

Four-quark exotic mesons

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Abstract: We review our investigations devoted to the analysis of the resonances $Z_c(3900)$, $Z_c(4430)$, $Z_c(4100)$, $X(4140)$, $X(4274)$, $a_1(1420)$, $Y(4660)$, $X(2100)$, $X(2239)$, and $Y(2175)$ discovered in various processes by Belle, BaBar, BESIII, D0, CDF, CMS, LHCb and COMPASS collaborations. These resonances are considered as serious candidates to four-quark (tetraquark) exotic mesons. We treat all of them as diquark-antidiquark states with relevant spin-parities, find their masses and couplings, as well as explore their dominant strong decay channels. Calculations are performed in the context of the QCD sum rule method. Thus, the spectroscopic parameters of the tetraquarks are evaluated using the two-point sum rules. For computations of the strong couplings $G_{TM_1M_2}$, corresponding to the vertices TM_1M_2 and necessary to find the partial widths of the strong decays $T \rightarrow M_1M_2$, we employ either the three-point or full/approximate versions of the QCD light-cone sum rules methods. Obtained results are compared with available experimental data, and with predictions of other theoretical studies.

Key words: Exotic states, tetraquarks, QCD sum rules, strong decays

1. Introduction

During last five decades the Quantum Chromodynamics (QCD) as the theory of strong interactions was successfully used to explore spectroscopic parameters and decay channels of hadrons, to analyze features of numerous exclusive and inclusive hadronic processes. The asymptotic freedom of QCD allowed ones to employ at high momentum transfers the perturbative methods of the quantum field theory. At relatively low momentum transfer, $Q^2 \sim 1 \text{ GeV}^2$, when the coupling of the strong interactions, $\alpha_s(Q^2)$, is large enough and nonperturbative effects become important, physicists invented and applied various models and approaches to investigate hadronic processes. Now the QCD, appeared from merging of the parton model and nonabelian quantum field theory of colored quarks and gluons, is a part of the standard model (SM) of elementary particles. It is worth noting that despite numerous attempts of various experimental collaborations to find particles and interactions beyond the standard model, all observed experimental processes and measured quantities can be explained within framework of this theory.

In accordance with a contemporary paradigm, conventional mesons and baryons have quark-antiquark $q\bar{q}$ and three-quark (antiquark) $qq'q''$ structures, respectively. The electromagnetic, weak and strong interactions of these particles can be explored in the context of SM. But fundamental principles of the QCD do not forbid

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existence of multi-quark hadrons, i.e., particles made of four, five, six, etc. quarks. Apart from pure theoretical interest, multi-quark systems attracted interests of researches as possible cures to treat old standing problems of conventional hadron spectroscopy. Actually, a hypothesis about multi-quark nature some of known particles was connected namely with evident problems of quark-antiquark model of mesons. In fact, in the ordinary picture the nonet of scalar mesons are 1^3P_0 quark-antiquark states. But different and independent calculations prove that 1^3P_0 states are heavier than 1 GeV. Therefore, only the isoscalar $f_0(1370)$ and $f_0(1710)$, isovector $a_0(1450)$ or isospinor $K_0^*(1430)$ mesons can be identified as members of the 1^3P_0 multiplet. Because the masses of mesons from the light scalar nonet are below 1 GeV, during a long time the mesons $f_0(500)$, $f_0(980)$, $K_0^*(800)$, and $a_0(980)$ were subject of controversial theoretical hypothesis and suggestions. To describe unusual properties of light mesons R. Jaffe assumed that they are composed of four valence quarks [1]. Within this paradigm problems with low masses, and a mass hierarchy inside the light nonet seem found their solutions. The current status of these theoretical studies can be found in Refs. [2–5].

Another interesting result about multi-quark hadrons with important consequences was obtained also by R. Jaffe [6]. He considered six-quark (dibaryon or hexaquark) states built of only light u , d , and s quarks that belong to flavor group $SU_f(3)$. Using for analysis the MIT quark-bag model, Jaffe predicted existence of a H-dibaryon, i.e., a flavor-singlet and neutral six-quark $uuddss$ bound state with isospin-spin-parity $I(J^P) = 0(0^+)$. This double-strange six-quark structure with mass 2150 MeV lies 80 MeV below the $2m_\Lambda = 2230$ MeV threshold and is stable against strong decays. It can transform through weak interactions, which means that mean lifetime of H-dibaryon, $\tau \approx 10^{-10}$ s, is considerably longer than that of most ordinary hadrons. It is remarkable that the hexaquark $uuddss$ may be considered as a candidate to dark matter provided its mass satisfies some constraints [7–10].

Theoretical studies of stable four-quark configurations meanwhile were continued using available methods of high energy physics. The four-quark mesons or tetraquarks composed of heavy bb or cc diquarks and light antiquarks were considered as true candidates to such states. The class of exotic mesons $QQ\bar{Q}\bar{Q}$ and $QQ\bar{q}\bar{q}'$ was investigated in Refs. [11–13], where a potential model with additive two-particle interactions was utilized to find stable tetraquarks. In the framework of this method it was proved that tetraquarks $QQ\bar{q}\bar{q}$ may bind to stable states if the ratio m_Q/m_q is large. The similar conclusion was drawn in Ref. [14], where a restriction on the confining potential was its finiteness at small two-particle distances. It was found there, that the isoscalar axial-vector tetraquark $T_{bb;\bar{u}\bar{d}}^-$ resides below the threshold necessary to create B mesons, and therefore can transform only through weak decays. But the tetraquarks $T_{cc;\bar{q}\bar{q}'}$ and $T_{bc;\bar{q}\bar{q}'}$ may form both unstable or stable compounds. The stability of structures $QQ\bar{q}\bar{q}$ in the limit $m_Q \rightarrow \infty$ was studied in Ref. [15], as well.

Progress of those years was not limited by qualitative analysis of four-quark bound states. Thus already at eighties of the last century investigations of tetraquarks and hybrid hadrons were put on basis of QCD-inspired nonperturbative methods, which allowed ones to perform quantitative analyses and made first predictions for their masses and other parameters [16–21]. But achievements of these theoretical investigations then were not accompanied by reliable experimental measurements, which negatively affected development of the field.

Situation changed after observation of the charmoniumlike state $X(3872)$ reported in 2003 by the Belle collaboration [22]. Existence of the narrow resonance $X(3872)$ was later verified by various collaborations such as D0, CDF and BaBar [23–25]. Discovery of charged resonances $Z_c(4430)$ and $Z_c(3900)$ had also important impact on physics of multi-quark mesons, because they could not be confused with neutral $\bar{c}c$ charmonia, and

were candidates to four-quark mesons. The $Z_c^\pm(4430)$ were observed in B meson decays $B \rightarrow K\psi'\pi^\pm$ by Belle as resonances in the $\psi'\pi^\pm$ invariant mass distributions [26]. The resonances $Z_c^+(4430)$ and $Z_c^-(4430)$ were fixed and investigated later again by Belle in the processes $B \rightarrow K\psi'\pi^+$ and $B^0 \rightarrow K^+\psi'\pi^-$ [27, 28], respectively. Evidence for $Z_c(4430)$ and its decay to $J/\psi\pi$ was found in reaction $\bar{B}^0 \rightarrow J/\psi K^-\pi^+$ by the same collaboration [29]. Along with masses and widths of these states Belle fixed also their quantum numbers $J^P = 1^+$ as a realistic assumption. The parameters of $Z_c^-(4430)$ were measured in decay $B^0 \rightarrow K^+\psi'\pi^-$ by the LHCb collaboration as well, where its spin-parity was clearly determined to be 1^+ [30, 31].

Another charged tetraquarks $Z_c^\pm(3900)$ were found in the process $e^+e^- \rightarrow J/\psi\pi^+\pi^-$ by BESIII as resonances in the $J/\psi\pi^\pm$ invariant mass distributions [32]. These structures were seen by Belle and CLEO [33, 34], as well. The BESIII announced also detection of a neutral $Z_c^0(3900)$ state in the process $e^+e^- \rightarrow \pi^0 Z_c^0 \rightarrow \pi^0\pi^0 J/\psi$ [35].

An important observation of last few years was made by D0, which reported about a structure $X(5568)$ in a chain of transformations $X(5568) \rightarrow B_s^0\pi^\pm$, $B_s^0 \rightarrow J/\psi\phi$, $J/\psi \rightarrow \mu^+\mu^-$, $\phi \rightarrow K^+K^-$ [36]. It was noted that $X(5568)$ is first discovered exotic meson which is composed of four different quarks. Indeed, from the decay channels $X(5568) \rightarrow B_s^0\pi^\pm$ it is easy to conclude that $X(5568)$ contains b, s, u, d quarks. The resonance $X(5568)$ is a scalar particle with the positive charge conjugation parity $J^{PC} = 0^{++}$, its mass and width are equal to $m = 5567.8 \pm 2.9(\text{stat})_{-1.9}^{+0.9}(\text{syst})$ MeV and $\Gamma = 21.9 \pm 6.4(\text{stat})_{-2.5}^{+5.0}(\text{syst})$ MeV, respectively. But, very soon LHCb announced results of analyses of pp collision data at energies 7 TeV and 8 TeV collected at CERN [37]. The LHCb could not find evidence for a resonant structure in the $B_s^0\pi^\pm$ invariant mass distributions at the energies less than 5700 MeV. Stated differently, a status of the resonance $X(5568)$, probably composed of four different quarks is controversial, and necessitates further experimental studies. The exotic state named $X(5568)$ deserves to be looked for by other collaborations, and maybe, in other hadronic processes.

There are new experimental results on different resonances which may be considered as exotic mesons. Thus, recently LHCb rediscovered resonances $X(4140)$ and $X(4274)$ in the $J/\psi\phi$ invariant mass distribution by analyzed the exclusive decay $B^+ \rightarrow J/\psi\phi K^+$ [38, 39]. It reported on detection of heavy resonances $X(4500)$ and $X(4700)$ in the same $J/\psi\phi$ channel as well. Besides, LHCb fixed the spin-parities of these resonances. It turned out that $X(4140)$ and $X(4274)$ are axial-vector states $J^{PC} = 1^{++}$, whereas $X(4500)$ and $X(4700)$ are scalar particles with $J^{PC} = 0^{++}$. The first two states were already discovered by CDF in the decays $B^\pm \rightarrow J/\psi\phi K^\pm$ [40], and confirmed later by CMS and D0 [41, 42], respectively. Hence they are old members of tetraquarks' family, whereas last two heavy states were seen for the first time. The resonances X may belong to a group of hidden-charm exotic mesons. From their decay modes, it is also evident that they have to contain a strange $s\bar{s}$ component as candidates for tetraquarks. In other words, the quark content of the states X is presumably $c\bar{c}s\bar{s}$.

The family of vector resonances $\{Y\}$, which are candidates for tetraquarks, contains at least four hidden-charm particles with the quantum numbers $J^{PC} = 1^{--}$. One of them, the resonance $Y(4660)$ for the first time was detected by Belle via initial-state radiation in the process $e^+e^- \rightarrow \gamma_{\text{ISR}}\psi' \pi^+\pi^-$ as one of two resonant structures in the $\psi'\pi^+\pi^-$ invariant mass distribution [43, 44]. The second state observed in this experiment was labeled $Y(4360)$. The analyses of Refs. [43, 44] proved that these resonances cannot be identified with known charmonia. The state $Y(4630)$, which is traditionally identified with $Y(4660)$, was seen in the process

$e^+e^- \rightarrow \Lambda_c^+\Lambda_c^-$ as a peak in the $\Lambda_c^+\Lambda_c^-$ invariant mass distribution [45]. The BaBar studied the same process $e^+e^- \rightarrow \gamma_{\text{ISR}}\psi'\pi^+\pi^-$ and independently confirmed appearance of two resonant structures in the $\pi^+\pi^-\psi'$ invariant mass distribution [46]. Masses and widths of these structures allowed BaBar to identify them with resonances $Y(4660)$ and $Y(4360)$, respectively. Apart from these two resonances, there are states $Y(4260)$ and $Y(4390)$ which can also be considered as members of $\{Y\}$ family.

Among new resonances it is worth noting the state $Z_c^-(4100)$ discovered also by LHCb in the decay $B^0 \rightarrow K^+\eta_c\pi^-$ [47]. In this article, it was noted that the spin-parity of this structure is $J^P = 0^+$ or $J^P = 1^-$: both assignments are consistent with the data. From analysis of the decay $Z_c^-(4100) \rightarrow \eta_c\pi^-$ it is clear that $Z_c^-(4100)$ may be composed of quarks $cd\bar{c}\bar{u}$, and is probably another member of the family of charged Z -resonances with the same quark content: let us emphasize that the well-known resonances $Z_c^\pm(4430)$ and $Z_c^\pm(3900)$ have also the $cd\bar{c}\bar{u}$ or $cu\bar{c}\bar{d}$ contents.

In the present work, we review our theoretical works devoted to investigations of these and other resonances as candidates to exotic four-quark mesons. All investigations in our original articles were carried out in framework of the QCD sum rule approach, which is an effective nonperturbative method to study exclusive hadronic processes [48, 49]. The spectroscopic parameters of tetraquarks were calculated by means of the QCD two-point sum rule method. Their decays can be explored using other versions of the sum rule method. It is known that tetraquarks decay dominantly to two conventional mesons via strong interactions. Widths of these processes are determined by strong couplings describing vertices of initial and final particles. Therefore, strong couplings are key components of relevant investigations, and they can be extracted either from the QCD three-point sum rule approach or light-cone sum rule (LCSR) method [50].

Calculation of the strong couplings corresponding to tetraquark-meson-meson vertices in the framework of the LCSR method requires additional technical recipes. The reason is that a tetraquark contains four valence quarks, and light-cone expansion of the relevant nonlocal correlation function leads to expressions with local matrix elements of one of a final meson. Then the four-momentum conservation in a such strong vertex can be satisfied by setting the four-momentum of this meson equal to zero, i.e., by treating it a "soft" particle. Difficulties appeared due to such approximation can be evaded using a soft-meson technique of the LCSR method [51, 52]. Let us note that in the case of three-meson vertices a soft limit is an approximation to full LCSR correlation functions, whereas for vertices with one tetraquark this approach is an only way to compute them. For analyses of four-quark systems the soft-meson approximation was adjusted in Ref. [53], and successfully applied to study decays of various tetraquarks. The full version of the LCSR method is restored when exploring strong vertices of two tetraquarks and a meson [54]. In the present review all of these methods will be used to evaluate strong couplings of exotic and conventional mesons.

Detailed information on exotic resonances XYZ including a history of the problem, as well as experimental and theoretical achievements of last years are collected in numerous interesting reviews [55–66].

