to understand this issue explicitly, let us consider $\bar{\xi}^\mu$, which denotes the Killing vector field of $\bar{g}$. Contraction of (4) with $\bar{\xi}^\mu$ and integration of the result over Σ, which denotes the hypersurface of the spacetime manifold $\mathscr{M}$, gives

$$\int_\Sigma d^{n-1}x\,\sqrt{\bar{\gamma}}\,\bar{\xi}_\mu\,(\mathscr{E}^{\mu\nu})^{(1)}(k) = -\int_\Sigma d^{n-1}x\,\sqrt{\bar{\gamma}}\,\bar{\xi}_\mu\,(\mathscr{E}^{\mu\nu})^{(2)}(h,h), \tag{5}$$

where we have used the background inverse metric and the metric to raise and lower the indices, respectively, and $\bar{\gamma}$ denotes the metric of the hypersurface. Once we are given the field equations, we can express the left-hand side of (5) as a pure divergence of an antisymmetric field $F^{\mu\nu}$:

$$\sqrt{\bar{\gamma}}\bar{\xi}_\mu\,(\mathscr{E}^{\mu\nu})^{(1)}(k) = \partial_\mu\left(\sqrt{\bar{\gamma}}F^{\mu\nu}\right). \tag{6}$$

When the left-hand side of (5) is expressed in terms of the perturbation of the metric tensor, it is called the Abbott–Deser–Tekin (ADT) current (or charges) [1, 2] and it is an extension of the Abbott–Deser–Misner (ADM) [3] charges of flat spacetime. Substituting the last expression in (5), we conclude that the right-hand side, which is known as the Taub charge [4], must also be expressed as a pure boundary. Then one ends up with the equality of the Taub and ADT charges:

$$Q_{ADT} := \int_{\partial\Sigma} d\Sigma_\mu\,\sqrt{\bar{\sigma}}\,\hat{n}_\nu\,\bar{\xi}_\mu\,F^{\mu\nu} = -\int_\Sigma d^{n-1}x\,\sqrt{\bar{\gamma}}\,\bar{\xi}_\mu\,(\mathscr{E}^{\mu\nu})^{(2)}(h,h) =: -Q_{Taub}, \tag{7}$$

where $\partial\Sigma$ denotes the boundary of the hypersurface Σ, $\bar{\sigma}$ is the pull-back metric on it, and $\hat{n}_\nu$ is the outward unit normal vector on $\partial\Sigma$. If the background spacetime has no boundary, one arrives at the integral constraint on h as:

$$\int_\Sigma d^{n-1}x\,\sqrt{\bar{\gamma}}\,\bar{\xi}_\mu\,(\mathscr{E}^{\mu\nu})^{(2)}(h,h) = 0. \tag{8}$$

When this integral constraint is satisfied, we say that $\bar{g}$ is linearization stable and the perturbation h can be used to obtain an exact solution, but if this is not the case the background solution has linearization instability and we cannot improve it to get an exact solution. In other words, $\bar{g}$ is an isolated solution. This issue was studied for Einstein's theory in [5–11], summarized in [12, 13], and it was extended to the generic gravity theories recently in [14, 15] and to chiral gravity in [16]. For the cosmological Einstein theory it was shown that $\sqrt{\bar{\gamma}}\,\bar{\xi}_\mu\,(\mathscr{E}^{\mu\nu})^{(2)}(h,h)$ cannot be expressed as a pure boundary [17]; it has an additional bulk part, which becomes a constraint on the linear order expansion of the metric tensor. The constraint in Einstein's theory reads as:

$$\frac{1}{\Lambda}\int_{\Sigma} d^{n-1}x\,\sqrt{\bar{\gamma}}\,\bar{\xi}_\mu(\Gamma^{\beta}_{\nu\rho})^{(1)}\bar{\nabla}^\rho\bar{\xi}^\sigma(\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(1)} = 0. \tag{9}$$

Below, in Section 2, we consider the cosmological Einstein gravity and give the Abbott–Deser (AD) formula of the conserved charges [1] for background Einstein spacetimes, and we summarize the new formulation [18, 19] to construct the conserved charges. Then we give the linear order perturbation of the new formula and its behavior under gauge transformations for (anti) de Sitter background spacetime. In Section 3, we summarize the second-order expansion of the new formula and construct the gauge transformation of the result. In Section 4, we discuss the results in terms of second-order gauge-invariant perturbation theory of Nakamura [20–23], which is a useful technique to construct the relevant quantities as gauge-variant and -invariant parts explicitly. Since the computations are somewhat lengthy, we relegate them to the Appendices.

2. Cosmological Einstein theory at first order

The linear order expansion of the cosmological Einstein tensor[1] about a generic background is:

$$(\mathcal{G}_{\mu\nu})^{(1)} := (R_{\mu\nu})^{(1)} - \frac{1}{2}\bar{g}_{\mu\nu}(R)^{(1)} - \frac{1}{2}h_{\mu\nu}\bar{R} + \Lambda h_{\mu\nu}. \tag{10}$$

This background tensor can be written as two parts [2, 24]:

$$(\mathcal{G}^{\mu\nu})^{(1)} = \bar{\nabla}_\alpha\bar{\nabla}_\beta K^{\mu\alpha\nu\beta} + X^{\mu\nu}, \tag{11}$$

with

$$X^{\mu\nu} \equiv \frac{1}{2}\left(h^{\mu\alpha}\bar{R}_\alpha{}^\nu - \bar{R}^{\mu\alpha\nu\beta}h_{\alpha\beta}\right) + \frac{1}{2}\bar{g}^{\mu\nu}h^{\rho\sigma}\bar{R}_{\rho\sigma} + \Lambda h^{\mu\nu} - \frac{1}{2}h^{\mu\nu}\bar{R}, \tag{12}$$

and

$$K^{\mu\alpha\nu\beta} \equiv \frac{1}{2}\left(\bar{g}^{\alpha\nu}\tilde{h}^{\mu\beta} + \bar{g}^{\mu\beta}\tilde{h}^{\alpha\nu} - \bar{g}^{\alpha\beta}\tilde{h}^{\mu\nu} - \bar{g}^{\mu\nu}\tilde{h}^{\alpha\beta}\right). \tag{13}$$

Here $\tilde{h}^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}h$. Suppose that the background spacetime has one Killing vector field at least, say $\bar{\xi}_\nu$. Contraction of the background Killing vector $\bar{\xi}_\nu$ with $(\mathcal{G}^{\mu\nu})^{(1)}$ yields:

$$\bar{\xi}_\nu\,(\mathcal{G}^{\mu\nu})^{(1)} = \bar{\nabla}_\alpha\left(\bar{\xi}_\nu\bar{\nabla}_\beta K^{\mu\alpha\nu\beta} - K^{\mu\beta\nu\alpha}\bar{\nabla}_\beta\bar{\xi}_\nu\right) + K^{\mu\alpha\nu\beta}\bar{R}^{\rho}_{\beta\alpha\nu}\bar{\xi}_\rho + X^{\mu\nu}\bar{\xi}_\nu, \tag{14}$$

where the last two terms vanish for a background Einstein spacetime and, therefore, the current can be written as pure divergence:

$$\bar{\xi}_\nu\,(\mathcal{G}^{\mu\nu})^{(1)} = \bar{\nabla}_\mu\bar{\nabla}_\alpha\left(\bar{\xi}_\nu\bar{\nabla}_\beta K^{\mu\alpha\nu\beta} - K^{\mu\beta\nu\alpha}\bar{\nabla}_\beta\bar{\xi}_\nu\right) := \bar{\nabla}_\mu F^{\mu\nu}. \tag{15}$$

One natural question is to ask how this expression changes when one changes the coordinates on the background spacetime. Let the vector field X be the generator of the small diffeomorphism. This equation does not change

[1]The details of the computations are given in Appendix A.

since $\delta_X \left(\mathcal{G}^{\mu\nu}\right)^{(1)} = \mathscr{L}_X \bar{\mathcal{G}}^{\mu\nu}$, which vanishes for the background Einstein spaces. Although the result is gauge-invariant, the antisymmetric tensor $F^{\mu\nu}$ as defined (15) is gauge-invariant only up to a boundary. The change of $F^{\mu\nu}$ under gauge transformations is complicated and was given in [19]. On the other hand, for (anti) de Sitter background spacetime it is possible to express the current in a completely gauge-invariant way [18, 19], starting from the second Bianchi identity on the Riemann tensor:

$$\nabla_\nu R_{\sigma\beta\mu\rho} + \nabla_\sigma R_{\beta\nu\mu\rho} + \nabla_\beta R_{\nu\sigma\mu\rho} = 0. \tag{16}$$

Using the contracted Bianchi identity $\nabla_\mu \mathcal{G}^{\mu\nu} = 0$ and the metric compatibility $\nabla_\mu g_{\alpha\beta} = 0$, and carrying out the $g^{\nu\rho}$ multiplication, one can construct a divergence-free rank-four tensor (let us denote it as $\mathcal{P}^{\nu\mu}{}_{\beta\sigma}$), which has additional properties. The $\mathcal{P}$-tensor satisfies the symmetry properties of the Riemann tensor; it vanishes for the background (anti) de Sitter space, $\bar{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma} = 0$, and contraction of its indices yields the cosmological Einstein tensor, $\mathcal{P}^{\mu}{}_{\sigma} := \mathcal{P}^{\nu\mu}{}_{\nu\sigma} = (3-n)\mathcal{G}^{\mu}_{\sigma}$. Explicitly, the $\mathcal{P}$-tensor reads as:

$$\mathcal{P}^{\nu\mu}{}_{\beta\sigma} := R^{\nu\mu}{}_{\beta\sigma} + \delta^\nu_\sigma \mathcal{G}^\mu_\beta - \delta^\nu_\beta \mathcal{G}^\mu_\sigma + \delta^\mu_\beta \mathcal{G}^\nu_\sigma - \delta^\mu_\sigma \mathcal{G}^\nu_\beta + \left(\frac{R}{2} - \frac{\Lambda(n+1)}{n-1}\right)\left(\delta^\nu_\sigma \delta^\mu_\beta - \delta^\nu_\beta \delta^\mu_\sigma\right). \tag{17}$$

