Turkish Journal of Physics

http://journals.tubitak.gov.tr/physics/

Research Article

Turk J Phys
(2020) 44: 1 – 9

doi:10.3906/fiz-1906-20

Analysis of chaotic and regular behavior of Matinyan–Yang–Mills–Higgs Hamiltonian system

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Received: 28.06.2019 • **Accepted/Published Online:** 16.10.2019 • **Final Version:** 12.02.2020

Abstract: In this study we analyze the Matinyan–Yang–Mills–Higgs (MYMH) system, based on semiclassical solutions to a Yang–Mills model, using Poincaré surfaces of section and the method of averaging. To investigate the possible chaotic behavior for the system, we simulate the trajectories of the system and calculate the Lyapunov exponents. We observe that the system displays weakly chaotic behavior. We search for the existence of approximately conserved quantities for the system using the method of averaging. In this way, we show the existence of four fixed points where period orbits exist.

Key words: Hamiltonian system, chaos theory, Poincaré section, method of averaging

1. Introduction

Hamiltonian formalism allows us to understand many physical phenomena very well in classical physics, since it treats the coordinates and momenta on equal footing. Furthermore, equations derived from most of the Hamiltonian systems include nonlinear terms, so they are usually nonintegrable. Nonlinear Hamiltonian systems may display irregular or chaotic behavior with infinitesimal changes in system parameters, including energy, which attracts the interest of many scientists working on nonlinear phenomena.

In the literature, the best known examples of chaotic Hamiltonian systems are the Fermi–Pasta–Ulam (FPU) problem [1] and the 3-particle Toda lattice problem. For these two example systems, the whole system may be integrable, but its truncations may not be so. The aim of the FPU problem is to simulate a system of particles interacting with its next neighbors where the description of interaction is nonlinear. To justify the relationship between equipartition of energy and the degrees of freedom of the system, a one-dimensional anharmonic chain of 32 or 64 particles with fixed ends was studied in [1]. In the FPU simulation, the observation of nonequipartition regions demonstrates the transition from order to chaos. In addition, the dependence of the largest Lyapunov exponents on the chosen energy density ϵ was studied in [2] and there exists a threshold value of ϵ where the trajectory of the motion displays chaotic behavior rather than the expected regular motion. The direct evidence for dependence of chaotic motion in N-dimensional Hamiltonian systems on energy density ϵ, where $N >> 2$, was given in [3]. The Toda lattice is also the subject of many studies of nonlinear dynamics. This describes the motion of a chain of particles with next-neighbor interactions [4, 5]. The Toda lattice is an integrable system where the exact solution exists. The complex and regular behavior of the Toda lattice system was numerically studied in [6]; chaotic behavior with interaction of resonance zones was observed in [7]. Another

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well-known example of chaos is the Hénon–Heiles system, which is a truncation of the Toda lattice. In [8], the influence of the total orbital energy on the trajectory of the system was shown and the transition from regular to chaotic motion was demonstrated. In a later study [9], the same author discussed the complete escape dynamics of the Henon–Heéiles system using numerical simulations and related them to the corresponding escape periods of the orbits. In [10], a possible relation between an approximately conserved quantity in the Hénon–Heiles problem and integrals of the Toda lattice was studied. In [11, 12] and [13, 14], the Hénon–Heiles system with modified potential (actually a pp-waves model) was studied and a fractal method was used to characterize the initial conditions of the system, for which the system displayed chaotic behavior at different energy levels.

To analyze a Hamiltonian system, especially the transition from order to chaos, many analytic methods are used. In this study, we have used Poincaré section analysis and the method of averaging. For the case of weakly perturbed Hamiltonian systems, Poincaré sections are very widely used to detect the integrability of the system where the exact solution does not exist. For example, in [15, 16], Poincaré maps were used to show that for electrovac, non-Abelian plane wave backgrounds and Kundt type III spacetimes, chaotic behavior exists under some conditions. The second method that we use in this study is the method of averaging, an analytic method used for weakly perturbed and multiple periodic systems. This method gives a reasonable approximation to the global behavior of nonlinear dynamical systems, especially where classical perturbation theory remains insufficient.