This review is organized in the following form: In Sec. 2, we investigate charged axial-vector resonances $Z_c(3900)$ and $Z_c(4430)$ by treating them as exotic mesons $cu\bar{c}\bar{d}$. In our approach we consider $Z_c(4430)$ as a radial excitation of the ground-state particle $Z_c(3900)$. Apart from the spectroscopic parameters of these resonances, we calculate their full widths by exploring strong decays $Z_c(3900), Z_c(4430) \rightarrow J/\psi\pi, \psi'\pi, \eta'_c\rho,$ and $\eta_c\rho$. In the next section we model the resonance $Z_c^-(4100)$ as a scalar tetraquark $cd\bar{c}\bar{u}$, and find its mass and coupling. The full width of $Z_c^-(4100)$ is evaluated by taking into account the strong decays $Z_c^-(4100) \rightarrow \eta_c\pi^-$, $\eta'_c\pi^-, D^0D^-$, and $J/\psi\rho^-$. Section 4 is devoted to analysis of the resonances $X(4140)$ and $X(4274)$ as

tetraquarks $c\bar{s}\bar{s}$ with $J^{PC} = 1^{++}$ and color-triplet and color-sextet organization of constituent diquarks, respectively. We also consider their decay modes $X(4140) \rightarrow J/\psi\phi$ and $X(4274) \rightarrow J/\psi\phi$. Our analysis demonstrate that parameters of $X(4140)$ are compatible with LHCb data, while prediction for the full width of $X(4274)$ exceeds experiment data. In Sec. 5 we investigate the tetraquark $([us][\bar{u}\bar{s}] - [ds][\bar{d}\bar{s}])/\sqrt{2}$ with spin-parities $J^{PC} = 1^{++}$ and find its parameters. The decays of this state to final mesons $f_0(980)\pi^0$, $K^{*\pm}K^\mp$, $K^{*0}\bar{K}^0$ and $\bar{K}^{*0}K^0$ are also investigated. Obtained results allow us to interpret this tetraquark as the axial-vector resonance $a_1(1420)$. The resonance $Y(4660)$ is explored in Sec. 6 as a vector tetraquark $[cs][\bar{c}\bar{s}]$ with internal structure $C\gamma_5 \otimes \gamma_5\gamma_\mu C$. We calculate the mass and coupling of this state, and investigate decay channels $Y \rightarrow J/\psi f_0(500)$, $\psi' f_0(500)$, $J/\psi f_0(980)$ and $\psi' f_0(980)$. The resonances $X(2100)$ and $X(2239)$ and their structures, spectroscopic parameters, and decay modes are considered in Sec. 7. Section 8 is reserved for analysis of the resonance $Y(2175)$, which is interpreted as the vector tetraquark $\tilde{Y} = [su][\bar{s}\bar{u}]$. We evaluate its spectroscopic parameters, and explore strong decays $\tilde{Y} \rightarrow \phi f_0(980)$, $\tilde{Y} \rightarrow \phi\eta$, and $\tilde{Y} \rightarrow \phi\eta'$. The section 9 contains our brief concluding notes. In Appendix 10, we provide expressions of quark propagators which have been used in calculations.

2. The resonances $Z_c(3900)$ and $Z_c(4430)$

The parameters of $Z^-(4430)$ were measured by the LHCb collaboration in the $B^0 \rightarrow K^+\psi'\pi^-$ decay

$$M = (4475 \pm 7_{-25}^{+15}) \text{ MeV}, \Gamma = (172 \pm 13_{-34}^{+37}) \text{ MeV}, \quad (1)$$

where its spin-parity was definitely fixed to be $J^P = 1^+$ [30, 31]. Another charged tetraquarks $Z_c^\pm(3900)$ were discovered by BESIII

$$M = (3899.0 \pm 3.6 \pm 4.9) \text{ MeV}, \Gamma = (46 \pm 10 \pm 20) \text{ MeV}, \quad (2)$$

and have the spin-parity $J^P = 1^+$ [32].

Theoretical investigations of the resonances $Z_c(3900)$ and $Z_c(4430)$ (in this section Z_c and Z , respectively) embrace plethora of models and computational methods [60, 62]. The goal of these studies is to understand internal quark-gluon structures of the states Z_c and Z , to find their spectroscopic parameters, and partial widths of relevant decay channels. Thus, Z was examined as a diquark-antidiquark [67–74] or a meson molecule state [75–79], a threshold effect [80], and a hadrocharmonium composite [81]. A situation around of the resonance Z_c does not differ significantly from studies which try to describe properties of Z . In fact, there are publications, in which Z_c is treated as the tightly bound diquark-antidiquark [53, 82–84], as a molecule built of conventional mesons [85–93], or as a threshold cusp [94, 95].

The intriguing assumption was made in Ref. [72], in which the authors interpreted Z_c and Z as the ground state and first radial excitation of the same tetraquark. This suggestion was justified by observation that dominant decay modes of these resonances are

$$Z_c^\pm \rightarrow J/\psi\pi^\pm, \quad Z^\pm \rightarrow \psi'\pi^\pm, \quad (3)$$

and that a mass splitting between $1S$ and $2S$ vector charmonia $m_{\psi'} - m_{J/\psi}$ is approximately equal to a mass gap $m_Z - m_{Z_c}$. This idea was realized in the diquark-antidiquark model in Refs. [73, 74], where the authors evaluated masses and current couplings (pole residues) of Z_c and Z . Within this scheme decay modes of the

resonances Z_c and Z were considered in Ref. [74]: these processes contain important dynamical information on structures of particles under discussion. The analysis performed in these works seems confirm a suggestion about their ground state and excited natures.

The mass and decay constant (or current coupling) are parameters of ordinary and exotic mesons, which have to be measured and evaluated primarily. As usual, all theoretical models suggested to describe the internal organization of tetraquarks and explain their features begin from evaluation of these parameters. Only after successful comparison of a theoretical result for the mass with existing experimental information a model may be accepted and used for further analysis of a tetraquark candidate. But for reliable conclusions on the structure of discovered resonances, one needs additional information. Experimental collaborations measure not only masses of resonances, but their full widths as well. They also determine spins and parities of these structures.

Because an overwhelming number of models predict correctly the masses of the resonances Z_c and Z , there is a necessity to compute full widths of these structures. In all fairness, there are publications in which decays of Z^\pm were analyzed as well. Indeed, within a phenomenological Lagrangian approach and a molecule picture decays $Z^\pm \rightarrow J/\psi\pi^\pm; \psi'\pi^\pm$ were studied in Ref. [78]. Unfortunately, in this article Z^\pm were treated as pseudoscalar or vector particles ruled out by new measurements. The decay modes $Z^+ \rightarrow J/\psi\pi^+; \psi'\pi^+$ were reanalyzed in context of the covariant quark model in Ref. [79].

In Refs. [82] and [53] the authors studied decays of the resonance Z_c by modeling it as a diquark-antidiquark state with the quantum numbers $J^{PC} = 1^{+-}$. In Ref. [82] partial widths of the decays $Z_c^+ \rightarrow J/\psi\pi^+, \eta_c\rho$, and $D^+\bar{D}^{*0}$ were computed by employing the three-point sum rule approach. The light cone sum rule method and a technique of soft-meson approximation were used to evaluate widths of processes $Z_c^+ \rightarrow J/\psi\pi^+, \eta_c\rho$ in Ref. [53].

Decays of the resonances Z_c^\pm were also investigated in the context of alternative approaches [79, 87, 89]. In fact, processes $Z_c \rightarrow J/\psi\pi, \psi'\pi, h_c(1P)\pi$ were considered in Ref. [87] using the phenomenological Lagrangian approach and modeling Z_c as an axial-vector meson molecule $\bar{D}D$. In the context of the same model radiative and leptonic decays $Z_c^+ \rightarrow J/\psi\pi^+\gamma$ and $J/\psi\pi^+l^+l^-, l = (e, \mu)$ were analyzed in Ref. [89]. The covariant quark model was employed to calculate partial widths of the channels $Z_c^+ \rightarrow J/\psi\pi^+, \eta_c\rho^+, \bar{D}^0 D^{*+}$, and $\bar{D}^{*0} D^+$ in Ref. [79]. Let us note also Ref. [93], in which the decay $Z_c \rightarrow h_c\pi$ was explored in the light front model.

In this section, we evaluate spectroscopic parameters of the resonances Z_c and Z , and investigate their decay channels by suggesting that Z_c and Z are a ground state and radial excitation of the tetraquark with $J^{PC} = 1^{+-}$, respectively. In other words, we treat them as $1S$ and $2S$ axial-vector members of the $[cu][\bar{c}\bar{d}]$ multiplet and present results of Ref. [74].

2.1. The masses and couplings of the tetraquarks Z_c and Z

The QCD two-point sum rule method is one of best approaches to calculate the spectroscopic parameters of the resonances Z_c and Z . We find the masses and couplings of positively charged tetraquarks $cu\bar{c}\bar{d}$, but due to the exact chiral limit accepted throughout this review, parameters of resonances with negative charges do not differ from them.

Starting point to extract the mass and coupling of the tetraquarks Z_c and Z is the correlation function

$$\Pi_{\mu\nu}(p) = i \int d^4x e^{ipx} \langle 0 | \mathcal{T} \{ J_\mu^Z(x) J_\nu^{Z\dagger}(0) \} | 0 \rangle. \quad (4)$$

Here, $J_\mu^Z(x)$ is the interpolating current for these tetraquarks: it corresponds to axial-vector particle $J^{PC} = 1^{+-}$ and is given by the expression

$$J_\mu^Z(x) = \frac{\epsilon\tilde{\epsilon}}{\sqrt{2}} \left\{ [u_a^T(x)C\gamma_5c_b(x)] [\bar{d}_d(x)\gamma_\mu C\bar{c}_e^T(x)] - [u_a^T(x)C\gamma_\mu c_b(x)] [\bar{d}_d(x)\gamma_5 C\bar{c}_e^T(x)] \right\}, \quad (5)$$

where the notations $\epsilon = \epsilon_{abc}$ and $\tilde{\epsilon} = \epsilon_{dec}$ are introduced. In Eq. (5) a, b, c, d, e are color indices, whereas C is the charge conjugation operator.

In these calculations, we accept the "ground-state+radially excited state+continuum" scheme, and carry out ordinary and well-known calculations: we find the physical side of the sum rules by inserting into $\Pi_{\mu\nu}(p)$ a full set of relevant states, separating contributions of the resonances Z_c and Z , and performing the integration over x . As a result, for $\Pi_{\mu\nu}^{\text{Phys}}(p)$ we obtain

$$\Pi_{\mu\nu}^{\text{Phys}}(p) = \frac{\langle 0|J_\mu^Z|Z_c(p)\rangle\langle Z_c(p)|J_\nu^{Z\dagger}|0\rangle}{m_{Z_c}^2 - p^2} + \frac{\langle 0|J_\mu^Z|Z(p)\rangle\langle Z(p)|J_\nu^{Z\dagger}|0\rangle}{m_Z^2 - p^2} + \dots, \quad (6)$$

where m_{Z_c} and m_Z are the masses of Z_c and Z , respectively. Contributions to the correlation function originating from higher resonances and continuum states are denoted by dots.

In order to finish analysis of the phenomenological side, we introduce the couplings f_{Z_c} and f_Z through matrix elements

$$\langle 0|J_\mu^Z|Z_c\rangle = f_{Z_c}m_{Z_c}\varepsilon_\mu, \quad \langle 0|J_\mu^Z|Z\rangle = f_Zm_Z\tilde{\varepsilon}_\mu, \quad (7)$$

where ε_μ and $\tilde{\varepsilon}_\mu$ are the polarization vectors of Z_c and Z , respectively. Then the function $\Pi_{\mu\nu}^{\text{Phys}}(p)$ can be written as

$$\Pi_{\mu\nu}^{\text{Phys}}(p) = \frac{m_{Z_c}^2 f_{Z_c}^2}{m_{Z_c}^2 - p^2} \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_{Z_c}^2} \right) + \frac{m_Z^2 f_Z^2}{m_Z^2 - p^2} \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2} \right) + \dots. \quad (8)$$

The Borel transformation applied to Eq. (8) yields

$$\mathcal{B}\Pi_{\mu\nu}^{\text{Phys}}(p) = m_{Z_c}^2 f_{Z_c}^2 e^{-m_{Z_c}^2/M^2} \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_{Z_c}^2} \right) + m_Z^2 f_Z^2 e^{-m_Z^2/M^2} \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2} \right) + \dots, \quad (9)$$

with M^2 being the Borel parameter.

The second component of the QCD sum rules is the correlation function $\Pi_{\mu\nu}^{\text{OPE}}(p)$ expressed in terms of quark propagators. It can be found after inserting the explicit expression of J_μ^Z into Eq. (4) and contracting heavy and light quark fields

$$\begin{aligned} \Pi_{\mu\nu}^{\text{OPE}}(p) = & -\frac{i}{2} \int d^4x e^{ipx} \epsilon\tilde{\epsilon}'\tilde{\epsilon}' \left\{ \text{Tr} \left[\gamma_\nu \tilde{S}_u^{aa'}(x) \gamma_5 S_c^{bb'}(x) \right] \text{Tr} \left[\gamma_\mu \tilde{S}_c^{e'e}(-x) \gamma_\nu S_d^{d'd}(-x) \right] \right. \\ & - \text{Tr} \left[\gamma_\mu \tilde{S}_c^{e'e}(-x) \gamma_5 S_d^{d'd}(-x) \right] \text{Tr} \left[\gamma_\nu \tilde{S}_u^{aa'}(x) \gamma_5 S_c^{bb'}(x) \right] - \text{Tr} \left[\gamma_5 \tilde{S}_u^{a'a}(x) \gamma_\mu S_c^{b'b}(x) \right] \\ & \left. \times \text{Tr} \left[\gamma_5 \tilde{S}_c^{e'e}(-x) \gamma_\nu S_d^{d'd}(-x) \right] + \text{Tr} \left[\gamma_\nu \tilde{S}_u^{aa'}(x) \gamma_\mu S_c^{bb'}(x) \right] \text{Tr} \left[\gamma_5 \tilde{S}_c^{e'e}(-x) \gamma_5 S_d^{d'd}(-x) \right] \right\}. \quad (10) \end{aligned}$$

Here

$$\tilde{S}_{c(q)}^{ab}(x) = C S_{c(q)}^{abT}(x) C, \quad (11)$$

and $S_{c(q)}^{ab}(x)$ are quark propagators: their explicit expressions are moved to Appendix 10.

The function $\Pi_{\mu\nu}^{\text{OPE}}(p)$ has the following decomposition over the Lorentz structures

$$\Pi_{\mu\nu}^{\text{OPE}}(p) = \Pi^{\text{OPE}}(p^2)g_{\mu\nu} + \tilde{\Pi}^{\text{OPE}}(p^2)p_{\mu}p_{\nu}, \quad (12)$$

where $\Pi^{\text{OPE}}(p^2)$ and $\tilde{\Pi}^{\text{OPE}}(p^2)$ are corresponding invariant amplitudes.

The QCD sum rules for the parameters of Z can be found by equating invariant amplitudes of the same structures in $\Pi_{\mu\nu}^{\text{Phys}}(p)$ and $\Pi_{\mu\nu}^{\text{OPE}}(p)$. For our purposes terms proportional to $g_{\mu\nu}$ are convenient structures, and we employ them in further calculations.

The invariant amplitude $\Pi^{\text{Phys}}(p^2)$ corresponding to structure $g_{\mu\nu}$ has a simple form. The similar function $\Pi^{\text{OPE}}(p^2)$ can be written down as the dispersion integral

$$\Pi^{\text{OPE}}(p^2) = \int_{4m_c^2}^{\infty} ds \frac{\rho^{\text{OPE}}(s)}{s - p^2}, \quad (13)$$

where the two-point spectral density is denoted by $\rho^{\text{OPE}}(s)$. It is equal to the imaginary part of the correlation function $\sim g_{\mu\nu}$, and can be obtained by means of well-known prescriptions. Let us note that calculations have been performed by taking into account various vacuum condensates up to dimension eight. We omit here the details of computations, and do not write down explicitly $\rho^{\text{OPE}}(s)$.

To suppress contributions of higher resonances and continuum states, we apply the Borel transformation on the variable p^2 to both sides of QCD sum rule's equality, and subtract them by using the assumption on the quark-hadron duality. After some operations one gets the sum rules for the parameters of the excited Z state:

$$m_Z^2 = \frac{\int_{4m_c^2}^{s_0^*} \rho^{\text{OPE}}(s) s e^{-s/M^2} ds - f_{Z_c}^2 m_{Z_c}^4 e^{-m_{Z_c}^2/M^2}}{\int_{4m_c^2}^{s_0^*} \rho^{\text{OPE}}(s) e^{-s/M^2} ds - f_{Z_c}^2 m_{Z_c}^2 e^{-m_{Z_c}^2/M^2}}, \quad (14)$$

and

$$f_Z^2 = \frac{1}{m_Z^2} \left[\int_{4m_c^2}^{s_0^*} \rho^{\text{OPE}}(s) e^{(m_Z^2 - s)/M^2} ds - f_{Z_c}^2 m_{Z_c}^2 e^{(m_Z^2 - m_{Z_c}^2)/M^2} \right], \quad (15)$$

where s_0^* is the continuum threshold parameter, which separates contributions of the tetraquarks $Z_c + Z$ and higher resonances and continuum states from each another.