This tensor was used to give a new formulation of conserved charges in [18], and also the construction was improved for the extensions of the Einstein gravity in [19]. Let us summarize how one can construct the conserved charges by using the $\mathcal{P}$-tensor. Consider the following exact equation:

$$\nabla_\nu(\mathcal{P}^{\nu\mu}{}_{\beta\sigma}\nabla^\beta \xi^\sigma) - \mathcal{P}^{\nu\mu}{}_{\beta\sigma}\nabla_\nu\nabla^\beta\xi^\sigma = 0, \tag{18}$$

which is valid for all smooth g without using the field equations. Consider the background spacetime to be the n-dimensional (anti) de Sitter spacetime with the given relations:

$$\bar{R}_{\mu\alpha\nu\beta} = \frac{2}{(n-2)(n-1)}\Lambda\left(\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta} - \bar{g}_{\mu\beta}\bar{g}_{\alpha\nu}\right), \qquad \bar{R}_{\mu\nu} = \frac{2}{n-2}\Lambda\bar{g}_{\mu\nu}, \qquad \bar{R} = \frac{2n\Lambda}{n-2}. \tag{19}$$

First-order expansion of (18) about the background (anti) de Sitter spacetime gives:

$$\bar{\nabla}_\nu\left((\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(1)}\bar{\nabla}^\beta\bar{\xi}^\sigma\right) - (\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(1)}\bar{\nabla}_\nu\bar{\nabla}^\beta\bar{\xi}^\sigma = 0, \tag{20}$$

where at linear order expansion of the $\mathcal{P}$-tensor about the (anti) de Sitter spacetime reads as:

$$(\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(1)} = (R^{\nu\mu}{}_{\beta\sigma})^{(1)} + 2(\mathcal{G}^\mu_{[\beta})^{(1)}\delta^\nu_{\sigma]} + 2(\mathcal{G}^\nu_{[\sigma})^{(1)}\delta^\mu_{\beta]} + (R)^{(1)}\delta^\mu_{[\beta}\delta^\nu_{\sigma]}. \tag{21}$$

Substituting the linearized $\mathcal{P}$-tensor, assuming $\bar{\xi}^\mu$ to be Killing vector, and using the identity $\bar{\nabla}_\nu\bar{\nabla}_\beta\xi_\sigma = \bar{R}_{\lambda\nu\beta\sigma}\bar{\xi}^\lambda$, the linearized equation (20) becomes:

$$\bar{\xi}_\nu(\mathcal{G}^{\nu\mu})^{(1)} = c\bar{\nabla}_\nu\left((\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(1)}\bar{\nabla}^\beta\bar{\xi}^\sigma\right), \tag{22}$$

where we have defined $c = \frac{(n-1)(n-2)}{4\Lambda(n-3)}$. Since $(\mathcal{G}^{\mu\nu})^{(1)}$ and $(R)^{(1)}$ vanish on the boundary, the conserved charges of the cosmological Einstein theory can be written as:

$$Q = \frac{c}{2G\Omega_{n-2}}\int_{\partial\bar{\Sigma}} d^{n-2}x\,\sqrt{\bar{\sigma}}\,\bar{n}_\mu\bar{\sigma}_\nu\left(R^{\nu\mu}{}_{\beta\sigma}\right)^{(1)}\bar{\nabla}^\beta\bar{\xi}^\sigma, \tag{23}$$

where $\bar{\sigma}_\nu$ is the unit outward normal vector on the boundary of the hypersurface, $\partial\bar{\Sigma}$. For a general background spacetime, under a variation the linear order expansion of the Riemann tensor transforms as $\delta_X \left(R^{\nu\mu}{}_{\beta\sigma}\right)^{(1)} = \mathscr{L}_X \bar{R}^{\nu\mu}{}_{\beta\sigma}$, where the vector field X is the generator of the transformation. Since $\mathscr{L}_X \bar{R}^{\nu\mu}{}_{\beta\sigma}$ vanishes for (anti) de Sitter background (for more details see [19]), it turns out that the conserved charges are given with a gauge-invariant expression that involves the linearized Riemann tensor explicitly.

3. Cosmological Einstein gravity at second order

Here we summarize perturbations of the cosmological Einstein tensor at the second-order following [17]. After using the linearized equation $\bar{\nabla}_\nu(\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(1)} = 0$, the second-order perturbation of equation (18) about background (anti) de Sitter spacetime reduces to the divergence and nondivergence parts as:

$$\bar{\xi}^\nu(\mathcal{G}^\mu_\nu)^{(2)} = c\left(\bar{\nabla}_\nu\left(\bar{\nabla}^\beta\bar{\xi}^\sigma(T^{\nu\mu}{}_{\beta\sigma})^{(2)}\right) - 2(\Gamma^\beta_{\nu\rho})^{(1)}\bar{\nabla}^\rho\bar{\xi}^\sigma(\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(1)}\right), \tag{24}$$

where we have defined a second-order background tensor:

$$(T^{\nu\mu}{}_{\beta\sigma})^{(2)} := (\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(2)} + \frac{h}{2}(\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(1)}, \tag{25}$$

and the constant c is defined below (22). Once we use the cosmological Einstein gravity field equations explicitly, it is shown that the left-hand side of (24) cannot be written as a pure divergence term [17]. It turns out that the nondivergence part can involve some divergence terms, but it cannot be completely expressed as a boundary term. It is obvious that a compact hypersurface, Σ, which has no boundary, of the manifold $\mathscr{M}$, the nondivergence part of (24), becomes an integral constraint on the solutions to the linear order perturbation of the equations. Note that if the spacetime $\mathscr{M}$ has a compact hypersurface with a boundary, then we obtain the equality (7), which relates the solutions of the first-order linearized equations to the solutions of the second-order equations. If solutions to the first- and the second-order perturbed equations, say h and k, respectively, come from linearization of an exact solution g, then the integral constraint is automatically satisfied for a spacetime manifold $\mathscr{M}$, which has a compact hypersurface without a boundary. Similarly, if the spacetime $\mathscr{M}$ has a compact hypersurface with a boundary, the equality of the conserved charges (7) will also be satisfied. Otherwise, we say that $\bar{g}$ is linearization-unstable and the perturbation theory about it does not make sense.

4. Gauge-invariant perturbation theory

The second-order gauge-invariant perturbation theory was studied in detail in [21–23] and the existence of the two perturbation parameters was included in [20]. Gauge-invariant perturbation theory is a technique that allows to compute the tensor fields in terms of gauge-variant and -invariant terms. Of course, one cannot use this method on any arbitrary background spacetime since the main assumption of the theory is decomposing the first-order perturbation of the metric tensor as:

$$h_{\mu\nu} := \widetilde{h}_{\mu\nu} + \mathscr{L}_X\bar{g}_{\mu\nu}, \tag{26}$$

where $\widetilde{h}_{\mu\nu}$ denotes the gauge-invariant part, and the gauge-variant term $\mathscr{L}_X\bar{g}_{\mu\nu}$ denotes the Lie differentiation of the background metric with respect to vector field X, which is the generator of the gauge transformation.

In the following discussion, we denote the gauge-variant quantities with a tilde and the background quantities with a bar. If such a decomposition exists, one can express the linear order perturbation of any tensor field T as:

$$(T)^{(1)} = (\widetilde{T})^{(1)} + \mathscr{L}_X \bar{T}. \tag{27}$$

Expansion of the metric tensor at the second order can be expressed as:

$$k_{\mu\nu} := \frac{1}{2}\widetilde{k}_{\mu\nu} + \mathscr{L}_X h_{\mu\nu} + \frac{1}{2}\left(\mathscr{L}_Y - \mathscr{L}_X^2\right)\bar{g}_{\mu\nu}, \tag{28}$$

where Y, just like X, generates the gauge transformations. Using (26) and (28), the second-order perturbation of any generic tensor field T can be written as:

$$(T)^{(2)} = (\tilde{T})^{(2)} + \mathscr{L}_{\mathrm{X}}(T)^{(1)} + \frac{1}{2}\left(\mathscr{L}_{\mathrm{X}} - \mathscr{L}_{\mathrm{Y}}{}^2\right)\overline{T}. \tag{29}$$

Note that since the metric tensor involves irreducible gauge-invariant terms at the linear and the second orders, the gauge-invariant part of any generic tensor has the same form. Of course, the irreducible gauge-invariant part of the tensor field only includes $\widetilde{h}_{\mu\nu}$ and $\widetilde{k}_{\mu\nu}$. Details of the computations are given in Appendix C. Here we discuss the conserved charges, which are constructed by using the $\mathcal{P}$-tensor, in terms of the gauge-invariant perturbation theory. Let us consider the first-order linearized equation (22), which we can use to construct the conserved charges. Using the gauge-invariant perturbation theory, the left-hand side of (22) is gauge-invariant:

$$\bar{\xi}_\nu(\mathcal{G}^{\nu\mu})^{(1)} = \bar{\xi}_\nu\left((\tilde{\mathcal{G}}^{\nu\mu})^{(1)} + \mathscr{L}_X\bar{\mathcal{G}}^{\mu\nu}\right) = \bar{\xi}_\nu(\tilde{\mathcal{G}}^{\nu\mu})^{(1)}, \tag{30}$$

since we consider the (anti) de Sitter background spacetime, for which we have $\bar{\mathcal{G}}^{\mu\nu} = 0$. The right-hand side of (22) can be expressed as:

$$\bar{\nabla}_\nu\left((\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(1)}\bar{\nabla}^\beta\bar{\xi}^\sigma\right) = \bar{\nabla}_\nu\left(\left((\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)} + \mathscr{L}_X\bar{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma}\right)\bar{\nabla}^\beta\bar{\xi}^\sigma\right). \tag{31}$$