In this study, we investigate the Matinyan–Yang–Mills–Higgs (MYMH) Hamiltonian system by using both Poincaré section analysis and the method of averaging. This study is organized as follows: In the second section, we describe the Poincaré surface of sections and the method of averaging. In the third section, we simulate the trajectories of the MYMH system and give the Poincaré sections of the system for an energy where this system behaves chaotically. In the fourth section, conclusions and possible future work are stated.

2. Methodologies

2.1. Poincaré section (map) analysis

Poincaré section analysis is a very useful approach in investigating the phase space structure and time evolution of a dynamical system, especially for nearly integrable systems. This method has become very popular in the last decades in connection with the KAM theorem [17] for weakly perturbed Hamiltonian systems. More specifically, this method is very efficient if one wants to explore the system dynamics with two degrees of freedom. Consider the following Hamiltonian system:

$$H(p_1, p_2, q_1, q_2) = E, \tag{1}$$

where p_i and q_i (i=1,2) are generalized coordinates and momenta. This restricts the orbit to a three-dimensional hypersurface (or energy surface) in phase space. Suppose that the system has a second integral of motion I_2:

$$I_2(p_1, p_2; q_1, q_2) = C, \tag{2}$$

where C is a constant. Then the orbits of the system are embedded in a two-dimensional surface, which is the intersection of the three-dimensional energy surface E and second integral of motion C. For instance, we can express the generalized momentum p_1 with respect to generalized coordinates q_1 and q_2 as $p_1 = p_1(q_1, q_2, E, C)$. We then consider the surface where $q_2 = 0$; the trajectory of the motion embeds in a plane curve.

In general, suppose that for a given Hamiltonian of the form in Equation 1 it is not known whether or not the second integral I_2 exists. We can observe its existence by solving Hamilton's equations of motion numerically:

$$\begin{aligned} \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}, \\ \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i}, \end{aligned} \tag{3}$$

for $i = 1, 2$. Then the plot of p_2 vs. q_2 for initial values of $p_1 \geq 0$ and $q_1 = 0$ is the Poincaré section. If the system is integrable, the series of points seem to lie in a curve. If the system is nonintegrable, points on the trajectory seem to scatter around a finite region due to energy conservation.

2.2. Method of averaging

The averaging method is a useful classical method for analyzing weakly nonlinear problems. The origin of the method of averaging goes back to 1788 when the Lagrangian formulation of the gravitational three-body problem was treated as a perturbation of the two-body problem. However, up to the 1930s, no systematic proof of the method of averaging was given. Krylov and Bogoliubov first gave a systematic study of the method of averaging in [18, 19]. Then the method of averaging became one of the standard mathematical methods for approximate analysis of oscillatory processes in nonlinear dynamics. The importance of the averaging theory is that the averaged system would be a good candidate for approximation of the exact dynamics of the original system. This is very useful for cases that classical perturbation theory fail to explain.

The method of averaging can be used for dynamical systems in the following form:

$$\dot{x} = \epsilon f(x, t, \epsilon), \tag{4}$$

where $x \in U \subseteq \Re^n$, $\epsilon << 1$ and $f : \Re^n \times \Re \times \Re^+$ is C^r, $r \geq 1$, and the system has a period $\tau > 0$ in t, and U is bounded and open. According to this, values in the averaged system are defined as:

$$\dot{y} = \epsilon \frac{1}{\tau} \int_0^{\tau} f(y, t, 0) dt = \epsilon \bar{f}(y). \tag{5}$$

We have said that the averaged system is a good approximation of the original system, but it is necessary to show that qualitative properties of the solutions of the averaged system correspond to those of the original system. To justify this, we need to recall the averaging theorem given in [20]. A paraphrase of the relevant part of the theorem follows:

Theorem 2.1 (Averaging Theorem) *Equation (4) can be cast in the form of Equation (6) using the C^r near identity transformation $x = y + \epsilon w(y, t, \epsilon)$, under which Equation (4) becomes:*

$$\dot{y} = \epsilon f(y) + \epsilon^2 f_1(y, t, \epsilon), \tag{6}$$

where f_1 has period τ in t. Furthermore:

i For two solutions of Equation (4) and Equation (5), starting from initial conditions x_0, y_0, with $|x_0 - y_0| = O(\epsilon)$, then $|x_0 - y_0|$ remains $O(\epsilon)$ on a time scale $t \sim 1/\epsilon$.

ii Also, solutions lying in the stable manifold satisfy $|x_s(t) - y_s(t)| = O(\epsilon)$, if $|x_s(0) - y_s(0)| = O(\epsilon)$ for all times. If the solution lies in the unstable manifold, a similar result applies for the time interval $t \in [0, \epsilon)$.