We consider the mass and coupling of Z_c as input parameters in Eqs. (14) and (15). These parameters can be found from the sum rules

$$m_{Z_c}^2 = \frac{\int_{4m_c^2}^{s_0} ds \rho^{\text{OPE}}(s) s e^{-s/M^2}}{\int_{4m_c^2}^{s_0} ds \rho^{\text{OPE}}(s) e^{-s/M^2}}, \quad (16)$$

and

$$f_{Z_c}^2 = \frac{1}{m_{Z_c}^2} \int_{4m_c^2}^{s_0} ds \rho^{\text{OPE}}(s) e^{(m_{Z_c}^2 - s)/M^2}. \quad (17)$$

The expressions (16) and (17) correspond to the "ground-state + continuum" scheme when one includes the tetraquark Z into a class of "higher resonances". It is clear that $\rho^{\text{OPE}}(s)$ is the common spectral density, and

the continuum threshold should obey $s_0 < s_0^*$. Once calculated the parameters m_{Z_c} and f_{Z_c} of the tetraquark Z_c appear as an input information in the sum rules (14) and (15) for the tetraquark Z .

The sum rules obtained here depend on various vacuum condensates, which are input parameters in numerical computations. These sum rules contain also the mass of c quark. The quark, gluon, and mixed vacuum condensates, as well as masses of the quarks are well known:

$$\begin{aligned}
 \langle \bar{q}q \rangle &= -(0.24 \pm 0.01)^3 \text{ GeV}^3, \quad \langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle, \quad \langle \bar{q}g_s\sigma Gq \rangle = m_0^2 \langle \bar{q}q \rangle, \\
 \langle \bar{s}g_s\sigma Gs \rangle &= m_0^2 \langle \bar{s}s \rangle, \quad m_0^2 = (0.8 \pm 0.1) \text{ GeV}^2, \quad \langle \frac{\alpha_s G^2}{\pi} \rangle = (0.012 \pm 0.004) \text{ GeV}^4, \\
 \langle g_s^3 G^3 \rangle &= (0.57 \pm 0.29) \text{ GeV}^6, \quad m_s = 93_{-5}^{+11} \text{ MeV}, \quad m_c = 1.27 \pm 0.2 \text{ GeV}, \\
 m_b &= 4.18_{-0.02}^{+0.03} \text{ GeV}.
 \end{aligned} \tag{18}$$

The masses and couplings of the tetraquarks depend on auxiliary parameters M^2 and $s_0(s_0^*)$, which have to satisfy constraints of sum rule computations. It means that edges of the working windows for the Borel parameter should be fixed by convergence of the operator product expansion (OPE) and restriction imposed on the pole contribution (PC). Additionally, extracted quantities should be stable while the parameter M^2 is varied within this region. Analysis carried out by taking into account these conditions allows one to extract regions of the parameters M^2 and s_0 , where aforementioned constraints are fulfilled. Our predictions are collected in Table 1, where we present not only parameters of the resonances Z and Z_c , but write down also windows for M^2 and $s_0(s_0^*)$ used to extract them. One can see that agreement between m_{Z_c} and experimental data is excellent. It also confirms our previous prediction for m_{Z_c} made in Ref. [53]. Result for m_Z is less than the corresponding LHCb datum, but it is still compatible with measurements provided one takes into account errors of calculations.

Table 1. The masses and current couplings of the resonances Z_c and Z .

Resonance	Z_c	Z
M^2 (GeV ²)	3 – 6	3 – 6
$s_0(s_0^*)$ (GeV ²)	4.2 ² – 4.4 ²	4.8 ² – 5.2 ²
m_Z (MeV)	3901 ⁺¹²⁵ ₋₁₄₈	4452 ⁺¹⁸² ₋₂₂₈
$f_Z \times 10^2$ (GeV ⁴)	0.42 ^{+0.07} _{-0.09}	1.48 ^{+0.31} _{-0.42}

2.2. Strong decays of the tetraquarks Z_c and Z

The masses of Z_c and Z obtained above should be employed to distinguish from each another their kinematically allowed and forbidden decay modes. Moreover, parameters of these resonances enter as input information to sum rules for strong couplings corresponding to vertices $Z_c M_h M_l$ and $Z M_h M_l$, and are also embedded into formulas for decay widths.

The tetraquarks Z_c and Z can dissociate to conventional mesons through different ways. We consider only their decays to mesons $J/\psi\pi$, $\psi'\pi$, and $\eta_c\rho$, $\eta'_c\rho$. One can find masses and decay constants of these mesons in Table 2, and easily check that these processes are kinematically allowed modes.

In our treatment, the tetraquark Z is the first radial excitation of Z_c . It is clear that ψ' and $\eta'_c \equiv \eta_c(2S)$ are first radial excitations of the mesons J/ψ and η_c , respectively. Therefore, in framework of the QCD sum

rule method, we have to analyze decays $Z_c, Z \rightarrow J/\psi\pi, \psi'\pi$ and $Z_c, Z \rightarrow \eta_c\rho, \eta'_c\rho$ in a correlated form. The reason is that, in the QCD sum rules particles are modeled by interpolating currents which couple both to their ground states and excitations.

Table 2. Masses and decay constants of the conventional mesons.

Parameters	Values (MeV)
$m_{J/\psi}$	3096.900 ± 0.006
$f_{J/\psi}$	411 ± 7
$m_{\psi'}$	3686.097 ± 0.025
$f_{\psi'}$	279 ± 8
m_{η_c}	2983.4 ± 0.5
f_{η_c}	404
$m_{\eta'_c}$	3686.2 ± 1.2
$f_{\eta'_c}$	331
m_π	139.57018 ± 0.00035
f_π	131.5
m_ρ	775.26 ± 0.25
f_ρ	216 ± 3

2.2.1. Decays $Z_c, Z \rightarrow J/\psi\pi, \psi'\pi$

In order to calculate partial widths of the decays $Z_c \rightarrow J/\psi\pi, \psi'\pi$ and $Z \rightarrow J/\psi\pi, \psi'\pi$, we begin from analysis of the correlation function

$$\Pi_{\mu\nu}(p, q) = i \int d^4x e^{ipx} \langle \pi(q) | \mathcal{T} \{ J_\mu^\psi(x) J_\nu^{Z\dagger}(0) \} | 0 \rangle, \quad (19)$$

where

$$J_\mu^\psi(x) = \bar{c}_i(x) \gamma_\mu c_i(x), \quad (20)$$

and ψ is one of J/ψ and ψ' mesons. The current $J_\nu^Z(x)$ is defined by Eq. (5), and $p' = p + q$ and p, q are the momenta of initial and final particles, respectively. As we have just emphasized above the interpolating currents $J_\nu^Z(x)$ and $J_\mu^\psi(x)$ couple to Z_c, Z and $J/\psi, \psi'$, respectively. Therefore, the correlation function $\Pi_{\mu\nu}^{\text{Phys}}(p, q)$, necessary for our purposes, contains four terms

$$\begin{aligned} \Pi_{\mu\nu}^{\text{Phys}}(p, q) = & \sum_{\psi=J/\psi, \psi'} \left[\frac{\langle 0 | J_\mu^\psi | \psi(p) \rangle \langle \psi(p) \pi(q) | Z_c(p') \rangle \langle Z_c(p') | J_\nu^{Z\dagger} | 0 \rangle}{p^2 - m_\psi^2} \right. \\ & \left. + \frac{\langle 0 | J_\mu^\psi | \psi(p) \rangle \langle \psi(p) \pi(q) | Z(p') \rangle \langle Z(p') | J_\nu^{Z\dagger} | 0 \rangle}{p^2 - m_\psi^2} \right] + \dots \end{aligned} \quad (21)$$

To find the correlation function, we use the matrix elements

$$\langle 0 | J_\mu^\psi | \psi(p) \rangle = f_\psi m_\psi \varepsilon_\mu, \quad \langle Z_c(p') | J_\nu^{Z\dagger} | 0 \rangle = f_{Z_c} m_{Z_c} \varepsilon_\nu^{t*}, \quad \langle Z(p') | J_\nu^{Z\dagger} | 0 \rangle = f_Z m_Z \tilde{\varepsilon}_\nu^{t*}, \quad (22)$$

with m_ψ , f_ψ , and ε_μ being the mass, decay constant, and polarization vector of J/ψ or ψ' mesons. Accordingly, ε'_ν and $\tilde{\varepsilon}'_\nu$ stand for the polarization vectors of the states Z_c and Z , respectively. We model the vertices in the forms

$$\begin{aligned}\langle\psi(p)\pi(q)|Z_c(p')\rangle &= g_{Z_c\psi\pi}[(p\cdot p')(\varepsilon^*\cdot\varepsilon')-(p\cdot\varepsilon')(p'\cdot\varepsilon^*)], \\ \langle\psi(p)\pi(q)|Z(p')\rangle &= g_{Z\psi\pi}[(p\cdot p')(\varepsilon^*\cdot\tilde{\varepsilon}')-(p\cdot\tilde{\varepsilon}')(p'\cdot\varepsilon^*)],\end{aligned}\quad (23)$$

where $g_{Z_c\psi\pi}$ and $g_{Z\psi\pi}$ are the strong couplings, that have to be evaluated from the sum rules. After some transformations, we get for $\Pi_{\mu\nu}^{\text{Phys}}(p, q)$ the expression

$$\begin{aligned}\Pi_{\mu\nu}^{\text{Phys}}(p, q) &= \sum_{\psi=J/\psi, \psi'} \left[\frac{f_\psi f_{Z_c} m_{Z_c} m_\psi g_{Z_c\psi\pi}}{(p'^2 - m_{Z_c}^2)(p^2 - m_\psi^2)} \left(\frac{m_{Z_c}^2 + m_\psi^2}{2} g_{\mu\nu} - p'_\mu p_\nu \right) \right. \\ &\quad \left. + \frac{f_\psi f_Z m_Z m_\psi g_{Z\psi\pi}}{(p'^2 - m_Z^2)(p^2 - m_\psi^2)} \left(\frac{m_Z^2 + m_\psi^2}{2} g_{\mu\nu} - p'_\mu p_\nu \right) \right] + \dots\end{aligned}\quad (24)$$

It is convenient to proceed by choosing structures $\sim g_{\mu\nu}$ and corresponding invariant amplitudes.

To derive the second ingredient of the sum rule $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$, we express the correlation function (19) in terms of the quark propagators, and find

$$\begin{aligned}\Pi_{\mu\nu}^{\text{OPE}}(p, q) &= \int d^4x e^{ipx} \frac{\tilde{\epsilon}\tilde{\epsilon}}{\sqrt{2}} \left[\gamma_5 \tilde{S}_c^{ib}(x) \gamma_\mu \tilde{S}_c^{ei}(-x) \gamma_\nu + \gamma_\nu \tilde{S}_c^{ib}(x) \gamma_\mu \tilde{S}_c^{ei}(-x) \gamma_5 \right]_{\alpha\beta} \\ &\quad \times \langle\pi(q)|\bar{u}_\alpha^a(0)d_\beta^d(0)|0\rangle,\end{aligned}\quad (25)$$

where α and β are the spinor indices.

In order to continue, we expand $\bar{u}_\alpha^a(0)d_\beta^d(0)$ over the full set of Dirac matrices Γ^j and project them onto the color-singlet states by employing the formula

$$\bar{u}_\alpha^a d_\beta^d \rightarrow \frac{1}{12} \Gamma_{\beta\alpha}^j \delta_{ad} (\bar{u} \Gamma^j d), \quad (26)$$

where Γ^j

$$\Gamma^j = \mathbf{1}, \gamma_5, \gamma_\lambda, i\gamma_5\gamma_\lambda, \sigma_{\lambda\rho}/\sqrt{2}. \quad (27)$$

Then the matrix elements $\langle\pi(q)|\bar{u}_\alpha^a(0)d_\beta^d(0)|0\rangle$ transform in accordance with the scheme

$$\langle\pi(q)|\bar{u}_\alpha^a(0)d_\beta^d(0)|0\rangle \rightarrow \frac{1}{12} \Gamma_{\beta\alpha}^j \delta_{ad} \langle\pi(q)|\bar{u}(0)\Gamma^j d(0)|0\rangle. \quad (28)$$

It is seen that the correlation function $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$ depends on local matrix elements of the pion. This is typical situation for the LCSR method when one of particles is a tetraquark. For such tetraquark-meson-meson vertices, the four-momentum conservation requires equating a momentum one of final mesons, in the case under discussion of the pion, to $q = 0$ [53]. This constraint has to be taken into account also in the phenomenological side of the sum rule. At vertices of ordinary two-quark mesons, in general $q \neq 0$, and only as some approximation

one sets q equal to zero. A limit $q = 0$ in the conventional LCSR is known as the soft-meson approximation [51]. Contrary, tetraquark-meson-meson vertices can be explored in the framework of the LCSR method only if $q = 0$. An important conclusion made in Ref. [51] states that for the strong couplings of ordinary mesons the full LCSR method and its soft-meson version lead to numerically close predictions.

Having inserted Eq. (28) into the correlation function, we perform the summation over color and calculate traces over Lorentz indices. Relevant prescriptions were explained in a detailed form in Ref. [53], hence we do not concentrate here on these questions. These manipulations allow us to determine local matrix elements of the pion that contribute to $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$, and find the spectral density $\rho^{\text{OPE}}(s)$ as the imaginary part of $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$. It appears that the matrix element of the pion

$$\langle 0 | \bar{d}(0) i\gamma_5 u(0) | \pi(q) \rangle = f_\pi \mu_\pi, \quad (29)$$

where $\mu_\pi = m_\pi^2 / (m_u + m_d)$, contributes to $\rho^{\text{OPE}}(s)$.

To calculate $\rho^{\text{OPE}}(s)$, we choose in $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$ the structure $\sim g_{\mu\nu}$, and get

$$\rho^{\text{OPE}}(s) = \frac{f_\pi \mu_\pi}{12\sqrt{2}} [\rho^{\text{pert.}}(s) + \rho^{\text{n.-pert.}}(s)]. \quad (30)$$

The spectral density $\rho^{\text{OPE}}(s)$ consists of two components. Thus, its perturbative part $\rho^{\text{pert.}}(s)$ has a simple form and was computed in Ref. [53]

$$\rho^{\text{OPE}}(s) = \frac{(s + 2m_c^2)\sqrt{s(s - 4m_c^2)}}{\pi^2 s}. \quad (31)$$

The $\rho^{\text{n.-pert.}}(s)$ is a nonperturbative component of the spectral density, which includes terms up to eighth dimension: $\rho^{\text{n.-pert.}}(s)$ is given by the formula

$$\rho^{\text{n.-pert.}}(s) = \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle m_c^2 \int_0^1 f_1(z, s) dz + \left\langle g_s^3 G^3 \right\rangle \int_0^1 f_2(z, s) dz + \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle^2 m_c^2 \int_0^1 f_3(z, s) dz. \quad (32)$$

Explicit expressions of functions $f_1(z, s)$, $f_2(z, s)$, and $f_3(z, s)$ were written down in Appendix of Ref. [74].