This reduces to

$$\bar{\nabla}_\nu\left((\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(1)}\bar{\nabla}^\beta\bar{\xi}^\sigma\right) = \bar{\nabla}_\nu\left((\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)}\bar{\nabla}^\beta\bar{\xi}^\sigma\right) \tag{32}$$

by using the vanishing of the $\mathcal{P}$-tensor for the (anti) de Sitter background spacetime, $\bar{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma} = 0$. Thus, similar to the usual perturbation theory, the current is gauge-invariant. At the second order, the left-hand side of the equation (24) is gauge-invariant, since we have

$$(\mathcal{G}^\mu_\nu)^{(2)} = (\widetilde{\mathcal{G}}^\mu_\nu)^{(2)} + \mathscr{L}_X(\mathcal{G}^\mu_\nu)^{(1)} + \frac{1}{2}\left(\mathscr{L}_Y - \mathscr{L}_X^2\right)\bar{\mathcal{G}}^\mu_\nu, \tag{33}$$

which becomes

$$(\mathcal{G}^\mu_\nu)^{(2)} = (\widetilde{\mathcal{G}}^\mu_\nu)^{(2)}, \tag{34}$$

where we used $(\mathcal{G}^\mu_\nu)^{(1)} = 0 = \bar{\mathcal{G}}^\mu_\nu$ in (anti) de Sitter background spacetime. Now let us construct the right-hand side of (24). For the second order expansion of the $\mathcal{P}$-tensor, we get:

$$(\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(2)} = (\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(2)} + \mathscr{L}_X(\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(1)} + \frac{1}{2}\left(\mathscr{L}_Y - \mathscr{L}_X^2\right)\bar{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma}, \tag{35}$$

where the last term vanishes at the (anti) de Sitter background spacetime and so we obtain

$$(\mathcal{P}^{\nu\mu}{}_{\beta\sigma})^{(2)} = (\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(2)} + \mathscr{L}_X(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)}. \tag{36}$$

Inserting the results in (24) we can write:

$$\bar{\xi}^{\nu}(\widetilde{\mathcal{G}}^{\mu}_{\nu})^{(2)} = c\bar{\nabla}_{\nu}\left(\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(2)} + \bar{\nabla}^{\beta}\bar{\xi}^{\sigma}\mathscr{L}_X(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)} + \frac{h}{2}\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)}\right)$$
$$- 2c(\Gamma^{\beta}_{\nu\rho})^{(1)}\bar{\nabla}^{\rho}\bar{\xi}^{\sigma}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)}, \tag{37}$$

where the first terms on the right-hand and the left-hand side are in a gauge-invariant form. Then let us concentrate on the gauge-variant terms. The second term reads:

$$\bar{\nabla}_{\nu}\left(\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}\mathscr{L}_X(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)}\right) = (\bar{\nabla}_{\nu}\bar{\nabla}^{\beta}\bar{\xi}^{\sigma})\mathscr{L}_X(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)} + \bar{\nabla}^{\beta}\bar{\xi}^{\sigma}\bar{\nabla}_{\nu}\mathscr{L}_X(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)}, \tag{38}$$

where the first term vanishes after using the identity $\bar{\nabla}_{\nu}\bar{\nabla}^{\beta}\bar{\xi}^{\sigma} = \bar{R}_{\lambda\nu}{}^{\beta\sigma}\bar{\xi}^{\lambda}$, and then we obtain:

$$\bar{\nabla}_{\nu}\left(\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}\mathscr{L}_X(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)}\right) = \bar{\nabla}^{\beta}\bar{\xi}^{\sigma}\bar{\nabla}_{\nu}\mathscr{L}_X(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)}. \tag{39}$$

Using identity (74) in Appendix B, we get

$$\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}\bar{\nabla}_{\nu}\mathscr{L}_X(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)} = \bar{\nabla}^{\beta}\bar{\xi}^{\sigma}\Bigg(\mathscr{L}_X\bar{\nabla}_{\nu}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)} - \delta_X(\Gamma^{\nu}_{\nu\lambda})^{(1)}(\widetilde{\mathcal{P}}^{\lambda\mu}{}_{\beta\sigma})^{(1)}$$
$$+ 2\delta_X(\Gamma^{\lambda}_{\nu\beta})^{(1)}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\lambda\sigma})^{(1)}\Bigg). \tag{40}$$

Thus, one has

$$\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}\bar{\nabla}_{\nu}\mathscr{L}_X(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)} = \bar{\nabla}^{\beta}\bar{\xi}^{\sigma}\left(-\delta_X(\Gamma^{\nu}_{\nu\lambda})^{(1)}(\widetilde{\mathcal{P}}^{\lambda\mu}{}_{\beta\sigma})^{(1)} + 2\delta_X(\Gamma^{\lambda}_{\nu\beta})^{(1)}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\lambda\sigma})^{(1)}\right), \tag{41}$$

where we have used the first-order linearization of $\nabla_{\nu}\mathcal{P}^{\nu\mu}{}_{\beta\sigma} = 0$ about the (anti) de Sitter background metric. Substituting the results in (37) and using the decomposition of the first-order expansion of the metric tensor (26), we arrive at:

$$\bar{\xi}^{\nu}(\widetilde{\mathcal{G}}^{\mu}_{\nu})^{(2)} = c\bar{\nabla}_{\nu}\left(\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(2)} + \frac{z}{2}\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)} + \bar{\nabla}_{\rho}X^{\rho}\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)}\right)$$
$$- c\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}\delta_X(\Gamma^{\nu}_{\nu\lambda})^{(1)}(\widetilde{\mathcal{P}}^{\lambda\mu}{}_{\beta\sigma})^{(1)} + 2c(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\lambda\sigma})^{(1)}\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}\left(\delta_X(\Gamma^{\lambda}_{\nu\beta})^{(1)} - (\Gamma^{\lambda}_{\nu\beta})^{(1)}\right), \tag{42}$$

where the last two terms together form a gauge-invariant combination from the decomposition of the Christoffel connection:

$$(\Gamma^{\lambda}_{\nu\beta})^{(1)} - \delta_X(\Gamma^{\lambda}_{\nu\beta})^{(1)} = (\widetilde{\Gamma}^{\lambda}_{\nu\beta})^{(1)}. \tag{43}$$

Also, after a straightforward calculation one has:

$$\bar{\nabla}_{\nu}\left(\bar{\nabla}_{\rho}X^{\rho}\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)}\right) - \bar{\nabla}^{\beta}\bar{\xi}^{\sigma}\delta_X(\Gamma^{\nu}_{\nu\lambda})^{(1)}(\widetilde{\mathcal{P}}^{\lambda\mu}{}_{\beta\sigma})^{(1)} = 0, \tag{44}$$

which proves the vanishing of the gauge-variant terms. Collecting the pieces together, one arrives at:

$$\bar{\xi}^{\nu}(\widetilde{\mathcal{G}}^{\mu}_{\nu})^{(2)} = c\bar{\nabla}_{\nu}\left(\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(2)} + \frac{\widetilde{h}}{2}\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\beta\sigma})^{(1)}\right) - 2c\bar{\nabla}^{\beta}\bar{\xi}^{\sigma}(\widetilde{\mathcal{P}}^{\nu\mu}{}_{\lambda\sigma})^{(1)}(\widetilde{\Gamma}^{\lambda}_{\nu\beta})^{(1)}, \tag{45}$$

where the result involves divergence and nondivergence terms; $\widetilde{h}$ refers to the gauge-variant trace of the linear order perturbation of the metric. The second-order cosmological Einstein tensor is gauge-invariant in this formulation, and so are the conserved charges, which differs from the usual perturbation theory. For the compact hypersurfaces without a boundary, vanishing of the last term becomes an integral constraint on solutions of the first-order linearized equations.

5. Results and conclusions

The general covariance principle introduces a large gauge degree of freedom since in general relativity there is no preferred coordinate system. In perturbation theory, computing gauge-invariant results plays an important role since the gauge-variant results can include some unphysical parts, which depend on our choice of the coordinate system. On the other hand, the second-order gauge-invariant perturbation theory allows to compute the gauge-invariant parts of the relevant expressions. In this technique one can construct the relevant quantities as gauge-variant and -gauge-invariant parts, so there is no further need to discuss the gauge invariance since the quantities involve all information that we need.

In cosmological Einstein theory, construction of the gauge-invariant conserved charges is generally done by using the explicit form of the field equations. The current does not have to be a gauge-invariant quantity. Of course, finding a gauge-invariant current is more valuable since one only has the physical terms in this case. At the first order, starting with the second Bianchi identity, one can compute a gauge-invariant current where the Riemann tensor is involved explicitly. At the second order neither the cosmological Einstein tensor nor the conserved charges are gauge-invariant, where the gauge-variant expressions can be expressed as boundary terms.

In gauge-invariant perturbation theory, at the first order one has gauge-invariant current and conserved charges as expected. At the second order, one has a gauge-invariant cosmological Einstein tensor. Thus, the conserved charges and the current are all gauge-invariant in this theory.