Proof of this theorem can be found in [20], but the conclusion (ii) says that this theorem can be applied to approximate stable (or unstable) manifolds in bounded sets and for studying the Poincaré section of the system of interest.

3. Analysis of MYMH system

The MYMH system has been used for modeling the onset and limited suppression of chaotic behavior in the semiclassical Yang–Mills system [21]. This has become necessary considering our recent understanding of the stability of the universe and the mechanism for the onset of instability and subsequent stabilization during initial stages of expansion of the universe. The MYMH is a Hamiltonian system with two degrees of freedom:

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{1}{2}x^2y^2 + \frac{g^2}{2}(x^2 + y^2) - \frac{1}{2}y^2 + \frac{1}{4}py^4. \tag{7}$$

Here g and p are real constant parameters and x, y and p_x, p_y are generalized coordinates and corresponding momenta, respectively. The term $\frac{g^2}{2}(x^2 + y^2)$ is the harmonic oscillatory term, which has the potential to have a stabilizing effect on the system and lead the system to a structure of limited spatial extent. The second important term is the Higgs term: $\frac{1}{4}py^4$. Detailed analysis of the Higgs term, discussed very intuitively in [21], reveals that for high energy values, it leads the system to display chaotic characteristics. It is also important to note that there is no sharp transition from chaotic regions to ordered regions by small changes in energy values. The equations of motion are:

$$\begin{aligned} \ddot{x} &= -xy^2 - g^2x, \\ \ddot{y} &= -x^2y - g^2y - py^3. \end{aligned} \tag{8}$$

First, we try to investigate the effect of parameter g to ascertain its role in stabilizing the system. In Figures 1a–1f, Poincaré maps of the MYMH system in the $y - p_y$ plane are given for $E = 0.05(p = 1.5)$ with changing values of the parameter g.

Figure 1a is the Poincaré map of the system, in the absence of g (g=0), and it shows that the fixed point $(0,0,0,0)$ behaves as a source-like point and there exist multiple closed curves so that we have irregular motion. When we increase the value of parameter g from 0 to 0.5 (Figure 1c), the behavior of the system changes and we observe a closed orbit around the origin in the Poincaré section. For higher values of parameter g, the system reaches a structurally stable cycle of limited spatial extent, which can be seen in Figure 1e and Figure 1f. In light of these results from the Poincaré section analysis of our system, parameter g has a significant effect on the stability and time evolution of the system and the harmonic oscillatory term has a stabilizing effect on the Hamiltonian. Note that for each simulation result, we have checked that energy is conserved. Up to this point, we have observed that for low energy values, our system behaves chaotically by looking at Poincaré sections. To check this result, we have looked at the Lyapunov exponent spectrum of the system (Figures 2a–2f) and their corresponding largest Lyapunov exponents for each value of parameter g given by the Wolf algorithm in the Table.

In Figure 2a and Figure 2b, the Lyapunov exponent spectrum of the system for $g = 0.0$ and $g = 0.25$ is given, and in both cases, there exist one positive and one negative exponent since there is a conserved quantity in this Hamiltonian system, and the second positive exponent and its negative partner seems to be a pair of zero exponents [22]. In addition to this, since there are two positive exponents, chaos can be observed for these values of parameter g. Furthermore, for increasing values of parameter g, the largest Lyapunov exponent becomes close to zero so that the system is going to be in a structure of limited spatial extent. To analyze the Hamiltonian system (Equation 7) by the method of averaging, we need to introduce a small perturbation