Having found $\rho^{\text{OPE}}(s)$, we now are ready to calculate the phenomenological side of the sum rule in the soft-meson approximation. Because in the soft limit $p' = p$, the invariant amplitude in Eq. (21) depends solely on variable p^2 and has the form

$$\begin{aligned} \Pi^{\text{Phys}}(p^2) &= \frac{f_{J/\psi} f_{Z_c} m_{Z_c} m_{J/\psi} m_1^2}{(p^2 - m_1^2)^2} g_{Z_c J/\psi \pi} + \frac{f_{\psi'} f_{Z_c} m_{Z_c} m_{\psi'} m_2^2}{(p^2 - m_2^2)^2} g_{Z_c \psi' \pi} + \frac{f_{J/\psi} f_Z m_Z m_{J/\psi} m_3^2}{(p^2 - m_3^2)^2} g_{Z J/\psi \pi} \\ &+ \frac{f_{\psi'} f_Z m_Z m_{\psi'} m_4^2}{(p^2 - m_4^2)^2} g_{Z \psi' \pi} + \dots, \end{aligned} \quad (33)$$

where

$$m_1^2 = (m_{Z_c}^2 + m_{J/\psi}^2)/2, m_2^2 = (m_{Z_c}^2 + m_{\psi'}^2)/2, m_3^2 = (m_Z^2 + m_{J/\psi}^2)/2, m_4^2 = (m_Z^2 + m_{\psi'}^2)/2.$$

In the soft-meson limit the physical side of the sum rules has complicated content. Thus, besides $g_{Z_c J/\psi\pi}$ it contains also other strong couplings, i.e., terms that remain unsuppressed even after the Borel transformation [51]. To exclude them from $\Pi^{\text{Phys}}(p^2)$ one has to act by the operator

$$\mathcal{P}(M^2, m^2) = \left(1 - M^2 \frac{d}{dM^2}\right) M^2 e^{m^2/M^2}, \quad (34)$$

to both sides of sum rules [52]. In our studies, in order to evaluate strong couplings and calculate decay widths of various tetraquarks, we benefited from this technique (see, Ref. [53], as an example). But unsuppressed terms come from vertices of excited states of initial (final) particles, i.e., from vertices $ZJ/\psi\pi$, $Z\psi'\pi$ and $Z_c\psi'\pi$. In other words, contributions considered as contaminations while one investigates a vertex of ground-state particles become a subject of analysis in the present case. Because, in general, $\Pi^{\text{Phys}}(p^2)$ contains four terms, and at the first stage of analyses, is the sum of two contributions, we do not apply the operator \mathcal{P} to present sum rules.

We proceed by following recipes of the previous subsection, i.e., we fix the parameter s_0 below threshold for the decays $Z \rightarrow J/\psi\pi$ and $Z \rightarrow \psi'\pi$. Then in the considering range of $s \in [0, s_0]$ only first two terms in Eq. (33) should be explicitly taken into account: last two terms are automatically included into a "higher resonances and continuum". The one-variable Borel transformation applied to remaining two terms is the first step to derive a sum rule equality. Afterwards, we equate the physical and QCD sides of the sum rule, and carry out the continuum subtraction in accordance with the hadron-quark duality hypothesis

$$f_{Z_c} m_{Z_c} \left[f_{J/\psi} m_{J/\psi} m_1^2 g_{Z_c J/\psi\pi} e^{-m_1^2/M^2} + f_{\psi'} m_{\psi'} m_2^2 g_{Z_c \psi'\pi} e^{-m_2^2/M^2} \right] = \int_{4m_c^2}^{s_0} ds e^{-s/M^2} \rho^{\text{QCD}}(s). \quad (35)$$

But this expression is not enough to determine two unknown variables $g_{Z_c \psi'\pi}$ and $g_{Z_c J/\psi\pi}$. The second equality is obtained from Eq. (35) by applying the operator $d/d(-1/M^2)$ to its both sides. The equality derived by this way, and the master expression (35) allows us to extract sum rules for the couplings $g_{Z_c \psi'\pi}$ and $g_{Z_c J/\psi\pi}$. They are necessary to compute partial width of the decays $Z_c \rightarrow \psi'\pi$ and $Z_c \rightarrow J/\psi\pi$, and appear as input parameters in the next sum rules.

The sum rules for the couplings $g_{Z\psi'\pi}$ and $g_{ZJ/\psi\pi}$ are found by choosing $\sqrt{s_0^*} = m_Z + (0.5 - 0.7)$ GeV. Such choice for s_0^* is motivated by observation that a mass splitting in a tetraquark multiplet is approximately 0.5 - 0.7 GeV. For $s \in [0, s_0^*]$ the processes $Z \rightarrow J/\psi\pi$ and $Z \rightarrow \psi'\pi$ have to be taken into account as well. In other words, in this step of studies all terms in Eq. (33) have to be explicitly taken into account. We derive sum rules for the couplings $g_{Z\psi'\pi}$ and $g_{ZJ/\psi\pi}$ by repeating manipulations explained above and using two other couplings as input parameters.

We evaluate the width of the decay $Z \rightarrow \psi\pi$ by utilizing of the formula

$$\Gamma(Z \rightarrow \psi\pi) = \frac{g_{Z\psi\pi}^2 m_\psi^2}{24\pi} \lambda(m_Z, m_\psi, m_\pi) \left[3 + \frac{2\lambda^2(m_Z, m_\psi, m_\pi)}{m_\psi^2} \right], \quad (36)$$

where

$$\lambda(a, b, c) = \frac{\sqrt{a^4 + b^4 + c^4 - 2(a^2b^2 + a^2c^2 + b^2c^2)}}{2a}. \quad (37)$$

The equation (36) is valid for all four decay channels, where $Z = Z_c$ or Z , and $\psi = J/\psi$ or ψ' , respectively.

It is clear that apart from couplings $g_{Z\psi\pi}$ the partial width of the processes $Z \rightarrow \psi\pi$ contains parameters of initial and final particles. The spectroscopic parameters of the tetraquarks Z and Z_c have been calculated in this section. Masses and decay constants of mesons J/ψ , ψ' , π are presented in Ref. [96]. All these information are collected in Table 2, where we also write down spectroscopic parameters of the mesons η_c , η'_c and ρ , which will be used below to explore another decay channels of Z and Z_c . Let us note that decay constants f_{η_c} and $f_{\eta'_c}$ are borrowed from Ref. [97].

The working windows for the Borel and continuum threshold parameters used to evaluate strong couplings do not differ from ones employed for analysis of the masses and current couplings. Another problem, which should be considered, is contributions to the sum rules arising from excited terms. It is known, that dominant contribution to the sum rules is generated by a ground-state term. In the case under analysis, besides the strong coupling of the ground-state particles, we evaluate couplings of one or two radially excited particles as well. The sum rules for these couplings may lead to reliable predictions provided their effects and contributions are sizeable. This question can be analyzed by exploring the pole contribution to the sum rules

$$\text{PC} = \frac{\int_0^{s_0} ds \rho^{\text{OPE}}(s) e^{-s/M^2}}{\int_0^\infty ds \rho^{\text{OPE}}(s) e^{-s/M^2}}. \quad (38)$$

Choosing $s_0 = 4.2^2 \text{ GeV}^2$ and fixing $M^2 = 4.5 \text{ GeV}^2$ we find $\text{PC} = 0.81$, which is generated by the terms proportional to couplings $g_{Z_c J/\psi\pi}$ and $g_{Z_c \psi'\pi}$. At the next phase of analysis, we fix $s_0 \equiv s_0^*$ and get $\text{PC} = 0.95$, which now embraces effects of all four terms. In other words, contributions of terms $\sim g_{Z J/\psi\pi}$ and $\sim g_{Z \psi'\pi}$ amount to 14% part of the sum rules. We see that, effects of terms connected directly with decays of Z are small, nevertheless $g_{Z J/\psi\pi}$ and $g_{Z \psi'\pi}$ are extracted from full expressions, which contain contributions of four terms, and therefore their evaluations are founded on reliable basis. It is also seen that an effect of the "higher excited states and continuum" does not exceed 5% of PC, which means that contaminations arising from excited states higher than the resonance Z are negligible.

Numerical values of couplings g are sensitive to parameters M^2 and s_0 , nevertheless theoretical uncertainties of g generated by variations of M^2 and s_0 remain within limits typical for sum rule computations. These uncertainties and ones arising from other parameters form the full theoretical errors of numerical analysis.

Our computations for $g_{Z\psi'\pi}$ and width of the corresponding decay $Z \rightarrow \psi'\pi$ yield

$$g_{Z\psi'\pi} = (0.58 \pm 0.16) \text{ GeV}^{-1}, \quad \Gamma(Z \rightarrow \psi'\pi) = (129.7 \pm 37.6) \text{ MeV}. \quad (39)$$

The coupling $g_{Z J/\psi\pi}$ and width of the process $Z \rightarrow J/\psi\pi$ are found as

$$g_{Z J/\psi\pi} = (0.24 \pm 0.06) \text{ GeV}^{-1}, \quad \Gamma(Z \rightarrow J/\psi\pi) = (27.4 \pm 7.1) \text{ MeV}. \quad (40)$$

Predictions obtained for all of strong couplings, and for the partial width of corresponding decay channels are presented in Table 3.

2.2.2. Decays $Z_c, Z \rightarrow \eta'_c \rho, \eta_c \rho$

The Z_c and Z decay also to final mesons $\eta_c \rho$ and $\eta'_c \rho$. Because the decay $Z_c \rightarrow \eta'_c \rho$ is kinematically forbidden, in this subsection we have three channels $Z \rightarrow \eta'_c \rho$, $Z \rightarrow \eta_c \rho$ and $Z_c \rightarrow \eta_c \rho$ to be studied. Let us note that present analysis differs in some aspects from prescriptions explained above.

Table 3. The strong coupling g and width of the $Z(Z_c) \rightarrow \psi'(J/\psi)\pi$ decay channels.

Channels	$Z \rightarrow \psi'\pi$	$Z \rightarrow J/\psi\pi$	$Z_c \rightarrow \psi'\pi$	$Z_c \rightarrow J/\psi\pi$
g (GeV $^{-1}$)	0.58 ± 0.16	0.24 ± 0.06	0.29 ± 0.08	0.38 ± 0.11
Γ (MeV)	129.7 ± 37.6	27.4 ± 7.1	7.1 ± 1.9	39.9 ± 9.3

As usual, we consider the correlation function

$$\Pi_\nu(p, q) = i \int d^4x e^{ipx} \langle \rho(q) | \mathcal{T} \{ J^{\eta_c}(x) J_\nu^{Z\dagger}(0) \} | 0 \rangle, \quad (41)$$

where $\eta_c \equiv \eta_c$, η'_c , and the current $J^{\eta_c}(x)$ is defined as

$$J^{\eta_c}(x) = \bar{c}_i(x) i \gamma_5 c_i(x). \quad (42)$$

To express the correlation function in terms of involved particles' physical parameters, we use the matrix elements

$$\langle 0 | J^{\eta_c} | \eta_c(p) \rangle = \frac{f_{\eta_c} m_{\eta_c}^2}{2m_c}, \quad (43)$$

with m_{η_c} and f_{η_c} being the mass and decay constant of the meson η_c . The similar matrix element is also valid for the meson η'_c . The matrix elements of vertices are modeled in the forms

$$\langle \eta_c(p) \rho(q) | Z(p') \rangle = g_{Z\eta_c\rho} [(q \cdot \tilde{\varepsilon}')(p' \cdot \varepsilon^*) - (q \cdot p')(\varepsilon^* \cdot \tilde{\varepsilon}')], \quad (44)$$

and

$$\langle \eta_c(p) \rho(q) | Z_c(p') \rangle = g_{Z_c\eta_c\rho} [(q \cdot \varepsilon')(p' \cdot \varepsilon^*) - (q \cdot p')(\varepsilon^* \cdot \varepsilon')], \quad (45)$$

where q and ε are the momentum and polarization vector of the ρ -meson, respectively.

We write the phenomenological side of the sum rules $\Pi_\nu^{\text{Phys}}(p, q)$ in the form

$$\begin{aligned} \Pi_\nu^{\text{Phys}}(p, q) &= \frac{\langle 0 | J^{\eta_c} | \eta_c(p) \rangle}{p^2 - m_{\eta_c}^2} \langle \eta_c(p) \rho(q) | Z(p') \rangle \frac{\langle Z_c(p') | J_\nu^Z | 0 \rangle}{p'^2 - m_{Z_c}^2} \\ &+ \sum_{\eta_c = \eta_c, \eta'_c} \frac{\langle 0 | J^{\eta_c} | \eta_c(p) \rangle}{p^2 - m_{\eta_c}^2} \langle \eta_c(p) \rho(q) | Z(p') \rangle \frac{\langle Z(p') | J_\nu^Z | 0 \rangle}{p'^2 - m_Z^2} + \dots \end{aligned} \quad (46)$$

It contains three terms, which can be simplified using matrix elements introduced above. The full expression of $\Pi_\nu^{\text{Phys}}(p, q)$ is cumbersome, therefore we write down only the invariant amplitude corresponding to the structure $\sim \epsilon_\nu^*$ in the limit $q \rightarrow 0$, which is employed in our analysis. This amplitude is given by the formula

$$\begin{aligned} \Pi^{\text{Phys}}(p^2) &= \frac{f_{\eta_c} f_{Z_c} m_{Z_c} m_{\eta_c}^2 g_{Z_c\eta_c\rho}}{4m_c (p^2 - \tilde{m}_1^2)^2} (m_{Z_c}^2 - m_{\eta_c}^2) + \frac{f_{\eta_c} f_Z m_Z m_{\eta_c}^2 g_{Z\eta_c\rho}}{4m_c (p^2 - \tilde{m}_2^2)^2} (m_Z^2 - m_{\eta_c}^2) \\ &+ \frac{f_{\eta'_c} f_Z m_Z m_{\eta'_c}^2 g_{Z\eta'_c\rho}}{4m_c (p^2 - \tilde{m}_3^2)^2} (m_Z^2 - m_{\eta'_c}^2) + \dots, \end{aligned} \quad (47)$$

where the notations $\tilde{m}_1^2 = (m_{Z_c}^2 + m_{\eta_c}^2)/2$, $\tilde{m}_2^2 = (m_Z^2 + m_{\eta_c}^2)/2$ and $\tilde{m}_3^2 = (m_Z^2 + m_{\eta'_c}^2)/2$ are introduced.

Computation of the correlation function $\Pi_\nu^{\text{OPE}}(p, q)$ using quark propagators leads to the expression

$$\begin{aligned} \Pi_\nu^{\text{OPE}}(p, q) &= -i \int d^4x e^{ipx} \frac{\epsilon\tilde{\epsilon}}{\sqrt{2}} \left[\gamma_5 \tilde{S}_c^{ib}(x) \gamma_5 \times \tilde{S}_c^{ei}(-x) \gamma_\nu + \gamma_\nu \tilde{S}_c^{ib}(x) \gamma_5 \tilde{S}_c^{ei}(-x) \gamma_5 \right]_{\alpha\beta} \\ &\times \langle \rho(q) | \bar{u}_\alpha^d(0) d_\beta^a(0) | 0 \rangle. \end{aligned} \quad (48)$$

In the $q \rightarrow 0$ limit the contributions to $\rho^{\text{OPE}}(s)$ come from the matrix elements [53]

$$\langle 0 | \bar{u}(0) \gamma_\mu d(0) | \rho(p, \lambda) \rangle = \epsilon_\mu^{(\lambda)} f_\rho m_\rho, \quad (49)$$

and

$$\langle 0 | \bar{u}(0) g \tilde{G}_{\mu\nu} \gamma_\nu \gamma_5 d(0) | \rho(p, \lambda) \rangle = f_\rho m_\rho^3 \epsilon_\mu^{(\lambda)} \zeta_{4\rho}. \quad (50)$$

These elements contain the ρ -meson's mass and decay constant m_ρ , and f_ρ , and Eq. (50) additionally depends on a normalization factor $\zeta_{4\rho}$ of the ρ -meson's twist-4 matrix element [98]. The numerical value of $\zeta_{4\rho}$ was estimated in Ref. [99] at the scale $\mu = 1$ GeV, and amounts to $\zeta_{4\rho} = 0.07 \pm 0.03$.