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Appendix A: Second-order perturbation theory

Here we give the explicit expressions of the perturbation theory about the background spacetime $\bar{g}$ up to the third-order terms by considering the following metric tensor decomposition:

$$g_{ab} := \bar{g}_{ab} + \lambda h_{ab} + \lambda^2 k_{ab}, \tag{46}$$

where λ is a small parameter, and h_{ab} and k_{ab} are the linear and the second-order metric tensor expansions, respectively. Using $g_{ab}g^{bc} = \delta_a^c$, we can compute the expansion of the inverse metric as:

$$g^{ab} = \bar{g}^{ab} - \lambda h^{ab} + \lambda^2 \left(h_c^a h^{cb} - k^{ab}\right). \tag{47}$$

Let us consider a generic tensor T. It can be perturbed about the background spacetime $\bar{g}$ as follows:

$$T = \bar{T} + \lambda \left(T\right)^{(1)} + \lambda^2 \left(T\right)^{(2)}. \tag{48}$$

The Christoffel symbol Γ^c_{ab},

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}\Big(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\Big), \tag{49}$$

is not a tensor quantity but it can be decomposed in the same way:

$$\Gamma^c_{ab} = \bar{\Gamma}^c_{ab} + \lambda (\Gamma^c_{ab})^{(1)} + \lambda^2 (\Gamma^c_{ab})^{(2)}. \tag{50}$$

Inserting the given decompositions of the metric tensor and its inverse, we arrive at the linear order perturbation of the Christoffel symbol as:

$$(\Gamma^c_{ab})^{(1)} = \frac{1}{2}\left(\bar{\nabla}_a h_b^c + \bar{\nabla}_b h_a^c - \bar{\nabla}^c h_{ab}\right), \tag{51}$$

and the second-order perturbation as:

$$(\Gamma^c_{ab})^{(2)} = K^c_{ab} - h^c_d (\Gamma^d_{ab})^{(1)}, \tag{52}$$

where we have defined

$$K^c_{ab} = \frac{1}{2}\left(\bar{\nabla}_a k_b^c + \bar{\nabla}_b k_a^c - \bar{\nabla}^c k_{ab}\right). \tag{53}$$

We can write the linear order perturbation of the Riemann tensor as:

$$(R^a{}_{bcd})^{(1)} = \bar{\nabla}_c (\Gamma^a_{db})^{(1)} - \bar{\nabla}_d (\Gamma^a_{cb})^{(1)}, \tag{54}$$

and the second-order expansion of it as:

$$(R^a{}_{bcd})^{(2)} = \bar{\nabla}_c (\Gamma^a_{bd})^{(2)} - \bar{\nabla}_d (\Gamma^a_{bc})^{(2)} + (\Gamma^e_{bd})^{(1)} (\Gamma^a_{ce})^{(1)} - (\Gamma^e_{cb})^{(1)} (\Gamma^a_{de})^{(1)}, \tag{55}$$

which reduces to

$$(R^a{}_{bcd})^{(2)} = 2\bar{\nabla}_{[c} K^a_{d]b} - \bar{\nabla}_c \left(h^a_e (\Gamma^e_{bd})^{(1)}\right) + \bar{\nabla}_d \left(h^a_e (\Gamma^a_{bc})^{(1)}\right) + (\Gamma^e_{bd})^{(1)} (\Gamma^a_{ce})^{(1)} - (\Gamma^e_{cb})^{(1)} (\Gamma^a_{de})^{(1)}, \tag{56}$$

after using the second-order Christoffel connection given in (52). The first- and the second-order Ricci tensors are obtained from the contraction, $R_{ab} := R^c{}_{acb}$, and we get the linear order perturbation of the Ricci tensor:

$$(R_{ab})^{(1)} = \bar{\nabla}_c (\Gamma^c_{ab})^{(1)} - \bar{\nabla}_a (\Gamma^c_{cb})^{(1)}, \tag{57}$$

and the second-order Ricci tensor:

$$(R_{ab})^{(2)} = 2\bar{\nabla}_{[c}K^{c}_{a]b} - \bar{\nabla}_c\left(h^c_e(\Gamma^e_{ab})^{(1)}\right) + \bar{\nabla}_a\left(h^c_e(\Gamma^e_{cb})^{(1)}\right) + (\Gamma^e_{ab})^{(1)}(\Gamma^c_{ce})^{(1)} - (\Gamma^e_{ac})^{(1)}(\Gamma^c_{be})^{(1)}. \tag{58}$$

The first-order linearization of the scalar curvature becomes:

$$(R)^{(1)} = \bar{g}^{ab}(R_{ab})^{(1)} - \bar{R}_{ab}h^{ab}, \tag{59}$$

and Ricci scalar at the second order is:

$$(R)^{(2)} = \bar{R}_{ab}\left(h^a_c h^{bc} - k^{ab}\right) - (R_{ab})^{(1)}h^{ab} + \bar{g}^{ab}(R_{ab})^{(2)}. \tag{60}$$

The cosmological Einstein tensor,

$$\mathcal{G}_{ab} = R_{ab} - \frac{1}{2}g_{ab}R + \mathit{\Lambda} g_{ab}, \tag{61}$$

at first order yields

$$(\mathcal{G}_{ab})^{(1)} = (R_{ab})^{(1)} - \frac{1}{2}\bar{g}_{ab}(R)^{(1)} - \frac{1}{2}\bar{R}h_{ab} + \Lambda h_{ab}, \tag{62}$$

and at the second order becomes

$$(\mathcal{G}_{ab})^{(2)} = (R_{ab})^{(2)} - \frac{1}{2}\left(\bar{g}_{ab}(R)^{(2)} + h_{ab}(R)^{(1)} + k_{ab}\bar{R} + 2\Lambda k_{ab}\right). \tag{63}$$

Appendix B: Identities on Lie and covariant derivatives

The Lie derivative plays an important role in the second-order gauge-invariant perturbation theory and also in the usual gauge transformations generated by a vector field. Here we derive some useful identities heavily used in the computations. Since covariant and Lie derivatives do not commute, one needs to introduce the expressions in a compact way that appears when we change the order of these differentiations. In order to obtain the desired expressions, let us start with the Lie derivative of a rank-two tensor T:

$$\mathscr{L}_X T_{ab} = X^f\bar{\nabla}_f T_{ab} + T_{fb}\bar{\nabla}_a X^f + T_{fa}\bar{\nabla}_b X^f. \tag{64}$$

The covariant derivative of this expression yields:

$$\begin{aligned}\bar{\nabla}_c\mathscr{L}_X T_{ab} = {} & \bar{\nabla}_c X^f\bar{\nabla}_f T_{ab} + X^f\bar{\nabla}_c\bar{\nabla}_f T_{ab} + T_{fb}\bar{\nabla}_c\bar{\nabla}_a X^f + \bar{\nabla}_a X^f\bar{\nabla}_c T_{fb} \\ & + \bar{\nabla}_c\bar{\nabla}_b X^f T_{fa} + \bar{\nabla}_b X^f\bar{\nabla}_c T_{fa}.\end{aligned} \tag{65}$$

When we compute the derivatives changing the order we get:

$$\mathscr{L}_X\bar{\nabla}_c T_{ab} = X^f\bar{\nabla}_f\bar{\nabla}_c T_{ab} + \left(\bar{\nabla}_c X^f\right)\bar{\nabla}_f T_{ab} + \left(\bar{\nabla}_a X^f\right)\bar{\nabla}_c T_{fb} + \left(\bar{\nabla}_b X^f\right)\bar{\nabla}_c T_{af}, \tag{66}$$

and subtraction of the results yields:

$$\bar{\nabla}_c\mathscr{L}_X T_{ab} - \mathscr{L}_X\bar{\nabla}_c T_{ab} = X^f\left[\bar{\nabla}_c, \bar{\nabla}_f\right]T_{ab} + \left(\bar{\nabla}_c\bar{\nabla}_a X^f\right)T_{fb} + \left(\bar{\nabla}_c\bar{\nabla}_b X^f\right)T_{af}. \tag{67}$$

Using

$$\left[\bar{\nabla}_c, \bar{\nabla}_f\right]T_{ab} = \overline{R}_{cfa}{}^{e}T_{eb} + \overline{R}_{cfb}{}^{e}T_{ae}, \tag{68}$$

one can rewrite (67) as:

$$\bar{\nabla}_c \mathscr{L}_X T_{ab} = \mathscr{L}_X \bar{\nabla}_c T_{ab} + \left(\bar{\nabla}_c \bar{\nabla}_a X^e + \overline{R}_{cfa}{}^e X^f\right) T_{eb} + \left(\overline{R}_{cfb}{}^e X^f + \bar{\nabla}_c \bar{\nabla}_b X^e\right) T_{ae}. \tag{69}$$

We can relate the last expression with the gauge transformation of the first-order perturbation of the Christoffel connection as follows. Recall that under the gauge transformations generated by the vector field X, perturbation of the metric at the linear order transforms as $\delta_X h_{ab} = \bar{\nabla}_a X_b + \bar{\nabla}_b X_a = \mathscr{L}_X \bar{g}_{ab}$, and then the gauge transformation of the first-order expansion of the Christoffel symbol becomes:

$$\delta_X (\Gamma^c_{ab})^{(1)} = \frac{1}{2}\left(\bar{\nabla}_a \delta_X h^c_b + \bar{\nabla}_b \delta_X h^c_a - \bar{\nabla}^c \delta_X h_{ab}\right), \tag{70}$$

which can be rewritten as:

$$\delta_X (\Gamma^c_{ab})^{(1)} = \bar{\nabla}_a \bar{\nabla}_b X^c + \bar{R}^c{}_{bda} X^d. \tag{71}$$

Using the last expression, (69) can be expressed as:

$$\bar{\nabla}_c \mathscr{L}_X T_{ab} = \mathscr{L}_X \bar{\nabla}_c T_{ab} + \delta_X (\Gamma^e_{ca})^{(1)} T_{eb} + \delta_X (\Gamma^e_{cb})^{(1)} T_{ae}. \tag{72}$$

Similar computation for a $(1,1)$ tensor ends up with:

$$\bar{\nabla}_c \mathscr{L}_X T^a{}_b = \mathscr{L}_X \bar{\nabla}_c T^a{}_b + T^a{}_e \delta_X (\Gamma^e_{cb})^{(1)} - T^e{}_b \delta_X (\Gamma^a_{ce})^{(1)}. \tag{73}$$

We can extend the computation for a general (m,n) tensor as

$$\begin{aligned}
&\bar{\nabla}_c \mathscr{L}_X T^{a_1 a_2 \ldots a_m}{}_{b_1 b_2 \ldots b_n} = \mathscr{L}_X \bar{\nabla}_c T^{a_1 a_2 \ldots a_m}{}_{b_1 b_2 \ldots b_n} \\
&+ \delta_X (\Gamma^d_{cb_1})^{(1)} T^{a_1 a_2 \ldots a_m}{}_{d b_2 \ldots b_n} + \delta_X (\Gamma^d_{cb_2})^{(1)} T^{a_1 a_2 \ldots a_m}{}_{b_1 d \ldots b_n} + \ldots + \delta_X (\Gamma^d_{cb_n})^{(1)} T^{a_1 a_2 \ldots a_m}{}_{b_1 b_2 \ldots d} \\
&- \delta_X (\Gamma^{a_1}_{cd})^{(1)} T^{d a_2 \ldots a_m}{}_{b_1 b_2 \ldots b_n} - \delta_X (\Gamma^{a_2}_{cd})^{(1)} T^{a_1 d \ldots a_m}{}_{b_1 b_2 \ldots b_n} - \ldots - \delta_X (\Gamma^{a_m}_{cd})^{(1)} T^{a_1 a_2 \ldots d}{}_{b_1 b_2 \ldots b_n},
\end{aligned} \tag{74}$$

which simplifies the computations.