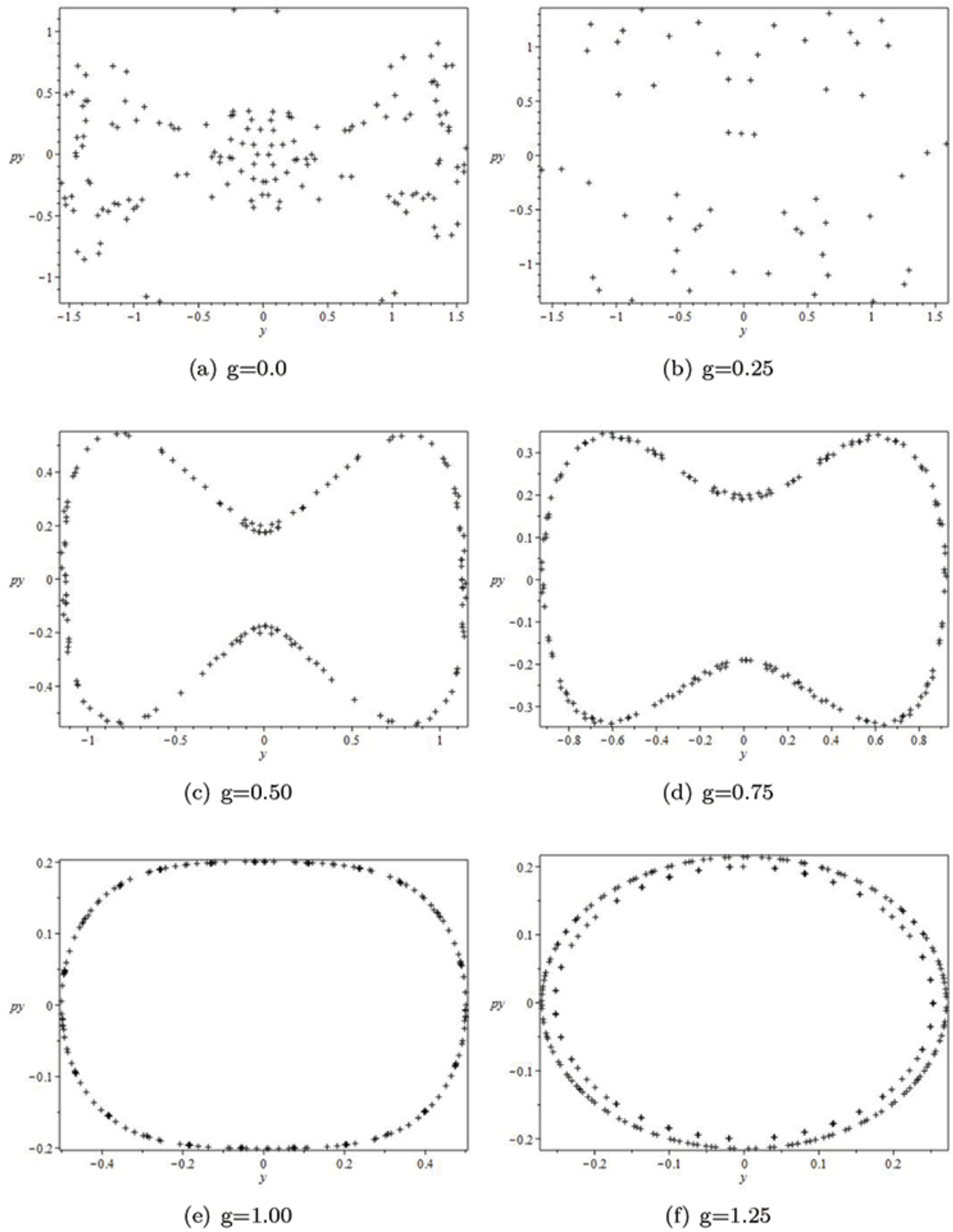


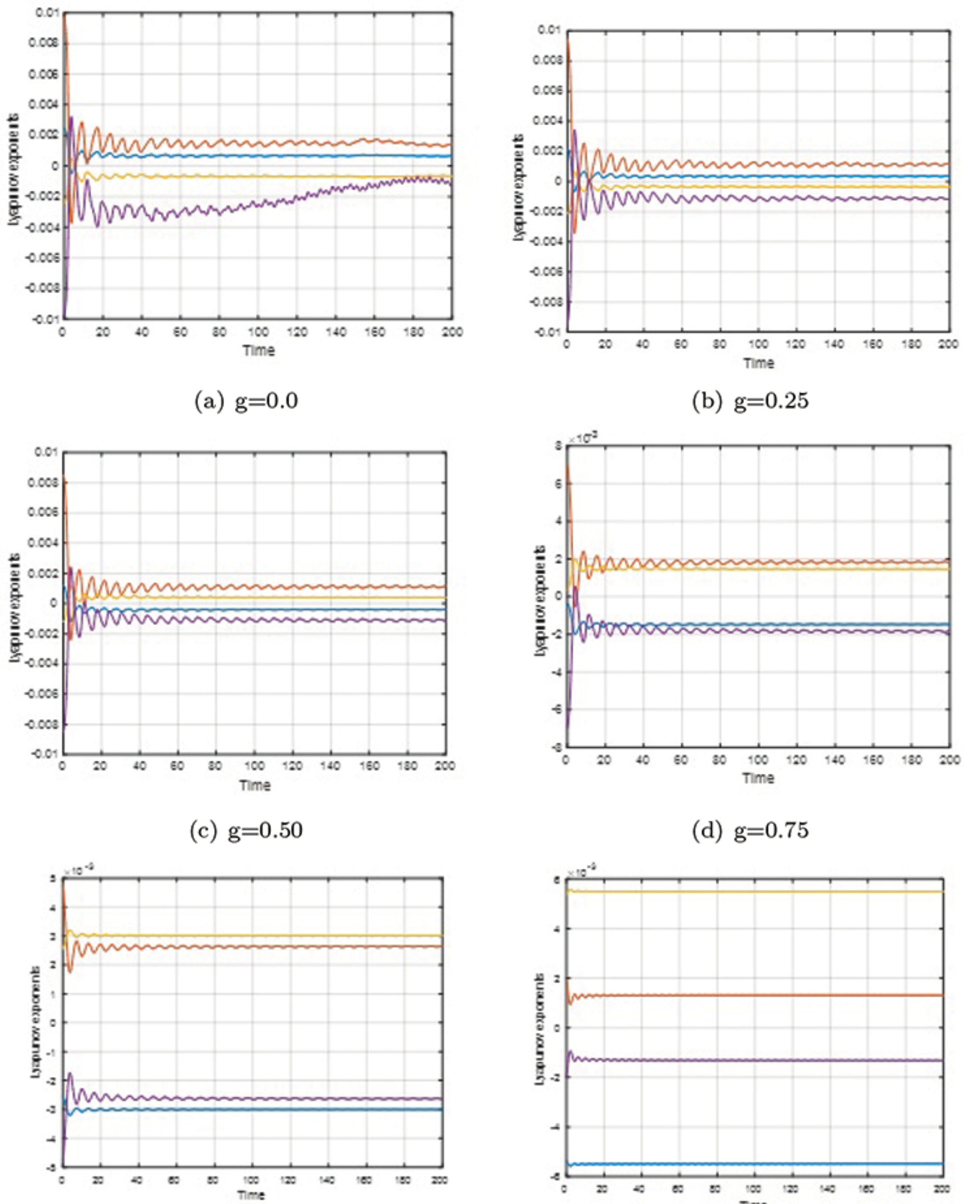
Figure 1. Poincaré maps of MYMH system at $E = 0.05$.

parameter ϵ by applying the following set of scale transformations to variables of the system:

$$\begin{aligned} x &= \sqrt{\epsilon}X, \\ y &= \sqrt{\epsilon}Y, \\ p_x &= \sqrt{\epsilon}P_x, \\ p_y &= \sqrt{\epsilon}P_y. \end{aligned} \tag{9}$$

Thus, the Hamiltonian (Equation 7) takes the following form:

$$H = \frac{1}{2}(P_x^2 + g^2P_y^2 + g^2X^2 + g^2Y^2) + \epsilon(\frac{1}{4}pY^4 + \frac{1}{2}X^2Y^2). \tag{10}$$



(a) g=0.0 (b) g=0.25 (c) g=0.50 (d) g=0.75 (e) g=1.00 (f) g=1.25

Figure 2. Lyapunov exponent spectrum of the system for different values of parameter g.