We derive the spectral density $\rho^{\text{OPE}}(s)$ in accordance with known recipes, and find

$$\rho^{\text{OPE}}(s) = \frac{f_\rho m_\rho}{8\sqrt{2}} \left[\frac{\sqrt{s(s-4m_c^2)}}{\pi^2} + \rho^{\text{n.-pert.}}(s) \right]. \quad (51)$$

The nonperturbative component of $\rho^{\text{OPE}}(s)$ is calculated with dimension-8 accuracy and has the following form

$$\rho^{\text{n.-pert.}}(s) = \frac{\zeta_{4\rho} m_\rho^2}{s} + \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle m_c^2 \int_0^1 \tilde{f}_1(z, s) dz + \left\langle g_s^3 G^3 \right\rangle \int_0^1 \tilde{f}_2(z, s) dz + \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle^2 m_c^2 \int_0^1 \tilde{f}_3(z, s) dz. \quad (52)$$

Explicit expressions of the functions $\tilde{f}_1(z, s)$, $\tilde{f}_2(z, s)$, and $\tilde{f}_3(z, s)$ can be found in Appendix of Ref. [74].

To obtain sum rules, we utilize again a prescription described above. At the first step, i.e., for $s \in [0, s_0]$ the physical side of the sum rule consists of a ground-state term. At this stage, we calculate the ground-state coupling $g_{Z_c \eta_c \rho}$, therefore to exclude effects of excited states from the physical side of the sum rule apply the operator $\mathcal{P}(M^2, \tilde{m}_1^2)$. Then, we find

$$g_{Z_c \eta_c \rho} = \frac{4m_c}{f_{\eta_c} f_{Z_c} m_{Z_c} m_{\eta_c}^2 (m_{Z_c}^2 - m_{\eta_c}^2)} \mathcal{P}(M^2, \tilde{m}_1^2) \int_{4m_c^2}^{s_0} ds e^{-s/M^2} \rho^{\text{OPE}}(s). \quad (53)$$

In the domain $s \in [0, s_0^*]$ all terms from Eq. (47) should be included into analysis, and, as a result, we get the expression with two additional couplings. Excited terms enter to this expression explicitly, and because our goal is to determine relevant couplings, in this situation we do not use the operator \mathcal{P} . The second equality can be found by applying the operator $d/d(-1/M^2)$ to both sides of the first expression. Solutions of these equations are sum rules for the couplings $g_{Z_c \eta_c \rho}$ and $g_{Z_c \eta'_c \rho}$. The width of the decays $Z \rightarrow \eta_c \rho$, $Z \rightarrow \eta'_c \rho$ and $Z_c \rightarrow \eta_c \rho$ after replacements $m_\pi \rightarrow m_{\eta_c}$ ($m_{\eta'_c}$) and $m_\psi \rightarrow m_\rho$ can be computed using Eq. (36).

For the coupling $g_{Z_c \eta_c \rho}$ and width of the decay $Z_c \rightarrow \eta_c \rho$, we get

$$g_{Z_c \eta_c \rho} = (1.28 \pm 0.32) \text{ GeV}^{-1}, \quad \Gamma(Z_c \rightarrow \eta_c \rho) = (20.28 \pm 5.17) \text{ MeV}. \quad (54)$$

The strong couplings $g_{Z\eta'_c\rho}$ and $g_{Z\eta_c\rho}$, and width of the decays $Z \rightarrow \eta'_c\rho$ and $Z \rightarrow \eta_c\rho$ are equal to

$$g_{Z\eta'_c\rho} = (0.81 \pm 0.20) \text{ GeV}^{-1}, \quad \Gamma(Z \rightarrow \eta'_c\rho) = (1.01 \pm 0.27) \text{ MeV}, \quad (55)$$

and

$$g_{Z\eta_c\rho} = (0.48 \pm 0.11) \text{ GeV}^{-1}, \quad \Gamma(Z \rightarrow \eta_c\rho) = (11.57 \pm 3.01) \text{ MeV}. \quad (56)$$

The processes $Z_c \rightarrow J/\psi\pi$ and $Z_c \rightarrow \eta_c\rho$ were considered in Ref. [53] using the QCD light-cone sum rule method and diquark-antidiquark type interpolating current. In Table 4, we compare the partial widths of these modes from Ref. [53] with results obtained in Ref. [74]. It is clear that these predictions are very close to each other. Stated differently, an iterative scheme used in this section led to results that are almost identical with predictions of Ref. [53]. This fact can be treated as a serious argument in favor of the used approach. The unessential discrepancies between two sets of results may be explained by accuracy of the spectral densities, which here have been calculated by taking into account condensates up to eight dimensions, whereas in Ref. [53] $\rho_\pi^{\text{OPE}}(s)$ and $\rho_\rho^{\text{OPE}}(s)$ contained only perturbative terms. Let us emphasize that, we have computed also the partial width of the decay $Z_c \rightarrow \psi'\pi$, which was omitted in Ref. [53].

It is evident that Z decays dominantly via the process $Z \rightarrow \psi'\pi$. The full width of Z saturated by two channels $Z \rightarrow \psi'\pi$ and $Z \rightarrow J/\psi\pi$ equals to $(157.1 \pm 38.3) \text{ MeV}$. This prediction is compatible with LHCb information (see, Eq. (1)), but is below the upper edge of the experimental data $\approx 212 \text{ MeV}$. Experimental data on the width of the decay $Z \rightarrow J/\psi\pi$ is limited by Belle report about product of branching fractions

$$\mathcal{B}(\bar{B}^0 \rightarrow K^- Z^+) \mathcal{B}(Z^+ \rightarrow J/\psi\pi) = (5.4_{-1.0-0.9}^{+4.0+1.1}) \times 10^{-6}. \quad (57)$$

By invoking similar experimental measurements for ψ' , it is possible to estimate a ratio

$$R_Z = \Gamma(Z \rightarrow \psi'\pi) / \Gamma(Z \rightarrow J/\psi\pi), \quad (58)$$

which was carried out in Ref. [79]. But, we are not going to draw strong conclusions from such computations. We think that, in the absence of direct measurements of $\Gamma(Z \rightarrow J/\psi\pi)$, an only reasonable way is to compute R_Z , which is equal to $R_Z = 4.73 \pm 1.84$.

Table 4. Predictions for decays of the resonance Z_c .

	$\Gamma(Z_c \rightarrow J/\psi\pi)$ (MeV)	$\Gamma(Z_c \rightarrow \psi'\pi)$ (MeV)	$\Gamma(Z_c \rightarrow \eta_c\rho)$ (MeV)
[74]	39.9 ± 9.3	7.1 ± 1.9	20.28 ± 5.17
[53]	41.9 ± 9.4	—	23.8 ± 4.9
[82]	29.1 ± 8.2	—	27.5 ± 8.5
[79]A	$27.9_{-5.0}^{+6.3}$	—	$35.7_{-5.2}^{+6.3}$
[79]B	1.8 ± 0.3	—	$3.2_{-0.4}^{+0.5}$
[87]	$10.43 - 23.89$	$1.28 - 2.94$	—

The decays of the resonances Z and Z_c were studied in Refs. [79, 82, 87]: some of these predictions are written down in Tables 4 and 5. Partial widths of decay modes $Z_c \rightarrow J/\psi\pi$, $Z_c \rightarrow \eta_c\rho$, $Z_c \rightarrow \bar{D}^0 D^*$ and

Table 5. The same as in Table 4, but for the resonance Z .

	$\Gamma(Z \rightarrow J/\psi\pi)$ (MeV)	$\Gamma(Z \rightarrow \psi'\pi)$ (MeV)	$\Gamma(Z \rightarrow \eta_c\rho)$ (MeV)	$\Gamma(Z \rightarrow \eta'_c\rho)$ (MeV)
[74]	27.4 ± 7.1	129.7 ± 37.6	11.57 ± 3.01	1.01 ± 0.27
[79]	26.9	120.6	—	—

$Z_c \rightarrow \bar{D}^{*0}D$ in the context of the three-point sum rule method and diquark-antidiquark picture for Z_c were calculated in Ref. [82]. Their predictions for first two channels are shown in Table 4.

The resonance Z_c was also treated in Ref. [79] both as diquark-antidiquark and molecule-type tetraquarks. Decays $Z_c \rightarrow J/\psi\pi$, and $Z_c \rightarrow \eta_c\rho$ were explored there using the covariant quark model. Partial widths of these processes were evaluated in the diquark-antidiquark picture using a size parameter $\Lambda_{Z_c} = 2.25 \pm 0.10$ GeV in their model (model A), and in a molecular-type structure with $\Lambda_{Z_c} = 3.3 \pm 0.1$ GeV (model B). Obtained results are presented in Table 4, as well.

In the context of the phenomenological Lagrangian method decays of the tetraquark Z_c were examined in Ref. [87]. The Z_c was considered there as hadronic molecules $\bar{D}D^*$ and \bar{D}^*D . In the case of the molecule's binding energy $\epsilon = 20$ MeV the authors estimated widths of different decay processes: some of obtained results are demonstrated in Table 4.

Decays of the resonance Z to $J/\psi\pi$ and $\psi'\pi$ were also studied in Ref. [79], where it was modeled as a diquark-antidiquark system. Results for the partial widths of these decays obtained at $\Lambda_{Z(4430)} = 2.4$ GeV, and estimates for $\Gamma(Z \rightarrow J/\psi\pi) + \Gamma(Z \rightarrow \psi'\pi) = 147.5$ MeV and $R_Z = 4.48$ are close to our predictions.

We have examined the tetraquark Z as first radial excitation of the diquark-antidiquark state Z_c . We evaluated the masses and full widths of the resonances Z_c and Z , and have found: $m_{Z_c} = 3901_{-148}^{+125}$ MeV, $\Gamma_{Z_c} = (67.3 \pm 10.8)$ MeV, and $m_Z = 4452_{-161}^{+132}$ MeV, $\Gamma_Z = (169.7 \pm 38.4)$ MeV, respectively. Predictions obtained here seem to support a suggestion about the excited nature of Z . But there are problems to be considered before making viable conclusions. Namely, there is necessity to improve our predictions for the full widths of tetraquarks Z_c and Z by studying their other decay modes. Experimental studies of the Z resonance's decay modes, including a direct measurement of $\Gamma(Z \rightarrow J/\psi\pi)$ may be helpful to confirm its nature as a radial excitation of the state Z_c .

3. The tetraquark $Z_c^-(4100)$

The tetraquark $Z_c^-(4100)$ was discovered by LHCb in $B^0 \rightarrow K^+\eta_c\pi^-$ decays as a resonance in the $\eta_c\pi^-$ mass distribution [47]. The mass and width of this new $Z_c^-(4100)$ state (in this section will be denoted \bar{Z}_c) were found equal to

$$m = 4096 \pm 20_{-22}^{+18} \text{ MeV}, \Gamma = 152 \pm 58_{-35}^{+60} \text{ MeV}. \quad (59)$$

In Ref. [47] the spin and parity of $Z_c^-(4100)$ were determined as well, and it was shown that assignments $J^P = 0^+$ or $J^P = 1^-$ do not contradict to the experimental data.

The theoretical articles, as usual, consider problems connected with the spin and possible decays of the resonance \bar{Z}_c [100–103]. Thus, sum rule calculations performed in Ref. [100] showed that \bar{Z}_c is probably a scalar tetraquark. The nature of \bar{Z}_c as a diquark-antidiquark state with $J^{PC} = 0^{++}$ was supported also in

Ref. [101]. The resonances \bar{Z}_c and $Z_c^-(4200)$ in the hadrocharmonium model were considered as the scalar η_c and vector J/ψ charmonia placed into a light-quark field with pion's quantum numbers [102]. Then, due to spin symmetry of charm quark, features of the particles \bar{Z}_c and $Z_c^-(4200)$, as well as their decay modes are connected by some relations.

Because the resonance \bar{Z}_c was seen in the decay $\bar{Z}_c \rightarrow \eta_c \pi^-$, it is natural to treat it as a scalar particle with quark content $c\bar{c}d\bar{u}$. Really, the decay $\bar{Z}_c \rightarrow \eta_c \pi^-$ is dominant S -wave mode for a scalar particle, but it turns to P -wave decay channel in the case of a vector tetraquark. The mass and coupling of the scalar tetraquark \bar{Z}_c built of $[cd][\bar{c}\bar{u}]$ diquark-antidiquark were computed in our paper [104]. There, we also explored decays of \bar{Z}_c and found its full width. The dominant strong decay of the resonance \bar{Z}_c is presumably the channel $Z_c \rightarrow \eta_c \pi^-$. But hidden-charm $\eta_c' \pi^-$, $J/\psi \rho^-$ and open-charm $D^0 D^-$ and $D^{*0} D^{*-}$ decays are also kinematically allowed S -wave channels of the resonance \bar{Z}_c . Below, we give detailed information about investigations of \bar{Z}_c based on our work [104].

3.1. Mass and coupling of the scalar tetraquark \bar{Z}_c

The most stable and lower lying scalar tetraquark can be built of scalar diquark $\epsilon^{ijk}[c_j^T C \gamma_5 d_k]$ and antidiquark $\epsilon^{imn}[\bar{c}_m \gamma_5 C \bar{u}_n^T]$ fields [55]. These two-quark states are color-antitriplet and -triplet configurations, respectively, and both are antisymmetric in flavor indices.

For scalar particles the two-point correlation function $\Pi(p)$ has a simple form and Lorentz structure: it is given by the following formula

$$\Pi(p) = i \int d^4x e^{ipx} \langle 0 | \mathcal{T} \{ J(x) J^\dagger(0) \} | 0 \rangle. \quad (60)$$

In expression above, the interpolating current for the tetraquark \bar{Z}_c is denoted by $J(x)$. In light of our suggestion about internal organization of \bar{Z}_c , the current $J(x)$ can be written in the form

$$J(x) = \epsilon \tilde{\epsilon} [c_j^T(x) C \gamma_5 d_k(x)] [\bar{c}_m(x) \gamma_5 C \bar{u}_n^T(x)], \quad (61)$$

where $\epsilon = \epsilon^{ijk}$, $\tilde{\epsilon} = \epsilon^{imn}$.

The sum rules for parameters of the tetraquark \bar{Z}_c can be extracted using the "ground-state + continuum" scheme. First of all, we need the phenomenological side of the sum rule $\Pi^{\text{Phys}}(p)$. For the scalar particle relevant invariant amplitude $\Pi^{\text{Phys}}(p^2) = m^2 f^2 / (m^2 - p^2)$ is simple function of the mass m and coupling f . At the next step, we have to determine the QCD side of the sum rules. In our case, it is given by the formula

$$\Pi^{\text{OPE}}(p) = i \int d^4x e^{ipx} \epsilon \tilde{\epsilon} \epsilon' \tilde{\epsilon}' \text{Tr} \left[\gamma_5 \tilde{S}_c^{jj'}(x) \gamma_5 S_d^{kk'}(x) \right] \text{Tr} \left[\gamma_5 \tilde{S}_u^{n'n}(-x) \gamma_5 S_c^{m'm}(-x) \right]. \quad (62)$$

For the mass m and coupling f of the tetraquark \bar{Z}_c after clear substitutions one can employ expressions (16) and (17). The relevant computations are carried out by taking into account nonperturbative terms up to dimension 10.

The sum rules for spectroscopic parameters of \bar{Z}_c contain various vacuum condensates, values of which have been presented in Eq. (18). The sum rules depend also on the Borel M^2 and continuum threshold

s_0 parameters: M^2 and s_0 are the auxiliary parameters and should be fixed in accordance with standard restrictions of the sum rule calculations. Thus, at the maximum of M^2 the pole contribution (38) should exceed some fixed value: as usual, for four-quark systems minimum of PC is approximately 0.15 – 0.2.

In the previous section, we have defined PC in terms of the spectral density, but in a general form it can be introduced through the ratio

$$\text{PC} = \frac{\Pi(M^2, s_0)}{\Pi(M^2, \infty)}, \quad (63)$$

where $\Pi(M^2, s_0)$ is the Borel transformed and subtracted invariant amplitude $\Pi^{\text{OPE}}(p^2)$. The minimum of the Borel parameter is determined from convergence of the operator product expansion, and can be extracted from analysis of the parameter

$$R(M^2) = \frac{\Pi^{\text{DimN}}(M^2, s_0)}{\Pi(M^2, s_0)}. \quad (64)$$

Here, $\Pi^{\text{DimN}}(M^2, s_0)$ is a contribution of the last term in expansion (or a sum of last few terms) to $\Pi(M^2, s_0)$. The parameter $R(M^2)$ should be small enough to guarantee a convergence of sum rules.