Appendix C: Second-order gauge-invariant perturbation theory

Here we summarize the results of the second-order gauge-invariant perturbation theory following [22]. The gauge transformation of a physical quantity T reads as:

$$T(p) = \bar{T}(\bar{p}) + \delta T(p), \tag{75}$$

where $T(p)$ denotes the physical quantity on spacetime $\mathscr{M}$ at point p, $\bar{T}(\bar{p})$ denotes the same quantity on the background spacetime $\mathscr{M}_0$ at point $\bar{p}$, and $\delta T(p)$ denotes the deviation of $T(p)$ from its background value $\bar{T}(\bar{p})$. We show the metric on $\mathscr{M}$ with g and the metric on the background spacetime $\mathscr{M}_0$ with $\bar{g}$. Let X and Y denote two different gauge choices and let ξ_1 and ξ_2 denote the generators of the gauge transformations. One can compute the following difference:

$$(T)^{(1)}_Y - (T)^{(1)}_X = \mathscr{L}_{\xi_1} \overline{T}, \tag{76}$$

where $(T)^{(1)}_Y$ is the first-order expansion of the physical quantity $T(p)$ in the gauge Y and $(T)^{(1)}_X$ denotes the same quantity in the gauge X. At the second order, expansion of the physical quantity $T(p)$ reads as:

$$(T)^{(2)}_Y - (T)^{(2)}_X = \mathscr{L}_{\xi_1} (T)^{(1)}_X + \left(\mathscr{L}_{\xi_2} + \mathscr{L}^2_{\xi_1}\right) \overline{T}, \tag{77}$$

which shows the difference of the perturbations under the change of the coordinate system. The generators ξ_1 and ξ_2 can be expressed as follows:

$$\xi_1 := Y - X \tag{78}$$

and

$$\xi_2 := [Y, X]\,. \tag{79}$$

Note that ξ_1 and ξ_2 may be different. Following Nakamura [22], we assume that the first-order metric perturbation can be expressed as gauge-variant and -invariant parts:

$$h_{ab} := \widetilde{h}_{ab} + \bar{\nabla}_a X_b + \bar{\nabla}_b X_a = \widetilde{h}_{ab} + \mathscr{L}_X \bar{g}_{ab}, \tag{80}$$

where $\widetilde{h}_{ab}$ is the gauge-invariant term and the $\mathscr{L}_X \bar{g}_{ab}$ denotes the gauge-variant part. From the gauge transformation given in (76), we can write

$$\delta_Y \widetilde{h}_{ab} - \delta_X \widetilde{h}_{ab} = 0, \tag{81}$$

which shows the gauge invariance of $\widetilde{h}_{ab}$. If we accept this decomposition, the second-order expansion of the metric tensor can be expressed as:

$$2k_{ab} := \widetilde{k}_{ab} + 2\mathscr{L}_X h_{ab} + \left(\mathscr{L}_Y - \mathscr{L}_X^2\right) \bar{g}_{ab}, \tag{82}$$

where $\widetilde{k}_{ab}$ is the gauge-invariant part and the additional terms are all gauge-variant. Using the given decompositions of the expansion of the metric at the first- and the second-order metric, the linear order expansion of a generic tensor field reads as:

$$(T)^{(1)} = (\widetilde{T})^{(1)} + \mathscr{L}_{\mathrm{X}} \overline{T}, \tag{83}$$

which means that the gauge-variant part of the tensor field is equivalent to the Lie derivative of this tensor field evaluated at the background spacetime. For the second-order perturbations, we obtain a similar expression as follows:

$$(T)^{(2)} = (\tilde{T})^{(2)} + \mathscr{L}_{\mathrm{X}}(T)^{(1)} + \frac{1}{2}\left(\mathscr{L}_{\mathrm{X}} - \mathscr{L}_{\mathrm{Y}}{}^2\right) \overline{T}. \tag{84}$$

Here $(\tilde{T})^{(2)}$ is the gauge-variant part of the second-order tensor $(T)^{(2)}$ and the remaining terms are gauge-variant. Using (80), the linear order perturbation of the Christoffel symbol (51) can be written as:

$$\begin{aligned}(\Gamma^c_{ab})^{(1)} = \frac{1}{2}\Bigg(&\bar{\nabla}_a(\widetilde{h}^c_b + \bar{\nabla}_b X^c + \bar{\nabla}^c X_b) + \bar{\nabla}_b(\widetilde{h}^c_a + \bar{\nabla}_a X^c + \bar{\nabla}^c X_a) \\ &-\bar{\nabla}^c(\widetilde{h}_{ab} + \bar{\nabla}_a X_b + \bar{\nabla}_b X_a)\Bigg).\end{aligned} \tag{85}$$

For simplicity, let us define a new gauge-invariant background tensor:

$$(\widetilde{\Gamma}^c_{ab})^{(1)} = \frac{1}{2}\left(\bar{\nabla}_a \widetilde{h}^c_b + \bar{\nabla}_b \widetilde{h}^c_a - \bar{\nabla}^c \widetilde{h}_{ab}\right). \tag{86}$$

Then we have

$$(\Gamma^c_{ab})^{(1)} = (\widetilde{\Gamma}^c_{ab})^{(1)} + \frac{1}{2}\left(2\bar{\nabla}_a\bar{\nabla}_b X^c + \left[\bar{\nabla}_a, \bar{\nabla}^c\right] X_b + \left[\bar{\nabla}_b, \bar{\nabla}_a\right] X^c + \left[\bar{\nabla}_b, \bar{\nabla}^c\right] X_a\right), \tag{87}$$

which reduces to

$$(\Gamma^c_{ab})^{(1)} = (\widetilde{\Gamma}^c_{ab})^{(1)} + \bar{\nabla}_a\bar{\nabla}_b X^c + \bar{R}^c{}_{bda} X^d, \tag{88}$$

where we used the identity $\left[\bar{\nabla}_a, \bar{\nabla}_b\right] X^c = \bar{R}_{ab}{}^{cd} X_d$, and also the first Bianchi identity $\bar{R}_{abcd} + \bar{R}_{bcad} + \bar{R}_{cabd} = 0$. Furthermore, from (71) we get:

$$(\Gamma_{ab}{}^c)^{(1)} = (\widetilde{\Gamma}_{ab}{}^c)^{(1)} + \delta_X(\Gamma_{ab}{}^c)^{(1)}, \tag{89}$$

which relates the first-order perturbation of the Christoffel connection with the usual gauge transformation of it. Similarly, the linear order expansion of the Riemann tensor (54) can be expressed as:

$$(R^a{}_{bcd})^{(1)} = 2\bar{\nabla}_{[c}(\widetilde{\Gamma}^a_{d]b})^{(1)} + \left[\bar{\nabla}_c, \bar{\nabla}_d\right]\bar{\nabla}_b X^a + \bar{R}^a{}_{bed}\bar{\nabla}_c X^e - \bar{R}^a{}_{bec}\bar{\nabla}_d X^e + X^e(\bar{\nabla}_c\bar{R}^a{}_{bed} - \bar{\nabla}_d\bar{R}^a{}_{bec}), \tag{90}$$

and it reduces to:

$$(R^a{}_{bcd})^{(1)} = 2\bar{\nabla}_{[c}(\widetilde{\Gamma}^a_{d]b})^{(1)} + X^e\bar{\nabla}_e\bar{R}^a{}_{bcd} + \bar{R}^a{}_{bed}\bar{\nabla}_c X^e + \bar{R}^a{}_{bce}\bar{\nabla}_d X^e + \bar{R}^a{}_{ecd}\bar{\nabla}_b X^e - \bar{R}^e{}_{bcd}\bar{\nabla}_e X^a, \tag{91}$$

after using the second Bianchi identity $\bar{\nabla}_a\bar{R}_{bcde} + \bar{\nabla}_b\bar{R}_{cade} + \bar{\nabla}_c\bar{R}_{abde} = 0$. Note that the gauge-variant part is obviously given as the Lie differentiation of the Riemann tensor evaluated at the background spacetime. Then the final expression becomes:

$$(R^a{}_{bcd})^{(1)} = 2\bar{\nabla}_{[c}(\widetilde{\Gamma}^a_{d]b})^{(1)} + \mathscr{L}_X\bar{R}^a{}_{bcd}, \tag{92}$$

which is consistent with the aim of the gauge-invariant perturbation theory. The first-order linearized Ricci tensor can be found from the contraction of the first and the third indices, $(R_{ab})^{(1)} := (R^c{}_{acb})^{(1)}$, so we have

$$(R_{ab})^{(1)} = 2\bar{\nabla}_{[c}(\widetilde{\Gamma}^c_{a]b})^{(1)} + \mathscr{L}_X\bar{R}_{ab}. \tag{93}$$

Since the first-order perturbation of the Christoffel symbol is a background tensor, one can raise and lower the indices with the background inverse metric and the metric, respectively. For an example, we use $(\Gamma_{acd})^{(1)} := \bar{g}_{bd}(\Gamma^b_{ac})^{(1)}$, where the up index is lowered as the last down index. The first-order linearized scalar curvature, by using (59) and the previous results, becomes:

$$(R)^{(1)} = 2\bar{\nabla}_{[b}(\widetilde{\Gamma}_{a]}{}^{ab})^{(1)} + \bar{g}^{ab}\mathscr{L}_X\bar{R}_{ab} - \bar{R}_{ab}(\widetilde{h}^{ab} - \mathscr{L}_X\bar{g}^{ab}). \tag{94}$$

Equivalently, it can be written as:

$$(R)^{(1)} = 2\bar{\nabla}_{[b}(\widetilde{\Gamma}_{a]}{}^{ab})^{(1)} - \bar{R}_{ab}\widetilde{h}^{ab} + \mathscr{L}_X(\bar{R}). \tag{95}$$

Inserting the corresponding expressions in the linear order perturbation of the cosmological Einstein tensor (62), we get:

$$(\mathcal{G}_{ab})^{(1)} = 2\bar{\nabla}_{[c}(\widetilde{\Gamma}_{a]b}{}^c)^{(1)} + \bar{g}_{ab}\bar{\nabla}_{[c}(\widetilde{\Gamma}_{d]}{}^{cd})^{(1)} + \frac{1}{2}\bar{g}_{ab}\bar{R}_{cd}\widetilde{h}^{cd} + \widetilde{h}_{ab}\left(\Lambda - \frac{1}{2}\bar{R}\right) + \mathscr{L}_X\bar{\mathcal{G}}_{ab}, \tag{96}$$

where only the last term is gauge-variant and it vanishes if $\bar{g}$ is a background solution. If this is the case, $(\mathcal{G}_{ab})^{(1)}$ becomes gauge-invariant.