Equation 10 has the same form of an isotropic harmonic oscillator plus quadratic potential for $g^2 = 1$. Instead of applying the averaging theorem directly to Equation 10, we introduce the coordinate transformation $(X, Y, P_x, P_y) \to (r, \theta, s, \phi) \in \Re^+ \times \mathbb{S}^1 \times \Re^+ \times \mathbb{S}^1$, where $\mathbb{S}^1$ is a circle:

$$\begin{aligned} X &= r\cos(\theta+\phi), \\ Y &= s\cos(\phi), \\ P_x &= r\sin(\theta+\phi), \\ P_y &= s\sin(\phi). \end{aligned} \tag{11}$$

Table 1. Lyapunov exponents of the system for different values of parameter g.

Parameter	Maximal Lyapunov exponents			
g	λ_1	λ_2	λ_3	λ_4
0.0	0.0015	0.0006	-0.0006	-0.0015
0.25	0.0011	0.00035	-0.00035	-0.0011
0.50	0.0012	0.00039	-0.00039	-0.0012
0.75	0.0018	0.0015	-0.0015	-0.0018
1.00	0.0030	0.0026	-0.0026	-0.0030
1.25	0.0055	0.0013	-0.0013	-0.0055

Note that this change of variable is valid under the condition $s, r > 0$). Under this change of variables, the equations of motion take the following forms:

$$\begin{aligned}
\dot{r} &= -4\epsilon rs\cos(\theta+\phi)\sin(\theta+\phi)\sin^2(\phi),\\
\dot{\theta} &= -2\epsilon r\sin^2(\theta+\phi)\sin^2(\phi) + 2\epsilon s\sin^2(\theta+\phi)\sin^2(\phi) - 2\epsilon sp\sin^4(\phi),\\
\dot{s} &= -4\epsilon rs\cos(\phi)\sin^2(\theta+\phi)\sin(\theta) - 4\epsilon ps^2\cos(\phi)\sin^3(\phi),\\
\dot{\phi} &= \epsilon(2r\sin^2(\theta+\phi)\sin^2(\phi) + 2ps\sin^4(\phi)) + 1.
\end{aligned} \tag{12}$$

To make the left-hand side of Equation 12 periodic to apply the averaging theorem, we change independent variable time t to ϕ. We divide Equation 12 by $\dot{\phi}$ to obtain periodicity and ignore the equation for $\dot{\phi}$, and the system (Equation 12) takes the following form:

$$\begin{aligned}
r' &= -4\epsilon rs\cos(\theta+\phi)\sin(\theta+\phi)sin^2(\phi) + O(\epsilon^2),\\
\theta' &= -2\epsilon r\sin^2(\theta+\phi)\sin^2(\phi) + 2\epsilon s\sin^2(\theta+\phi)\sin^2(\phi) - 2\epsilon sp\sin^4(\phi) + O(\epsilon^2),\\
s' &= -4\epsilon rs\cos(\phi)\sin^2(\theta+\phi)\sin(\theta) - 4\epsilon ps^2\cos(\phi)\sin^3(\phi) + O(\epsilon^2).
\end{aligned} \tag{13}$$

In Equation 13, primes refer to derivatives with respect to ϕ (note that we only consider the terms up to order ϵ in Equation 13 since we are interested in first-order averaging). It is important to state that the system defined by Equation 13 has a period 2π for variable ϕ. In order to find periodic orbits of the system (Equation 13), the Hamiltonian system is studied under fixed energy level $H(r,\theta,s,\phi) = E$. Under this restriction, the variable s can be described in terms of E, r, θ, and ϕ using the first integral:

$$H = r + s + \epsilon pr^2\sin^4(\theta+\phi) + 2\epsilon rs\sin^2(\theta+\phi), \tag{14}$$

and s:

$$s = \left(\frac{E - pr^2\epsilon\sin^4(\theta+\phi) - r}{2\epsilon\sin^2(\theta+\phi)\sin(\phi) + 1}\right). \tag{15}$$