The mass m and coupling f should not depend on the Borel parameter M^2 . But analyses demonstrate that m and f are sensitive to the choice of M^2 . There are also dependence on the continuum threshold parameter s_0 , but $\sqrt{s_0}$ determines a position of the first excitation of \bar{Z}_c and bears some information about a physical system. Therefore, M^2 should be fixed in such a way as to minimize a dependence of m and f on this parameter.

Computations demonstrate that regions for the parameters M^2 and s_0

$$M^2 \in [4, 6] \text{ GeV}^2, \quad s_0 \in [19, 21] \text{ GeV}^2, \quad (65)$$

satisfy all constraints of sum rule calculations. Indeed, at $M^2 = 6 \text{ GeV}^2$, we get $\text{PC} = 0.19$, and in the region $M^2 \in [4, 6] \text{ GeV}^2$ the pole contribution changes from 0.54 till 0.19. The low limit of the Borel parameter is fixed from Eq. (64), in which we choose $\text{DimN} = \text{Dim}(8 + 9 + 10)$. Then at $M^2 = 4 \text{ GeV}^2$ the parameter R becomes equal to $R(4 \text{ GeV}^2) = 0.02$ which guarantees the convergence of the sum rules. At $M^2 = 4 \text{ GeV}^2$ the perturbative contribution amounts to 83% of the full result overshooting nonperturbative terms.

For the mass and coupling of the tetraquark \bar{Z}_c our calculations yield

$$m = (4080 \pm 150) \text{ MeV}, \quad f = (0.58 \pm 0.12) \times 10^{-2} \text{ GeV}^4. \quad (66)$$

One can see that the mass of the scalar diquark-antidiquark state \bar{Z}_c is in excellent agreement with LHCb data.

The scalar tetraquark $[cu][\bar{c}\bar{d}]$ with the internal organization $C\gamma_5 \otimes \gamma_5 C$ was investigated in Ref. [105] as well. Using the mass $m = (3860 \pm 90) \text{ MeV}$ of this exotic state, the author interpreted it as a charged partner of the resonance $X^*(3860)$. The charmoniumlike state $X^*(3860)$ was seen by Belle [106] in the process $e^+e^- \rightarrow J/\psi D\bar{D}$, where D is one D^0 or D^+ mesons, and identified there with $\chi_{c0}(2P)$ meson. Comparing our result and prediction of Ref. [105], we find an overlapping region, but a difference 200 MeV between the central values of the masses is sizable. This difference probably stems from working windows for the parameters M^2 and s_0 used in computations, and also may be explained by fixed or evolved treatment of vacuum condensates.

3.2. Decays $\bar{Z}_c \rightarrow \eta_c \pi^-$ and $\bar{Z}_c \rightarrow \eta'_c \pi^-$

The strong decays of the resonance \bar{Z}_c form two groups of processes: the first of them contains decays with two pseudoscalar mesons in a final state, whereas the second group embraces decays to two vector mesons. The decays $\bar{Z}_c \rightarrow \eta_c \pi^-$ and $\bar{Z}_c \rightarrow \eta'_c \pi^-$ are from the first group of processes. The final phases of these processes are characterized by appearance of mesons η_c and η'_c , where the latter is a first radially excited state of the former one. In the QCD sum rule method such decays are explored in a correlated way. A suitable approach to analyze the decays $\bar{Z}_c \rightarrow \eta_c \pi^-$ and $\bar{Z}_c \rightarrow \eta'_c \pi^-$ is the QCD three-point sum rule method. The reason is that, this method allows one to get for the physical side of sum rules relatively simple expression. In fact, we are interested in extraction of sum rules for strong form factors $g_{\bar{Z}_c \eta_c \pi}(q^2)$, therefore in the context of standard operations should apply double Borel transformation over the momenta of particles \bar{Z}_c and η_c . The Borel transformation applied to physical side of the three-point sum rules suppresses contributions of higher resonances in these two channels, and eliminate contributions of pole-continuum transitions [51, 52]. The elimination of such terms is important for joint treatment of the form factors $g_{\bar{Z}_c \eta_c \pi}(q^2)$, because there is not a necessity to employ additional operators to remove contaminations from the phenomenological side. Nevertheless, in the pion channel still may survive contaminating terms corresponding to excited states of the pion [for the $NN\pi$ vertex, see discussions in Refs. [107, 108]]. To decrease ambiguities in extracting of the strong couplings at the vertices, it is possible to choose the pion on the mass shell, and consider one of remaining states (\bar{Z}_c or η_c) as an off-shell particle. This method was employed to investigate couplings of ordinary heavy-heavy-light mesons in Refs. [109, 110]. Form factors extracted by treating a light or one of heavy mesons off-shell may differ from each other considerably, but after extrapolating to the corresponding mass-shells give the same or negligibly different strong couplings.

The process $\bar{Z}_c \rightarrow J/\psi \rho^-$ belongs to the second group of \bar{Z}_c decays. We explore this decay using the LCSR method and soft-meson approximation. The LCSR method allows us to determine the strong coupling by evading extrapolating prescriptions and express $g_{\bar{Z}_c J/\psi \rho}$ in terms of the vacuum condensates and matrix elements of the ρ meson. The pole-continuum contributions surviving after a single Borel transformation in the physical side of sum rules, can be removed by employing well-known procedures [52].

The strong couplings $g_{\bar{Z}_c \eta_c \pi}$ and $g_{\bar{Z}_c \eta'_c \pi}$ can be found from analysis of the three-point correlation function

$$\Pi(p, p') = i^2 \int d^4x d^4y e^{-ipx} e^{ip'y} \langle 0 | \mathcal{T} \{ J^{\eta_c}(y) J^\pi(0) J^\dagger(x) \} | 0 \rangle, \quad (67)$$

where $J^{\eta_c}(y)$ is the interpolating current for η_c and η'_c mesons (42), and $J^\pi(0)$ is the interpolating current for the pion

$$J^\pi(x) = \bar{u}_b(x) i \gamma_5 d_b(x) \quad (68)$$

at $x = 0$, respectively.

The correlation function $\Pi(p, p')$ in terms of the physical parameters of involved particles has the form

$$\Pi^{\text{Phys}}(p, p') = \sum_{i=1}^2 \frac{\langle 0 | J^{\eta_c} | \eta_{ci}(p') \rangle \langle 0 | J^\pi | \pi(q) \rangle \langle \eta_{ci}(p') | \pi(q) | \bar{Z}_c(p) \rangle \langle \bar{Z}_c(p) | J^\dagger | 0 \rangle}{p'^2 - m_i^2} \frac{1}{q^2 - m_\pi^2} \frac{1}{p^2 - m^2} + \dots, \quad (69)$$

where m_π is the mass of the pion, and $m_1 \equiv m_{\eta_c}$, $m_2 = m_{\eta'_c}$ are the masses of the mesons η_c and η'_c , respectively. Their decay constants are denoted by $f_1 \equiv f_{\eta_c}$ and $f_2 \equiv f_{\eta'_c}$ and together with m_1 and m_2

determine the matrix elements $\langle 0|J^{\eta_c}|\eta_{ci}(p')\rangle$ [see, Eq. (43)]. The matrix element of the pion is also well known (29). In addition to this information the matrix elements of the vertices $\bar{Z}_c\eta_c\pi^-$ and $\bar{Z}_c\eta'_c\pi^-$ are required as well. For these purposes, we use

$$\langle \eta_{ci}(p')\pi(q)|\bar{Z}_c(p)\rangle = g_{\bar{Z}_c\eta_{ci}\pi}(p \cdot p'). \quad (70)$$

Here, the strong coupling $g_{\bar{Z}_c\eta_{c1}\pi}$ corresponds to the vertex $\bar{Z}_c\eta_c\pi^-$, whereas $g_{\bar{Z}_c\eta_{c2}\pi}$ describes $\bar{Z}_c\eta'_c\pi^-$.

After some manipulations for $\Pi^{\text{Phys}}(p, p')$ we find the following expression

$$\Pi^{\text{Phys}}(p, p') = \sum_{i=1}^2 \frac{g_{\bar{Z}_c\eta_{ci}\pi} m_i^2 f_i m f}{2m_c(p'^2 - m_i^2)(p^2 - m^2)} \frac{\mu_\pi f_\pi}{q^2 - m_\pi^2} (p \cdot p') + \dots \quad (71)$$

The $\Pi^{\text{Phys}}(p, p')$ has a simple Lorentz structure, hence the invariant amplitude $\Pi^{\text{Phys}}(p^2, p'^2)$ is equal to the sum of two terms in Eq. (71). The double Borel transformation of $\Pi^{\text{Phys}}(p^2, p'^2)$ over p^2 and p'^2 with the parameters M_1^2 and M_2^2 , respectively, constitutes a physical side in a sum rule equality.

The correlation function calculated in terms of the quark propagators is:

$$\Pi^{\text{OPE}}(p, p') = i^2 \int d^4x d^4y e^{-ipx} e^{ip'y} \epsilon \bar{\tau} \text{Tr} \left[\gamma_5 S_c^{aj}(y-x) \gamma_5 \tilde{S}_d^{bk}(-x) \gamma_5 \tilde{S}_u^{nb}(x) \gamma_5 S_c^{ma}(x-y) \right]. \quad (72)$$

The Borel transformation $\mathcal{B}\Pi^{\text{OPE}}(p^2, p'^2)$ of the amplitude $\Pi^{\text{OPE}}(p^2, p'^2)$ forms the QCD side of the sum rules. The first sum rule for $g_{\bar{Z}_c\eta_{c1}\pi}$ and $g_{\bar{Z}_c\eta_{c2}\pi}$ is obtained by equating Borel transformations of amplitudes $\Pi^{\text{Phys}}(p^2, p'^2)$ and $\Pi^{\text{OPE}}(p^2, p'^2)$ and performing the continuum subtractions.

The Borel transformed and subtracted amplitude $\Pi^{\text{OPE}}(p^2, p'^2)$ can be expressed using the spectral density $\rho_D(s, s', q^2)$ which is determined as an imaginary part of the correlation function $\Pi^{\text{OPE}}(p, p')$

$$\Pi(\mathbf{M}^2, \mathbf{s}_0, q^2) = \int_{4m_c^2}^{s_0} ds \int_{4m_c^2}^{s'_0} ds' \rho_D(s, s', q^2) e^{-s/M_1^2} e^{-s'/M_2^2}, \quad (73)$$

where $\mathbf{M}^2 = (M_1^2, M_2^2)$ and $\mathbf{s}_0 = (s_0, s'_0)$ are the Borel and continuum threshold parameters, respectively.

The second sum rule for the couplings $g_{\bar{Z}_c\eta_{c1}\pi}$ and $g_{\bar{Z}_c\eta_{c2}\pi}$ can be obtained by acting operators $d/d(-1/M_1^2)$ and/or $d/d(-1/M_2^2)$ on the first expression. These two expressions are enough to find $g_{\bar{Z}_c\eta_{c1}\pi}$ and $g_{\bar{Z}_c\eta_{c2}\pi}$. An alternative way is the master sum rule used repeatedly to evaluate the couplings $g_{\bar{Z}_c\eta_{c1}\pi}$ and $g_{\bar{Z}_c\eta_{c2}\pi}$. For these purposes, we choose the continuum threshold parameter $\sqrt{s'_0}$ that corresponds to the η_c channel below the mass of the radially excited state η'_c . In other words, we include η'_c into high resonances and get sum rule for the coupling of the ground-state meson η_c . At this phase of computations, the physical side of the sum rule (71) depends only on the coupling $g_{\bar{Z}_c\eta_{c1}\pi}$. This sum rule can be solved to find the coupling $g_{\bar{Z}_c\eta_{c1}\pi}$

$$g_{\bar{Z}_c\eta_{c1}\pi}(\mathbf{M}^2, \mathbf{s}_0^{(1)}, q^2) = \frac{\Pi(\mathbf{M}^2, \mathbf{s}_0^{(1)}, q^2) e^{m/M_1^2} e^{m_1^2/M_2^2}}{A_1}, \quad (74)$$

where

$$A_1 = \frac{m f m_1^2 f_1 \mu_\pi f_\pi}{4m_c(q^2 - m_\pi^2)} (m^2 + m_1^2 - q^2),$$

and $\mathbf{s}_0^{(1)} = (s_0, s'_0 \simeq m_2^2)$.

At the next stage, we move the continuum threshold $\sqrt{s'_0}$ to $m_2 + (0.5 - 0.8)$ GeV and employ the sum rule which now includes the ground-state meson η_c and its first radial excitation η'_c . The QCD side of this sum rule is determined by $\Pi(\mathbf{M}^2, \mathbf{s}_0^{(2)}, q^2)$, where $\mathbf{s}_0^{(2)} = (s_0, s'_0 \simeq [m_2 + (0.5 - 0.8)]^2)$. By substituting the obtained expression for $g_{Z_c \eta_{c1} \pi}$ into this sum rule, it is not difficult to evaluate the second coupling $g_{\bar{Z}_c \eta_{c2} \pi}$.

The couplings extracted by this manner, as usual, depend on the Borel and continuum threshold parameters, but are functions of q^2 as well. For simplicity of presentation, below we skip their dependence on the parameters, and denote strong couplings obtained by substitution $q^2 = -Q^2$ as $g_{\bar{Z}_c \eta_{c1} \pi}(Q^2)$ and $g_{\bar{Z}_c \eta_{c2} \pi}(Q^2)$. The widths of the decays under analysis depend on values of the couplings at the pion's mass shell $q^2 = m_\pi^2$. This region is not accessible to sum rule computations. The way out of this situation is to introduce extrapolating functions $F_{1(2)}(Q^2)$ which at $Q^2 > 0$ coincide with the sum rule's predictions, but can be easily used in the region $Q^2 < 0$ as well.

The strong couplings depend on the masses and decay constants of the final-state mesons, which are shown in Table 2. To perform numerical computations the Borel \mathbf{M}^2 and continuum threshold \mathbf{s}_0 parameters have to be specified as well. The parameters M_2^2, s'_0 in Eq. (74) are chosen as

$$M_2^2 \in [3, 4] \text{ GeV}^2, \quad s'_0 = 13 \text{ GeV}^2, \quad (75)$$

whereas in the sum rule for the second coupling $g_{\bar{Z}_c \eta_{c2} \pi}(Q^2)$, we employ

$$M_2^2 \in [3, 4] \text{ GeV}^2, \quad s'_0 \in [17, 19] \text{ GeV}^2. \quad (76)$$

We have noted above that at the pion mass-shell $Q^2 = -m_\pi^2$ the couplings can be evaluated using fit functions. For these purposes, we use exponential-type functions

$$F_i(Q^2) = F_0^i \exp \left[c_1^i \frac{Q^2}{m^2} + c_2^i \left(\frac{Q^2}{m^2} \right)^2 \right], \quad (77)$$

where F_0^i, c_1^i and c_2^i are free parameters. Our analysis allows us to fix these parameters: we get $F_0^1 = 0.49 \text{ GeV}^{-1}$, $c_1^1 = 27.64$ and $c_2^1 = -34.66$. Another set reads $F_0^2 = 0.39 \text{ GeV}^{-1}$, $c_1^2 = 28.13$ and $c_2^2 = -35.24$.