Now we compute the decompositions of the second-order tensors in terms of gauge-variant and -invariant parts. We can compute (53) by using (82) as:

$$K^c_{ab} = \frac{1}{4}(\bar{\nabla}_a \widetilde{k}^c_b + \bar{\nabla}_b \widetilde{k}^c_a - \bar{\nabla}^c \widetilde{k}_{ab}) \\ + \frac{1}{4}\bar{g}^{cd}\left(\bar{\nabla}_a \mathscr{L}_X \left(h_{bd} + \widetilde{h}_{bd}\right) + \bar{\nabla}_b \mathscr{L}_X \left(h_{ad} + \widetilde{h}_{ad}\right) - \bar{\nabla}_d \mathscr{L}_X \left(h_{ab} + \widetilde{h}_{ab}\right)\right) \\ + \frac{1}{4}\bar{g}^{cd}\left(\bar{\nabla}_a \mathscr{L}_Y \bar{g}_{bd} + \bar{\nabla}_b \mathscr{L}_Y \bar{g}_{ad} - \bar{\nabla}_d \mathscr{L}_Y \bar{g}_{ab}\right). \tag{97}$$

After defining a new gauge-invariant second-order background tensor,

$$\widetilde{K}^c_{ab} = \frac{1}{2}\left(\bar{\nabla}_a \widetilde{k}^c_b + \bar{\nabla}_b \widetilde{k}^c_a - \bar{\nabla}^c \widetilde{k}_{ab}\right), \tag{98}$$

we obtain

$$2K^c_{ab} = \widetilde{K}^c_{ab} + \frac{1}{2}\bar{g}^{cd}\mathscr{L}_X\left(\bar{\nabla}_a\left(h_{bd} + \widetilde{h}_{bd}\right) + \bar{\nabla}_b\left(h_{ad} + \widetilde{h}_{ad}\right) - \bar{\nabla}_d\left(h_{ab} + \widetilde{h}_{ab}\right)\right) \\ + \left(h^c_e + \widetilde{h}^c_e\right)\delta_X\left(\Gamma^e_{ab}\right)^{(1)} + \delta_Y\left(\Gamma^c_{ab}\right)^{(1)}. \tag{99}$$

Note that we have used the identity (72) given in Appendix B to get the last expression. After a straightforward calculation, the result reduces to

$$2K^c_{ab} = \widetilde{K}^c_{ab} + \mathscr{L}_X\left(\left(\Gamma^c_{ab}\right)^{(1)} + \left(\widetilde{\Gamma}^c_{ab}\right)^{(1)}\right) - \mathscr{L}_X\bar{g}^{cd}\left(\left(\Gamma_{abd}\right)^{(1)} + \left(\widetilde{\Gamma}_{abd}\right)^{(1)}\right) \\ + \left(h^c_e + \widetilde{h}^c_e\right)\delta_X\left(\Gamma^e_{ab}\right)^{(1)} + \delta_Y\left(\Gamma^e_{ab}\right)^{(1)}. \tag{100}$$

We can construct the following tensor:

$$4\bar{\nabla}_{[c}K^a_{d]b} = 2\bar{\nabla}_{[c}\widetilde{K}^a_{d]b} + \bar{\nabla}_c\left(\mathscr{L}_X\left((\Gamma^a_{bd})^{(1)} + (\widetilde{\Gamma}^a_{bd})^{(1)}\right)\right) - \bar{\nabla}_c\left(\mathscr{L}_X\bar{g}^{ea}\left((\Gamma_{bde})^{(1)} + (\widetilde{\Gamma}_{bde})^{(1)}\right)\right) \\ + \bar{\nabla}_c\left(\left(h^a_e + \widetilde{h}^a_e\right)\delta_X\left(\Gamma^e_{bd}\right)^{(1)}\right) + \bar{\nabla}_c\delta_Y\left(\Gamma^e_{bd}\right)^{(1)} \\ - \bar{\nabla}_d\left(\mathscr{L}_X\left((\Gamma^a_{bc})^{(1)} + (\widetilde{\Gamma}^a_{bc})^{(1)}\right)\right) + \bar{\nabla}_d\left(\mathscr{L}_X\bar{g}^{ea}\left((\Gamma_{bce})^{(1)} + (\widetilde{\Gamma}_{bce})^{(1)}\right)\right) \\ - \bar{\nabla}_d\left(\left(h^a_e + \widetilde{h}^a_e\right)\delta_X\left(\Gamma^e_{bc}\right)^{(1)}\right) - \bar{\nabla}_d\delta_Y\left(\Gamma^a_{bc}\right)^{(1)}, \tag{101}$$

to compute the Riemann tensor (56) at the second order. Using (74), it can be written as:

$$\begin{aligned}4\bar{\nabla}_{[c}K^{a}_{d]b} = 2\bar{\nabla}_{[c}\widetilde{K}^{a}_{d]b} &+ \mathscr{L}_X\left(\bar{\nabla}_c\big((\Gamma^a_{bd})^{(1)} + (\widetilde{\Gamma}^a_{bd})^{(1)}\big) - \bar{\nabla}_d\big((\Gamma^a_{bc})^{(1)} + (\widetilde{\Gamma}^a_{bc})^{(1)}\big)\right)\\
&+ (h^a_e + \widetilde{h}^a_e)\left(\bar{\nabla}_c\delta_X(\Gamma^e_{bd})^{(1)} - \bar{\nabla}_d\delta_X(\Gamma^e_{bc})^{(1)}\right) + \bar{\nabla}_c\delta_Y(\Gamma^a_{bd})^{(1)} - \bar{\nabla}_d\delta_Y(\Gamma^a_{bc})^{(1)}\\
&+ \left((\Gamma^a_{ed})^{(1)} + (\widetilde{\Gamma}^a_{ed})^{(1)} - \bar{\nabla}_d\big(h^a_e + \widetilde{h}^a_e\big)\right)\delta_X(\Gamma^e_{cb})^{(1)}\\
&- \left((\Gamma^a_{ec})^{(1)} + (\widetilde{\Gamma}^a_{ec})^{(1)} - \bar{\nabla}_c\big(h^a_e + \widetilde{h}^a_e\big)\right)\delta_X(\Gamma^e_{db})^{(1)}\\
&- \left((\Gamma^e_{bd})^{(1)} + (\widetilde{\Gamma}^e_{bd})^{(1)}\right)\left(\delta_X(\Gamma^a_{ce})^{(1)} - \bar{\nabla}_c\big(\bar{\nabla}^a X_e + \bar{\nabla}_e X^a\big)\right)\\
&+ \left((\Gamma^e_{bc})^{(1)} + (\widetilde{\Gamma}^e_{bc})^{(1)}\right)\left(\delta_X(\Gamma^a_{de})^{(1)} - \bar{\nabla}_d\big(\bar{\nabla}^a X_e + \bar{\nabla}_e X^a\big)\right)\\
&\quad - \mathscr{L}_X\bar{g}^{ea}\left(\bar{\nabla}_c\big((\Gamma_{bde})^{(1)} + (\widetilde{\Gamma}_{bde})^{(1)}\big) - \bar{\nabla}_d\big((\Gamma_{bce})^{(1)} + (\widetilde{\Gamma}_{bce})^{(1)}\big)\right).\end{aligned} \tag{102}$$

Since the last equation is complicated, we use the results given below to get a compact form. We have:

$$\bar{\nabla}_c\delta_Y\,(\Gamma^a_{bd})^{(1)} - \bar{\nabla}_d\delta_Y\,(\Gamma^a_{bc})^{(1)} = \mathscr{L}_Y\bar{R}^a{}_{bcd}, \tag{103}$$

and from (92),

$$\bar{\nabla}_c\delta_X\,(\Gamma^e_{bd})^{(1)} - \bar{\nabla}_d\delta_X\,(\Gamma^e_{bc})^{(1)} = \mathscr{L}_X\bar{R}^e{}_{bcd} = (R^e{}_{bcd})^{(1)} - 2\bar{\nabla}_{[c}(\widetilde{\Gamma}^e_{d]b})^{(1)}, \tag{104}$$

and

$$\mathscr{L}_X\left(\bar{\nabla}_c\big((\Gamma^a_{bd})^{(1)} + (\widetilde{\Gamma}^a_{bd})^{(1)}\big) - \bar{\nabla}_d\big((\Gamma^a_{bc})^{(1)} + (\widetilde{\Gamma}^a_{bc})^{(1)}\big)\right) = \mathscr{L}_X(2(R^a{}_{bcd})^{(1)} - \mathscr{L}_X\bar{R}^a{}_{bcd}), \tag{105}$$

and also

$$\begin{aligned}&\mathscr{L}_X\bar{g}^{ea}\left(\bar{\nabla}_c\big((\Gamma_{bde})^{(1)} + (\widetilde{\Gamma}_{bde})^{(1)}\big) - \bar{\nabla}_d\big((\Gamma_{bce})^{(1)} + (\widetilde{\Gamma}_{bce})^{(1)}\big)\right)\\
&\qquad = -(\bar{\nabla}^a X_e + \bar{\nabla}_e X^a)\left(2(R^e{}_{bcd})^{(1)} - \mathscr{L}_X\bar{R}^e{}_{bcd}\right),\end{aligned} \tag{106}$$

and

$$(\Gamma^a_{ed})^{(1)} + (\widetilde{\Gamma}^a_{ed})^{(1)} - \bar{\nabla}_d(h^a_e + \widetilde{h}^a_e) = -\left((\Gamma_d{}^a{}_e)^{(1)} + (\widetilde{\Gamma}_d{}^a{}_e)^{(1)}\right). \tag{107}$$