It is important to observe that as

$$\epsilon \to 0, \quad s \to (E - r). \tag{16}$$

Using this zero-order approximation on s, making a Taylor expansion of order ϵ, we get the following set of equations:

$$\begin{aligned} r^{'} &= 2\epsilon r(r-E)\sin(2(\theta+\phi))\sin^2(\phi)+O(\epsilon^2) \\ &= \epsilon F_{11}+O(\epsilon^2), \\ \theta^{'} &= 2\epsilon\sin^2(\phi)(E\sin^2(\theta+\phi-2r\sin^2(theta+\phi))-pE\sin^2\phi+pr\sin^2\phi)+O(\epsilon^2) \\ &= \epsilon F_{12}+O(\epsilon^2). \end{aligned} \tag{17}$$

Finally, we reach the canonical form and apply the averaging theorem. Averaging the functions F_{11} and F_{12} with respect to variable ϕ:

$$(f_{11}, f_{12}) = \int_0^{2\pi}(F_{11}, F_{12})d\phi, \tag{18}$$

gives the following:

$$\begin{aligned} f_{11} &= \sin(2\theta)\pi r(E-r), \\ f_{12} &= \tfrac{\pi}{2}(3E(1-a)-3r(p+2)+2\sin^2(\theta)(2r-E)), \end{aligned} \tag{19}$$

and then we find zeros (r^*,θ^*) of f_{11} and f_{12} and check that these points do not result in the zero of the Jacobian determinant:

$$det\left(\frac{\partial(f_{11}, f_{12})}{\partial(r,\theta)}\bigg|_{(r,\theta)=(r^*,\theta^*)}\right)\neq 0. \tag{20}$$

For the case where $f_{11}=0$, when $E=r$, s converges to zero as $\epsilon\to 0$. That is why the determinant of the Jacobian will always be zero. The zeros f_{12} occur at $r^*>0$. In summary, zeros of the determinant of the Jacobian occur at the following 4 points:

$$\begin{aligned} (r_1,\theta_1) &= \left(\tfrac{(E(p-1)}{(p-2)}, 0\right), \quad (r_2,\theta_2) = \left(\tfrac{(E(3p-1)}{(3p-2)}, \tfrac{\pi}{2}\right), \\ (r_3,\theta_3) &= \left(\tfrac{(E(3p-1)}{(3p-2)}, -\tfrac{\pi}{2}\right), \quad (r_4,\theta_4) = \left(\tfrac{(E(p-1)}{(p-2)}, \pi\right). \end{aligned} \tag{21}$$

For the case $h\neq 0$, the determinant of Jacobian at (r_i,θ_i), where i = 1,..,4, equals $(p-1)(2-p)\neq 0$.

Summarizing from the averaging theorem, the four solutions of (r^*,θ^*) of $f_{11}=f_{12}=0$ provide four periodic orbits of the system and, consequently, the Hamiltonian system is on the level $h\neq 0$ if $E(p-1)>0$ and $E(2-p)>0$ including the condition $(p-1)(2-p)\neq 0$.

4. Conclusion

In this study we have studied the MYHM system and have tried to investigate possible chaotic behavior of the system based on changing energy values and the effect of parameter g on the stability of the system. We observed that for small values of E, the system displays chaotic behavior where there exist at least two isolated islands around a fixed point $(0,0)$. When we increase the value of parameter g, the stabilizing effect of the harmonic oscillator term of the Hamiltonian becomes obvious. When we apply the method of averaging to the MYMH system, we observe the presence of four solutions with periodic orbits. In addition to the effect of parameter g, the effect of parameter p can be investigated using the averaging method with supporting numerical simulations. From the averaging analysis we demonstrated that the periodic orbits depend on the choice of the parameter p, so the numerical simulation for the range of values of parameter p will give us information about the nature of the expansion of the universe. In addition to this, further study will allow us to view alternative models against scalar cosmic acceleration. As an alternative in future works, we will try to find candidates for conserved quantities analytically by looking at Poisson brackets involving these quantities.

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