The strong couplings at the mass-shell are equal to

$$g_{\bar{Z}_c \eta_{c1} \pi}(-m_\pi^2) = (0.47 \pm 0.06) \text{ GeV}^{-1}, \quad g_{\bar{Z}_c \eta_{c2} \pi}(-m_\pi^2) = (0.38 \pm 0.05) \text{ GeV}^{-1}. \quad (78)$$

The widths of the decays $Z_c \rightarrow \eta_c \pi^-$ and $Z_c \rightarrow \eta'_c \pi^-$ can be evaluated by employing of the formula

$$\Gamma [\bar{Z}_c \rightarrow \eta_c(\text{IS})\pi^-] = \frac{g_{\bar{Z}_c \eta_{ci} \pi}^2 m_i^2}{8\pi} \lambda(m, m_i, m_\pi) \left[1 + \frac{\lambda^2(m, m_i, m_\pi)}{m_i^2} \right], \quad (79)$$

where $\text{I} \equiv i = 1, 2$. For the decay $\bar{Z}_c \rightarrow \eta_c \pi^-$ one has to set $g_{\bar{Z}_c \eta_{ci} \pi} \rightarrow g_{\bar{Z}_c \eta_{c1} \pi}$ and $m_i \rightarrow m_1$, whereas in the case of $\bar{Z}_c \rightarrow \eta'_c \pi^-$ quantities with subscript 2 have to be used.

Computations lead to the following predictions for the partial widths of the decay channels

$$\Gamma [\bar{Z}_c \rightarrow \eta_c \pi^-] = (81 \pm 17) \text{ MeV}, \quad \Gamma [\bar{Z}_c \rightarrow \eta'_c \pi^-] = (32 \pm 7) \text{ MeV}. \quad (80)$$

3.3. Decay $\bar{Z}_c \rightarrow D^0 D^-$

In this subsection we analyze S -wave decay of \bar{Z}_c to a pair of open-charm pseudoscalar mesons $\bar{Z}_c \rightarrow D^0 D^-$. The relevant three-point correlation function is given by the expression

$$\tilde{\Pi}(p, p') = i^2 \int d^4x d^4y e^{-ipx} e^{ip'y} \langle 0 | \mathcal{T} \{ J^D(y) J^{D^0}(0) J^\dagger(x) \} | 0 \rangle, \quad (81)$$

where we introduce the interpolating currents for the pseudoscalar mesons D^- and D^0

$$J^D(y) = \bar{c}_r(y) i \gamma_5 d_r(y), \quad J^{D^0}(0) = \bar{u}_s(0) i \gamma_5 c_s(0). \quad (82)$$

The correlation function $\Pi(p, p')$ written down using physical parameters of these mesons and tetraquark \bar{Z}_c takes the form

$$\tilde{\Pi}^{\text{Phys}}(p, p') = \frac{\langle 0 | J^D | D^-(p') \rangle \langle 0 | J^{D^0} | D^0(q) \rangle \langle D^-(p') D^0(q) | \bar{Z}_c(p) \rangle \langle \bar{Z}_c(p) | J^\dagger | 0 \rangle}{p'^2 - m_D^2} \frac{1}{q^2 - m_{D^0}^2} \frac{1}{p^2 - m^2} + \dots, \quad (83)$$

where m_D and m_{D^0} are masses of the mesons D^- and D^0 , respectively.

We continue analysis by using the matrix elements

$$\langle 0 | J^D | D^-(p') \rangle = \frac{f_D m_D^2}{m_c}, \quad \langle 0 | J^{D^0} | D^0(q) \rangle = \frac{f_{D^0} m_{D^0}^2}{m_c}, \quad \langle D^-(p') D^0(q) | \bar{Z}_c(p) \rangle = g_{\bar{Z}_c D D}(p \cdot p'). \quad (84)$$

Simple manipulations lead to

$$\tilde{\Pi}^{\text{Phys}}(p, p') = \frac{f_{D^0} m_{D^0}^2 f_D m_D^2}{m_c^2 (p'^2 - m_D^2) (q^2 - m_{D^0}^2)} \frac{m f}{p^2 - m^2} (p \cdot p') + \dots. \quad (85)$$

The same correlation function written down in terms of the quark propagators is

$$\tilde{\Pi}^{\text{OPE}}(p, p') = i^2 \int d^4x d^4y e^{-ipx} e^{ip'y} \epsilon \tilde{\text{Tr}} \left[\gamma_5 S_d^{rk}(y-x) \gamma_5 \tilde{S}_c^{sj}(-x) \gamma_5 \tilde{S}_u^{ns}(x) \gamma_5 S_c^{mr}(x-y) \right]. \quad (86)$$

The sum rule for the strong coupling $g_{\bar{Z}_c D D}$ can be expressed in a traditional form

$$g_{\bar{Z}_c D D}(\mathbf{M}^2, \mathbf{s}_0, q^2) = \frac{\tilde{\Pi}(\mathbf{M}^2, \mathbf{s}_0, q^2) e^{m/M_1^2} e^{m_D^2/M_2^2}}{B}, \quad (87)$$

where

$$B = \frac{f_D m_D^2 f_{D^0} m_{D^0}^2 m f}{2m_c^2 (q^2 - m_{D^0}^2)} (m^2 + m_D^2 - q^2).$$

Here, $\tilde{\Pi}(\mathbf{M}^2, \mathbf{s}_0, q^2)$ is the amplitude $\tilde{\Pi}^{\text{OPE}}(p^2, p'^2, q^2)$ after Borel transformation and subtraction procedures: it is expressible in term of the spectral density $\tilde{\rho}_D(s, s', q^2)$

$$\tilde{\Pi}(\mathbf{M}^2, \mathbf{s}_0, q^2) = \int_{4m_c^2}^{s_0} ds \int_{m_c^2}^{s'_0} ds' \tilde{\rho}_D(s, s', q^2) e^{-s/M_1^2} e^{-s'/M_2^2}. \quad (88)$$

The sum rule for $g_{\bar{Z}_c DD}$ depends on masses and decay constants of the mesons D^0 and D^- : for these parameters we utilize $m_{D^0} = (1864.83 \pm 0.05)$ MeV, $m_D = (1869.65 \pm 0.05)$ MeV and $f_D = f_{D^0} = (211.9 \pm 1.1)$ MeV, respectively. Restrictions on parameters \mathbf{M}^2 and \mathbf{s}_0 do not differ from ones considered above and are universal for such kind of computations. The M_1^2 and s_0 are varied within limits determined in the mass calculations (65). The parameters M_2^2 , s'_0 in Eq. (88) are

$$M_2^2 \in [3, 6] \text{ GeV}^2, \quad s'_0 \in [7, 9] \text{ GeV}^2. \quad (89)$$

Numerical computations of Eq. (88) with regions (65) lead to stable results for the form factor $g_{\bar{Z}_c DD}(\mathbf{M}^2, \mathbf{s}_0, q^2)$ at $q^2 < 0$. In what follows, we denote it $g_{\bar{Z}_c DD}(Q^2)$ by introducing $q^2 = -Q^2$ and omit parameters \mathbf{M}^2 and \mathbf{s}_0 .

The width of the decay $\bar{Z}_c \rightarrow D^0 D^-$ depends on the strong coupling $g_{\bar{Z}_c DD}$ at the mass shell of the meson D^0 . Therefore, we utilize the fit function $\tilde{F}(Q^2)$ from Eq. (77) with parameters $\tilde{F}_0 = 0.44 \text{ GeV}^{-1}$, $\tilde{c}_1 = 2.38$ and $\tilde{c}_2 = -1.61$. In figure 1 we depict $\tilde{F}(Q^2)$ and sum rule predictions for $g_{\bar{Z}_c DD}(Q^2)$ demonstrating very nice agreement between them.

The strong coupling at the mass shell $Q^2 = -m_{D^0}^2$ is

$$g_{\bar{Z}_c DD}(-m_{D^0}^2) = (0.25 \pm 0.05) \text{ GeV}^{-1}. \quad (90)$$

The width of the decay $\bar{Z}_c \rightarrow D^0 D^-$ is calculated employing Eq. (79) with necessary replacements, and by taking into account that $\lambda \Rightarrow \lambda(m, m_{D^0}, m_D)$.

The partial width of this decay reads

$$\Gamma[\bar{Z}_c \rightarrow D^0 D^-] = (19 \pm 5) \text{ MeV}. \quad (91)$$

This result will be employed to evaluate the full width of the tetraquark \bar{Z}_c .

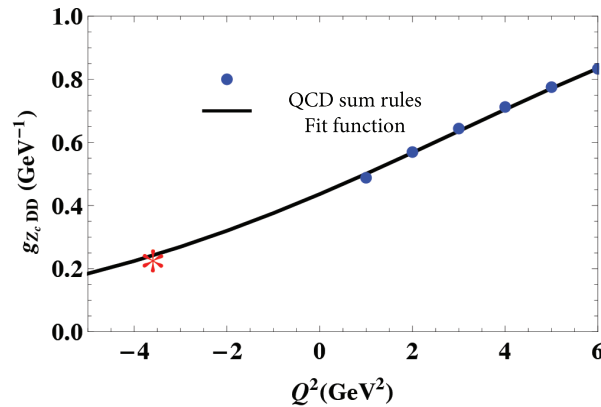


Figure 1. The sum rule predictions and fit function for the strong coupling $g_{\bar{Z}_c DD}(Q^2)$. The star marks the point $Q^2 = -m_{D^0}^2$.

3.4. Decay $\bar{Z}_c \rightarrow J/\psi\rho^-$

The scalar tetraquark \bar{Z}_c can decay to a pair of two vector mesons $J/\psi\rho^-$. In the context of the LCSR method this decay can be studied by means of the correlation function

$$\Pi_\mu(p, q) = i \int d^4x e^{ipx} \langle \rho(q) | \mathcal{T} \{ J_\mu^\psi(x) J^\dagger(0) \} | 0 \rangle, \quad (92)$$

where the interpolating current for the vector meson J/ψ is denoted by $J_\mu^\psi(x)$.

The correlation function $\Pi_\mu^{\text{Phys}}(p, q)$ in terms of the physical parameters of the tetraquark \bar{Z}_c , and mesons J/ψ and ρ is equal to

$$\Pi_\mu^{\text{Phys}}(p, q) = g_{Z_c J/\psi \rho} \frac{m_{J/\psi} f_{J/\psi}}{p^2 - m_{J/\psi}^2} \frac{m_f}{p^2 - m^2} \left[\frac{1}{2} (m^2 - m_{J/\psi}^2 - q^2) \varepsilon'_\mu - p \cdot \varepsilon' q_\mu \right] + \dots \quad (93)$$

It contains Lorentz structures proportional to ε'_μ and q_μ . We work with the structure $\sim \varepsilon'_\mu$ and label the corresponding invariant amplitude by $\Pi^{\text{Phys}}(p^2, q^2)$.

The second ingredient of the sum rule is the same correlation function $\Pi_\mu^{\text{OPE}}(p, q)$ expressed in terms of quark propagators

$$\Pi_\mu^{\text{OPE}}(p, q) = i^2 \int d^4x e^{ipx} \epsilon \tilde{\epsilon} \left[\gamma_5 \tilde{S}_c^{aj}(x) \gamma_\mu \tilde{S}_c^{ma}(-x) \gamma_5 \right]_{\alpha\beta} \langle \rho(q) | \bar{d}_\alpha^k(0) u_\beta^n(0) | 0 \rangle. \quad (94)$$

The $\Pi_\mu^{\text{OPE}}(p, q)$ contains two- and three-particle local matrix elements of the ρ -meson. Two of these elements (49) and (50) do not depend on the ρ meson momentum, whereas others are determined using momentum factors

$$\begin{aligned} \langle 0 | \bar{u} \sigma_{\mu\nu} d | \rho(q, \lambda) \rangle &= i f_\rho^T (\epsilon_\mu^{(\lambda)} q_\nu - \epsilon_\nu^{(\lambda)} q_\mu), \quad \langle 0 | \bar{u} g G_{\mu\nu} d | \rho(q, \lambda) \rangle = i f_\rho^T m_\rho^3 \zeta_4^T (\epsilon_\mu^{(\lambda)} q_\nu - \epsilon_\nu^{(\lambda)} q_\mu), \\ \langle 0 | \bar{u} g \tilde{G}_{\mu\nu} i \gamma_5 d | \rho(q, \lambda) \rangle &= i f_\rho^T m_\rho^3 \tilde{\zeta}_4^T (\epsilon_\mu^{(\lambda)} q_\nu - \epsilon_\nu^{(\lambda)} q_\mu). \end{aligned} \quad (95)$$

By substituting these matrix elements into the correlation function (94), carrying out the summation over color and calculating traces over Lorentz indices, we find local matrix elements of the ρ meson that contribute to $\Pi_\mu^{\text{OPE}}(p, q)$. It appears in the soft limit $q \rightarrow 0$ contributions to the invariant amplitude $\Pi^{\text{OPE}}(p^2)$ come from the matrix elements (49) and (50).

The Borel transformation of the amplitude $\Pi^{\text{OPE}}(p^2)$ is given by the formula

$$\Pi^{\text{OPE}}(M^2) = \int_{4m_c^2}^{\infty} ds \tilde{\rho}^{\text{OPE}}(s) e^{-s/M^2} + \Pi^{\text{OPE}(\text{tw}4)}(M^2), \quad (96)$$

with $\tilde{\rho}^{\text{OPE}}(s)$ and $\Pi^{\text{OPE}(\text{tw}4)}(M^2)$ being the spectral density and twist-4 contribution to $\Pi^{\text{OPE}}(M^2)$, respectively. Computation of $\tilde{\rho}^{\text{OPE}}(s)$ has been performed by taking into account condensates up to dimension six. The spectral density consists of the perturbative and nonperturbative components

$$\tilde{\rho}^{\text{OPE}}(s) = \frac{f_\rho m_\rho (s + 2m_c^2) \sqrt{s(s - 4m_c^2)}}{24\pi^2 s} + \rho^{\text{n.-pert.}}(s). \quad (97)$$

The nonperturbative part of the spectral density $\rho^{\text{n.-pert.}}(s)$ contains terms proportional to gluon condensates $\langle\alpha_s G^2/\pi\rangle$, $\langle\alpha_s G^2/\pi\rangle^2$ and $\langle g_s^3 G^3\rangle$: Here, we do not write down their expressions explicitly. The twist-4 term in Eq. (96) is equal to

$$\Pi^{\text{OPE(tw4)}}(M^2) = \frac{f_\rho m_\rho^3 \zeta_{4\rho} m_c^2}{8\pi} \int_0^1 d\alpha \frac{e^{-m_c^2/M^2 a(1-a)}}{a(1-a)}. \quad (98)$$

The sum rule for the strong coupling is given by the formula

$$g_{\bar{Z}_c J/\psi \rho} = \frac{2}{m_{J/\psi} f_{J/\psi} m f(m^2 - m_{J/\psi}^2)} \mathcal{P}(M^2, \tilde{m}^2) \Pi^{\text{OPE}}(M^2, s_0). \quad (99)$$

where $\mathcal{P}(M^2, \tilde{m}^2)$ is the operator in Eq. (34), and $\tilde{m}^2 = (m^2 + m_{J/\psi}^2)/2$. The width of the decay $\bar{Z}_c \rightarrow J/\psi \rho^-$ is determined by the expression

$$\Gamma(\bar{Z}_c \rightarrow J/\psi \rho^-) = \frac{g_{\bar{Z}_c J/\psi \rho}^2 m_\rho^2}{8\pi} \lambda(m, m_{J/\psi}, m_\rho) \left[3 + \frac{2\lambda^2(m, m_{J/\psi}, m_\rho)}{m_\rho^2} \right]. \quad (100)$$

Calculation of the sum rule Eq. (99) is done using M^2 and s_0 from Eq. (65). For the coupling $g_{\bar{Z}_c J/\psi \rho}$, we find

$$g_{\bar{Z}_c J/\psi \rho} = (0.56 \pm 0.07) \text{ GeV}^{-1}. \quad (101)$$

Then the width of the decay $\bar{Z}_c \rightarrow J/\psi \rho^-$ is

$$\Gamma[\bar{Z}_c \rightarrow J/\psi \rho^-] = (15 \pm 3) \text{ MeV}. \quad (102)$$

For the full width of the resonance \bar{Z}_c saturated by decay modes $\bar{Z}_c \rightarrow \eta_c \pi^-$, $\eta'_c \pi^-$, $D^0 D^-$ and $\bar{Z}_c \rightarrow J/\psi \rho^-$, we get

$$\Gamma = (147 \pm 19) \text{ MeV}. \quad (103)$$

Our predictions for the mass $m = (4080 \pm 150) \text{ MeV}$ and full width $\Gamma = (147 \pm 19) \text{ MeV}$ of the resonance \bar{Z}_c agree with LHCb data. Therefore, it is legitimate to interpret the charged resonance $Z_c^-(4100)$ as the scalar diquark-antidiquark $[cd][\bar{c}\bar{u}]$ with $C\gamma_5 \otimes \gamma_5 C$ structure. It is probably a member of charged Z -resonance multiplets that include also the axial-vector tetraquarks $Z_c^\pm(3900)$ and $Z_c^\pm(4330)$. The resonances $Z_c^\pm(4330)$ and $Z_c^\pm(3900)$ were discovered in the $\psi'\pi^\pm$ and $J/\psi\pi^\pm$ invariant mass distributions, whereas the neutral particle $Z_c^0(3900)$ was seen in the process $e^+e^- \rightarrow \pi^0\pi^0 J/\psi$. Since J/ψ and ψ' are vector mesons, and ψ' is the radial excitation of J/ψ , it is reasonable to treat $Z_c(4330)$ as first radial excitation of $Z_c(3900)$ (see, section 2). Then the resonance \bar{Z}_c fixed in the $\eta_c\pi^-$ channel can be considered as a scalar partner of these axial-vector tetraquarks. It is also meaningful to assume that a neutral member of this family $Z_c^0(4100)$ may be seen in the process $e^+e^- \rightarrow \pi^0\pi^0\eta_c$ with dominantly $\pi^0\pi^0$ mesons at the final state rather than $D\bar{D}$ ones.