Similarly, we have

$$(\Gamma^a_{ec})^{(1)} + (\widetilde{\Gamma}^a_{ec})^{(1)} - \bar{\nabla}_c(h^a_e + \widetilde{h}^a_e) = -\left((\Gamma_c{}^a{}_e)^{(1)} + (\widetilde{\Gamma}_c{}^a{}_e)^{(1)}\right) \tag{108}$$

and

$$\delta_X(\Gamma^a_{ce})^{(1)} - \bar{\nabla}_c(\bar{\nabla}^a X_e + \bar{\nabla}_e X^a) = -\delta_X\,(\Gamma_c{}^a{}_e)^{(1)}, \tag{109}$$

and also

$$\delta_X(\Gamma^a_{de})^{(1)} - \bar{\nabla}_d(\bar{\nabla}^a X_e + \bar{\nabla}_e X^a) = -\delta_X\,(\Gamma_d{}^a{}_e)^{(1)}. \tag{110}$$

Inserting the above results we obtain:

$$4\bar{\nabla}_{[c}K^a_{d]b} = 2\bar{\nabla}_{[c}\widetilde{K}^a_{d]b} + 2\mathscr{L}_X(R^a{}_{bcd})^{(1)} - \mathscr{L}^2_X\bar{R}^a{}_{bcd}$$
$$+ (h^a_e + \widetilde{h}^a_e)\mathscr{L}_X\bar{R}^e{}_{bcd} + \mathscr{L}_Y\bar{R}^a{}_{bcd} + (\bar{\nabla}^a X_e + \bar{\nabla}_e X^a)\left(2(R^e{}_{bcd})^{(1)} - \mathscr{L}_X\bar{R}^e{}_{bcd}\right)$$
$$- \left((\Gamma_d{}^a{}_e)^{(1)} + (\widetilde{\Gamma}_d{}^a{}_e)^{(1)}\right)\delta_X(\Gamma^e_{cb})^{(1)} + \left((\Gamma_c{}^a{}_e)^{(1)} + (\widetilde{\Gamma}_c{}^a{}_e)^{(1)}\right)\delta_X(\Gamma^e_{db})^{(1)}$$
$$+ \left((\Gamma^a_{bd})^{(1)} + (\widetilde{\Gamma}^e_{bd})^{(1)}\right)\delta_X(\Gamma_c{}^a{}_e)^{(1)} - \left((\Gamma^e_{bc})^{(1)} + (\widetilde{\Gamma}^e_{bc})^{(1)}\right)\delta_X(\Gamma_d{}^a{}_e)^{(1)}, \tag{111}$$

which can be rewritten as:

$$4\bar{\nabla}_{[c}K_{d]b}{}^a = 2\bar{\nabla}_{[c}\widetilde{K}_{d]b}{}^a - 4\widetilde{h}^a_e\bar{\nabla}_{[c}(\widetilde{\Gamma}_{d]b}{}^e)^{(1)} + 2(\widetilde{\Gamma}_d{}^a{}_e)^{(1)}(\widetilde{\Gamma}_{cb}{}^e)^{(1)} - 2(\widetilde{\Gamma}_c{}^a{}_e)^{(1)}(\widetilde{\Gamma}_{db}{}^e)^{(1)}$$
$$+2\mathscr{L}_X(R^a{}_{bcd})^{(1)} + \left(\mathscr{L}_Y - \mathscr{L}^2_X\right)\bar{R}^a{}_{bcd} + 2h^a_e(R^e{}_{bcd})^{(1)}$$
$$+2(\Gamma_c{}^a{}_e)^{(1)}(\Gamma_{db}{}^e)^{(1)} - 2(\Gamma_d{}^a{}_e)^{(1)}(\Gamma_{cb}{}^e)^{(1)}. \tag{112}$$

Using the last expression we can construct the Riemann tensor (56) at the second order as gauge-invariant and -variant quantities:

$$(R^a{}_{bcd})^{(2)} = \bar{\nabla}_{[c}\widetilde{K}^a_{d]b} - 2\widetilde{h}^a_e\bar{\nabla}_{[c}(\widetilde{\Gamma}^e_{d]b})^{(1)} + (\widetilde{\Gamma}_d{}^a{}_e)^{(1)}(\widetilde{\Gamma}^e_{cb})^{(1)} - (\widetilde{\Gamma}_c{}^a{}_e)^{(1)}(\widetilde{\Gamma}^e_{db})^{(1)}$$
$$+ \mathscr{L}_X(R^a{}_{bcd})^{(1)} + \frac{1}{2}\left(\mathscr{L}_Y - \mathscr{L}^2_X\right)\bar{R}^a{}_{bcd}, \tag{113}$$

where the second line shows the gauge-variant terms and this result is consistent with the aim of the gauge-invariant perturbation theory. Contraction of the indices yields the decomposition of the second-order Ricci tensor:

$$(R_{ab})^{(2)} = \bar{\nabla}_{[c}\widetilde{K}^c_{a]b} - 2\widetilde{h}^c_e\bar{\nabla}_{[c}(\widetilde{\Gamma}^e_{a]b})^{(1)} + (\widetilde{\Gamma}_a{}^c{}_e)^{(1)}(\widetilde{\Gamma}^e_{cb})^{(1)} - (\widetilde{\Gamma}_c{}^c{}_e)^{(1)}(\widetilde{\Gamma}^e_{ab})^{(1)}$$
$$+ \mathscr{L}_X(R_{ab})^{(1)} + \frac{1}{2}\left(\mathscr{L}_Y - \mathscr{L}^2_X\right)\bar{R}_{ab}. \tag{114}$$

The second-order Ricci scalar (60) becomes:

$$(R)^{(2)} = \bar{\nabla}_{[c}\widetilde{K}_{a]}{}^{ac} - 2\widetilde{h}^c_e\bar{\nabla}_{[c}(\widetilde{\Gamma}_{a]}{}^{ae})^{(1)} + 2(\widetilde{\Gamma}_{[c}{}^{ae})^{(1)}(\widetilde{\Gamma}_{a]}{}^c{}_e)^{(1)} + \bar{g}^{ab}\mathscr{L}_X(R_{ab})^{(1)}$$
$$+\frac{1}{2}\bar{g}^{ab}(\mathscr{L}_Y - \mathscr{L}^2_X)\bar{R}_{ab} - (\widetilde{h}^{ab} - \mathscr{L}_X\bar{g}^{ab})\left(2\bar{\nabla}_{[c}\widetilde{\Gamma}^c_{a]b} + \mathscr{L}_X\bar{R}_{ab}\right)$$
$$+(\widetilde{h}^{ac} - \mathscr{L}_X\bar{g}^{ac})\left(\widetilde{h}_{cb} + \mathscr{L}_X\bar{g}_{cb}\right)\bar{R}^b_a - \frac{1}{2}\left(\widetilde{k}_{ab} + 2\mathscr{L}_X h_{ab} + (\mathscr{L}_Y - \mathscr{L}^2_X)\bar{g}_{ab}\right)\bar{R}^{ab}, \tag{115}$$

which reduces to

$$(R)^{(2)} = \bar{\nabla}_{[c}\widetilde{K}_{a]}{}^{ac} - 2\widetilde{h}^c_e\bar{\nabla}_{[c}\widetilde{\Gamma}_{a]}{}^{ae} + 2\widetilde{\Gamma}_{[c}{}^{ae}\widetilde{\Gamma}_{a]}{}^c{}_e - 2\widetilde{h}^{ab}\bar{\nabla}_{[c}\widetilde{\Gamma}^c_{a]b} - \frac{1}{2}\widetilde{k}_{ab}\bar{R}^{ab} + \widetilde{h}^{ac}\widetilde{h}_{bc}\bar{R}^b_c$$
$$+\bar{g}^{ab}\mathscr{L}_X(R_{ab})^{(1)} + \frac{1}{2}\bar{g}^{ab}\big(\mathscr{L}_Y - \mathscr{L}^2_X\big)\bar{R}_{ab} - \bar{R}^{ab}\mathscr{L}_X h_{ab} - \frac{1}{2}\bar{R}^{ab}\big(\mathscr{L}_Y - \mathscr{L}^2_X\big)\bar{g}_{ab}$$
$$-\widetilde{h}^{ab}\mathscr{L}_X\bar{R}_{ab} + (R_{ab})^{(1)}\mathscr{L}_X\bar{g}^{ab} + \widetilde{h}^{ac}\bar{R}^b_a\mathscr{L}_X\bar{g}_{cb} - \widetilde{h}_{cb}\bar{R}^b_a\mathscr{L}_X\bar{g}^{ac} - \bar{R}^b_a\mathscr{L}_X\bar{g}^{ac}\mathscr{L}_X\bar{g}_{cb}. \tag{116}$$

Let us concentrate on the gauge-variant terms; we can write:

$$\overline{g}^{ab}\mathscr{L}_X(R_{ab})^{(1)} + (R_{ab})^{(1)}\mathscr{L}_X\overline{g}^{ab} - \bar{R}^{ab}\mathscr{L}_X h_{ab} = \mathscr{L}_X(R)^{(1)} + h_{ab}\mathscr{L}_X\bar{R}^{ab}, \tag{117}$$

and

$$\frac{1}{2}\overline{g}^{ab}(\mathscr{L}_Y - \mathscr{L}_X^2)\bar{R}_{ab} - \widetilde{h}^{ab}\mathscr{L}_X\bar{R}_{ab} - \frac{1}{2}\bar{R}^{ab}(\mathscr{L}_Y - \mathscr{L}_X^2)\overline{g}_{ab}$$
$$= \frac{1}{2}(\mathscr{L}_Y - \mathscr{L}_X^2)\bar{R} - h_{ab}\mathscr{L}_X\bar{R}^{ab} - 2h^{ab}\bar{R}_a^d\mathscr{L}_X\bar{g}_{db} - \bar{R}_d^a\mathscr{L}_X\bar{g}_{ca}\mathscr{L}_X\bar{g}^{dc}, \tag{118}$$

and also

$$\widetilde{h}^{ac}\bar{R}_a^b\mathscr{L}_X\overline{g}_{cb} - \widetilde{h}_{cb}\bar{R}_a^b\mathscr{L}_X\overline{g}^{ac} - \bar{R}_a^b\mathscr{L}_X\overline{g}^{ac}\mathscr{L}_X\overline{g}_{cb} = -\widetilde{h}_{cb}\bar{R}_a^b\mathscr{L}_X\overline{g}^{ac} + h^{ac}\bar{R}_a^b\mathscr{L}_X\overline{g}_{cb}. \tag{119}$$

Finally, the second-order scalar curvature yields:

$$(R)^{(2)} = \bar{\nabla}_{[c}\widetilde{K}_{a]}{}^{ac} - 2\widetilde{h}_e^c\bar{\nabla}_{[c}\widetilde{\Gamma}_{a]}{}^{ae} + \widetilde{\Gamma}_{[c}{}^{ae}\widetilde{\Gamma}_{a]}{}^c{}_e - 2\widetilde{h}^{ab}\bar{\nabla}_{[c}\widetilde{\Gamma}^c_{a]b} - \frac{1}{2}\bar{R}^{ab}\big(\widetilde{k}_{ab} - \widetilde{h}_a^c\widetilde{h}_{bc}\big)$$
$$+ \mathscr{L}_X(R)^{(1)} + \frac{1}{2}(\mathscr{L}_Y - \mathscr{L}_X^2)\bar{R}. \tag{120}$$

Now we can compute cosmological Einstein tensor (63) at the second order as gauge-variant and -invariant quantities. From the previous results we get:

$$(\mathcal{G}_{ab})^{(2)} = \bar{\nabla}_{[c}\widetilde{K}^c_{a]b} - 2\widetilde{h}_e{}^c\bar{\nabla}_{[c}\widetilde{\Gamma}^e_{a]b} + 2\widetilde{\Gamma}^e_{b[c}\widetilde{\Gamma}_{a]}{}^c{}_e + \mathscr{L}_X(R_{ab})^{(1)} + \frac{1}{2}(\mathscr{L}_Y - \mathscr{L}_X^2)\bar{R}_{ab}$$
$$- \frac{1}{2}\bar{g}_{ab}\Bigg(\bar{\nabla}_{[c}\widetilde{K}_{d]}{}^{dc} - 2\widetilde{h}_e^c\bar{\nabla}_{[c}\widetilde{\Gamma}_{d]}{}^{de} + \bar{R}^{cd}\big(\widetilde{h}^e{}_d\widetilde{h}_{ce} - \frac{1}{2}\widetilde{k}_{cd}\big)$$
$$+ 2\widetilde{\Gamma}^d{}_{[c}{}^e\widetilde{\Gamma}_{d]}{}^c{}_e - 2\widetilde{h}^{cd}\bar{\nabla}_{[e}\widetilde{\Gamma}^e_{c]d} + \mathscr{L}_X(R)^{(1)} + \frac{1}{2}(\mathscr{L}_Y - \mathscr{L}_X^2)\bar{R}\Bigg)$$
$$- \frac{1}{2}\left(\widetilde{h}_{ab} + \mathscr{L}_X\bar{g}_{ab}\right)\left(2\bar{\nabla}_{[c}\widetilde{\Gamma}_{d]}{}^{dc} - \bar{R}_{dc}\widetilde{h}^{dc} + \mathscr{L}_X\bar{R}\right)$$
$$+ \left(\widetilde{k}_{ab} + 2\mathscr{L}_X h_{ab} + (\mathscr{L}_Y - \mathscr{L}_X^2)\bar{g}_{ab}\right)\big(\frac{\Lambda}{2} - \frac{\bar{R}}{4}\big), \tag{121}$$

which reduces to

$$(\mathcal{G}_{ab})^{(2)} = \bar{\nabla}_{[c}\widetilde{K}^c_{a]b} - 2\widetilde{h}_e{}^c\bar{\nabla}_{[c}\widetilde{\Gamma}^e_{a]b} + 2\widetilde{\Gamma}^e_{b[c}\widetilde{\Gamma}_{a]}{}^c{}_e + \widetilde{k}_{ab}\big(\frac{\Lambda}{2} - \frac{\bar{R}}{4}\big) - \frac{1}{2}\widetilde{h}_{ab}\big(2\bar{\nabla}_{[c}\widetilde{\Gamma}_{d]}{}^{dc} - \bar{R}_{dc}\widetilde{h}^{dc}\big)$$
$$- \frac{1}{2}\bar{g}_{ab}\Bigg(\bar{\nabla}_{[c}\widetilde{K}_{d]}{}^{dc} - 2\widetilde{h}_e^c\bar{\nabla}_{[c}\widetilde{\Gamma}_{d]}{}^{de} + \bar{R}^{cd}\big(\widetilde{h}^e{}_d\widetilde{h}_{ce} - \frac{1}{2}\widetilde{k}_{cd}\big) + 2\widetilde{\Gamma}^d{}_{[c}{}^e\widetilde{\Gamma}_{d]}{}^c{}_e - 2\widetilde{h}^{cd}\bar{\nabla}_{[e}\widetilde{\Gamma}^e_{c]d}\Bigg)$$
$$+ \mathscr{L}_X(R_{ab})^{(1)} - \frac{1}{2}(R)^{(1)}\mathscr{L}_X\bar{g}_{ab} + \big(\Lambda - \frac{\bar{R}}{2}\big)\mathscr{L}_X h_{ab} - \frac{1}{2}\widetilde{h}_{ab}\mathscr{L}_X\bar{R} - \frac{1}{2}\bar{g}_{ab}\mathscr{L}_X(R)^{(1)}$$
$$- \frac{1}{4}\bar{g}_{ab}(\mathscr{L}_Y - \mathscr{L}_X^2)\bar{R} + \frac{1}{2}(\mathscr{L}_Y - \mathscr{L}_X^2)\overline{R}_{ab} + \big(\frac{\Lambda}{2} - \frac{\bar{R}}{4}\big)(\mathscr{L}_Y - \mathscr{L}_X^2)\bar{g}_{ab}, \tag{122}$$

where the first two lines denote the gauge-invariant part. Let us consider the gauge-variant terms. We can collect the third line as:

$$\mathscr{L}_X(R_{ab})^{(1)} - \frac{1}{2}(R)^{(1)}\mathscr{L}_X\bar{g}_{ab} + \big(\Lambda - \frac{\bar{R}}{2}\big)\mathscr{L}_X h_{ab} - \frac{1}{2}\widetilde{h}_{ab}\mathscr{L}_X\bar{R} - \frac{1}{2}\bar{g}_{ab}\mathscr{L}_X(R)^{(1)}$$
$$= \mathscr{L}_X(\mathcal{G}_{ab})^{(1)} + \frac{1}{2}\mathscr{L}_X\bar{g}_{ab}\mathscr{L}_X\bar{R} \tag{123}$$

and the terms on the last line yield:

$$-\frac{1}{4}\bar{g}_{ab}\left(\mathscr{L}_Y - \mathscr{L}_X^2\right)\bar{R} + \frac{1}{2}(\mathscr{L}_Y - \mathscr{L}_X^2)\overline{R}_{ab} + \big(\frac{\Lambda}{2} - \frac{\bar{R}}{4}\big)(\mathscr{L}_Y - \mathscr{L}_X^2)\bar{g}_{ab}$$
$$= \frac{1}{2}(\mathscr{L}_Y - \mathscr{L}_X^2)\bar{\mathcal{G}}_{ab} - \frac{1}{2}\mathscr{L}_X\bar{R}\mathscr{L}_X\bar{g}_{ab}. \tag{124}$$

Finally, we obtain the second-order cosmological Einstein tensor:

$$(\mathcal{G}_{ab})^{(2)} = \bar{\nabla}_{[c}\widetilde{K}^c_{a]b} - 2\widetilde{h}_e{}^c\bar{\nabla}_{[c}\widetilde{\Gamma}^e_{a]b} + 2\widetilde{\Gamma}^e_{b[c}\widetilde{\Gamma}_{a]}{}^c{}_e + \widetilde{k}_{ab}\big(\frac{\Lambda}{2} - \frac{\bar{R}}{4}\big) - \frac{1}{2}\widetilde{h}_{ab}\big(2\bar{\nabla}_{[c}\widetilde{\Gamma}_{d]}{}^{dc} - \bar{R}_{dc}\widetilde{h}^{dc}\big)$$
$$- \frac{1}{2}\bar{g}_{ab}\left(\bar{\nabla}_{[c}\widetilde{K}_{d]}{}^{dc} - 2\widetilde{h}^c_e\bar{\nabla}_{[c}\widetilde{\Gamma}_{d]}{}^{de} + \bar{R}^{cd}\left(\widetilde{h}^e{}_d\widetilde{h}_{ce} - \frac{1}{2}\widetilde{k}_{cd}\right) + 2\widetilde{\Gamma}^d{}_{[c}{}^e\widetilde{\Gamma}_{d]}{}^c{}_e - 2\widetilde{h}^{cd}\bar{\nabla}_{[e}\widetilde{\Gamma}^e_{c]d}\right)$$
$$+ \mathscr{L}_X(\mathcal{G}_{ab})^{(1)} + \frac{1}{2}\left(\mathscr{L}_Y - \mathscr{L}_X^2\right)\bar{\mathcal{G}}_{ab}, \tag{125}$$

where the gauge-variant terms vanish when $\bar{g}$ is the solution to the background equations, and h is the solution to the first-order perturbation of the equations. In this case we arrive at a pure gauge-invariant second-order cosmological Einstein tensor.