4. The resonances $X(4140)$ and $X(4274)$

Recently, after analyses of exclusive decays $B^+ \rightarrow J/\psi\phi K^+$, the LHCb confirmed existence of the resonances $X(4140)$ and $X(4274)$ in the $J/\psi\phi$ invariant mass distribution [38, 39]. In the same $J/\psi\phi$ channel LHCb

discovered heavy resonances $X(4500)$ and $X(4700)$ as well. The masses and decay widths of these resonances (in this section $X(4140) \Rightarrow X_1$, $X(4274) \Rightarrow X_2$, $X(4500) \Rightarrow X_3$ and $X(4700) \Rightarrow X_4$, respectively) in accordance with LHCb measurements are

$$\begin{aligned}
 X_1 : M &= 4146 \pm 4.5_{-2.8}^{+4.6} \text{ MeV}, \quad \Gamma = 83 \pm 21_{-14}^{+21} \text{ MeV}, \\
 X_2 : M &= 4273 \pm 8.3_{-3.6}^{+17.2} \text{ MeV}, \quad \Gamma = 56 \pm 11_{-11}^{+8} \text{ MeV}, \\
 X_3 : M &= 4506 \pm 11_{-15}^{+12} \text{ MeV}, \quad \Gamma = 92 \pm 21_{-20}^{+21} \text{ MeV}, \\
 X_4 : M &= 4704 \pm 10_{-24}^{+14} \text{ MeV}, \quad \Gamma = 120 \pm 31_{-33}^{+42} \text{ MeV}.
 \end{aligned} \tag{104}$$

The LHCb extracted also spins and PC-parities of these states. It appears that X_1 and X_2 are axial-vector resonances with $J^{\text{PC}} = 1^{++}$, whereas the X_3 and X_4 are scalar states $J^{\text{PC}} = 0^{++}$.

First experimental information on resonances X_1 and X_2 [40–42] stimulated appearance of different models to account for their properties. Thus, they were considered as meson molecules in Refs. [111–119]. The diquark-antidiquark picture was used in Refs. [120, 121] to model X_1 and X_2 . There are also competing approaches which consider them as dynamically generated resonances [122, 123] or coupled-channel effects [124].

After LHCb measurements the experimental situation around the resonances X_1 and X_2 became more clear. The reason is that LHCb removed from agenda an explanation of X_1 as 0^{++} or 2^{++} $D_s^{*+}D_s^{*-}$ molecular states. The LHCb also excluded interpretation of X_2 as a molecular bound-state and as a cusp. There were usual attempts to interpret X resonances as excitations of the ordinary charmonium or as dynamical effects. Indeed, by studying experimental information on processes $B \rightarrow K\chi_{c1}\pi^+\pi^-$ and $B \rightarrow KD\bar{D}$ by Belle and BaBar (see, Refs. [125] and [126]), the author of Ref. [127] identified the resonances X_1 and $Y(4080)$ with the P -wave charmonia $\chi_{c1}(3^3P_1)$ and $\chi_{c0}(3^3P_0)$, respectively.

Rescattering effects in the decay $B^+ \rightarrow J/\psi\phi K^+$ were investigated in Ref. [128], where the author tried to answer the question: can these effects simulate the discovered resonances X_1 , X_2 , X_3 and X_4 or not. In accordance with this analysis, rescattering of $D_s^{*+}D_s^-$ and $\psi'\phi$ mesons may be seen as structures X_1 and X_4 , respectively. At the same time, inclusion of X_2 and X_3 into this scheme is problematic, and hence they maybe are genuine four-quark states. But, the author did not rule out explanation of X_2 as the excited charmonium $\chi_{c1}(3^3P_1)$

The diquark-antidiquark and molecule pictures prevail over alternative models of X resonances, and constitute foundations for various studies to explain experimental information on these states [129–134]. Thus, the masses of the axial-vector diquark-antidiquark states $[cs][\bar{c}\bar{s}]$ with different spin-parities and color structures were calculated in Ref. [129]. Results obtained there for states $J^{\text{P}} = 1^+$ were used in Ref. [130] to interpret X_1 and X_2 as tetraquarks $[cs][\bar{c}\bar{s}]$ with opposite (i.e., color-triplet or -sextet constituent diquarks) color organizations. Within the same approach the resonances X_3 and X_4 were interpreted as D -wave excited states of X_1 and X_2 [130].

In the framework of the tetraquark model the resonances X_1 and X_2 were also explored in Refs. [131] and [132]. Results obtained in Ref. [131] excluded interpretation of X_1 as a tightly bound diquark-antidiquark state. The resonance X_2 was modeled as an octet-octet type molecule state, and it was demonstrated that the mass of X_2 agrees with LHCb results, while its width significantly exceeds the experimental data [132]. The resonance X_3 was examined as radial excitation of the scalar structure $X(3915)$, whereas X_4 was considered as the ground-state tetraquark $[cs][\bar{c}\bar{s}]$ composed of a vector diquark and antidiquark [133]. Let us note that the

resonance $X(3915)$ was detected by Belle in the $J/\psi\omega$ invariant mass distribution of the decay $B \rightarrow J/\psi\omega K$ [135], and also observed in the process $\gamma\gamma \rightarrow J/\psi\omega$ [136]. This structure was confirmed by BaBar in the same reaction $B \rightarrow J/\psi\omega K$ [137]. The $X(3915)$ was commonly considered as the scalar charmonium $\chi_{c0}(2^3P_0)$. But a lack of decay modes $\chi_{c0}(2P) \rightarrow D\bar{D}$ stimulated other assumptions. In fact, an alternative conjecture about the resonance $X(3915)$ was proposed in Ref. [138], where it was interpreted as the lightest scalar diquark-antidiquark state $[cs][\bar{c}\bar{s}]$. Exactly this structure was examined in Ref. [133] as the ground state of X_3 , and computations apparently support suggestions made on nature of the resonances X_3 and X_4 .

A plethora of charmoniumlike structures seen in numerous processes stimulated analysis of various diquark-antidiquark states, and led to suggestions about existence of different tetraquark multiplets (see, Refs. [139–141]). Thus, the resonances X were included into $1S$ and $2S$ multiplets of tetraquarks $[cs]_{s=0,1}[\bar{c}\bar{s}]_{s=0,1}$ built of color-triplet diquarks [140]. The X_1 was interpreted as $J^{\text{PC}} = 1^{++}$ particle of the $1S$ multiplet. The X_2 resonance is probably, an admixture of two states with the quantum numbers $J^{\text{PC}} = 0^{++}$ and $J^{\text{PC}} = 2^{++}$. The idea about mixing phenomenon is inspired by the fact, that in the multiplet of the tetraquarks composed of color-triplet diquarks, there is only one state with $J^{\text{PC}} = 1^{++}$. The heavy resonances X_3 and X_4 are included into the $2S$ multiplet as its $J^{\text{PC}} = 0^{++}$ members. But apart from the color triplet multiplets there may exist a multiplet of tetraquarks composed of color-sextet diquarks [139], which also contains a state with $J^{\text{PC}} = 1^{++}$. Stated differently, the multiplet of the tetraquarks with color-sextet diquarks doubles a number of states [139], and resonance X_2 may be identified with its $J^{\text{PC}} = 1^{++}$ member.

It is evident that assumptions on internal organization of the resonances X in the diquark-antidiquark model sometimes contradict to each other. In most of these studies, the spectroscopic parameters of these states were calculated using the QCD two-point sum rule method. Results of these computations obtained by employing various suggestions on interpolating currents are in agreement with existing experimental data. In some cases predictions of various articles coincide with each other as well. Stated differently, the masses and current couplings of exotic states do not give information enough to verify supposed models by comparing them with experimental data or/and alternative theoretical models. In such cases additional useful information can be extracted from studies of exotic states' decay channels. The spectroscopic parameters and strong decays of X_1 and X_2 were explored in Ref. [134], in which they were considered as tetraquarks made of color-triplet and -sextet diquarks, respectively. Below, we present results of this analysis.

4.1. Parameters of the resonances X_1 and X_2

The masses and couplings of the resonances X_1 and X_2 can be calculated by utilizing the QCD two-point sum rule method. Relevant sum rules can be extracted from analysis of the correlation function (4), where $J_\mu(x)$ is the interpolating current of the state X with the spin-parities $J^{\text{PC}} = 1^{++}$.

According to Ref. [130], the resonances X_1 and X_2 have the same quantum numbers, but different internal color structures. This means that colorless particles X_1 and X_2 are built of color-triplet and color-sextet diquarks, respectively. We pursue this suggestion and investigate X_1 and X_2 using the QCD sum rule method and currents of different color organization. Namely, we suggest that the current

$$J_\mu^1 = s_a^T C \gamma_5 c_b (\bar{s}_a \gamma_\mu C \bar{c}_b^T - \bar{s}_b \gamma_\mu C \bar{c}_a^T) + s_a^T C \gamma_\mu c_b (\bar{s}_a \gamma_5 C \bar{c}_b^T - \bar{s}_b \gamma_5 C \bar{c}_a^T), \quad (105)$$

which has the color structure $[\bar{\mathbf{3}}_c]_{cs} \otimes [\mathbf{3}_c]_{\bar{c}\bar{s}}$, presumably describes the resonance X_1 , whereas

$$J_\mu^2 = s_a^T C \gamma_5 c_b (\bar{s}_a \gamma_\mu C \bar{c}_b^T + \bar{s}_b \gamma_\mu C \bar{c}_a^T) + s_a^T C \gamma_\mu c_b (\bar{s}_a \gamma_5 C \bar{c}_b^T + \bar{s}_b \gamma_5 C \bar{c}_a^T), \quad (106)$$

with the color-symmetric diquark and antidiquark $[\mathbf{6}_c]_{cs} \otimes [\bar{\mathbf{6}}_c]_{\bar{c}\bar{s}}$ fields corresponds to the tetraquark X_2 .

In order to derive required sum rules, we find an expression of the correlator in terms of the physical parameters of the state X . In the case of a single particle X the Borel transformation of the phenomenological side of the sum rules takes the simple form

$$\mathcal{B}\Pi_{\mu\nu}^{\text{Phys}}(q) = m_X^2 f_X^2 e^{-m_X^2/M^2} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_X^2} \right) + \dots, \quad (107)$$

with m_X and f_X being the mass and coupling of the state X .

The QCD side of the sum rule should be expressed in terms of quark propagators. For these purposes, we contract c and s quark fields, and get for the correlation function $\Pi_{\mu\nu}^{\text{OPE}}(q)$ the expression (for definiteness, we provide explicitly results for J_μ^1):

$$\begin{aligned} \Pi_{\mu\nu}^{\text{OPE}}(q) = & -i \int d^4x e^{iqx} \tilde{\epsilon} \epsilon' \tilde{\epsilon}' \left\{ \text{Tr} \left[\gamma_\mu \tilde{S}_c^{n'n}(-x) \gamma_\nu S_s^{m'm}(-x) \right] \text{Tr} \left[\gamma_5 \tilde{S}_s^{aa'}(x) \gamma_5 S_c^{bb'}(x) \right] \right. \\ & + \text{Tr} \left[\gamma_\mu \tilde{S}_c^{n'n}(-x) \gamma_5 S_s^{m'm}(-x) \right] \text{Tr} \left[\gamma_\nu \tilde{S}_s^{aa'}(x) \gamma_5 S_c^{bb'}(x) \right] + \text{Tr} \left[\gamma_5 \tilde{S}_c^{n'n}(-x) \gamma_\nu S_s^{m'm}(-x) \right] \\ & \left. \times \text{Tr} \left[\gamma_5 \tilde{S}_s^{aa'}(x) \gamma_\mu S_c^{bb'}(x) \right] + \text{Tr} \left[\gamma_5 \tilde{S}_c^{n'n}(-x) \gamma_5 S_s^{m'm}(-x) \right] \text{Tr} \left[\gamma_\nu \tilde{S}_s^{aa'}(x) \gamma_\mu S_c^{bb'}(x) \right] \right\}, \quad (108) \end{aligned}$$

where $\epsilon = \epsilon^{cab}$, $\tilde{\epsilon} = \epsilon^{cmn}$ and $\epsilon' = \epsilon^{c'a'b'}$, $\tilde{\epsilon}' = \epsilon^{c'm'n'}$.

The spectroscopic parameters of the tetraquarks X can be calculated using the sum rules (16) and (17) after substituting $4m_c^2$, m_{Z_c} , and f_{Z_c} by $4(m_c + m_s)^2$, m_X and f_X .

The two-point spectral density $\rho^{\text{OPE}}(s)$ necessary for calculations can be derived using methods presented already in the literature (see, for example, Ref. [53]). Therefore, we do not detail here these usual and routine computations. Our predictions for parameters of the resonances X_1 and X_2 are collected in Table 6, where we also present working regions for M^2 and s_0 . In the working regions of the Borel and continuum threshold parameters the pole contribution is equal to 0.23, which is typical for the sum rule calculations involving tetraquarks. To keep under control convergence of the operator product expansion, we find a contribution of each term with fixed dimension: in the working regions convergence of OPE is satisfied. Let us only note that a contribution of the dimension-8 term to the whole result does not overshoot 1%.

Table 6. The masses and couplings of the tetraquarks X_1 and X_2 .

X	X_1	X_2
M^2 (GeV ²)	4 – 6	4 – 6
s_0 (GeV ²)	20 – 22	21 – 23
m_X (MeV)	4183 ± 115	4264 ± 117
f_X (GeV ⁴)	$(0.94 \pm 0.16) \times 10^{-2}$	$(1.51 \pm 0.21) \times 10^{-2}$

In Figures 2 and 3, we depict the parameters of the tetraquark X_1 as functions of M^2 and s_0 . It is clear that m_{X_1} and f_{X_1} are sensitive to the choice of these parameters. But, while effects of M^2 and s_0 on

the mass m_{X_1} are weak, a dependence of f_{X_1} on the Borel and continuum threshold parameters is noticeable. These effects combined with uncertainties of other input parameters generate errors of sum rule calculations. The theoretical errors of calculations are presented in Table 6 as well. The similar analysis and conclusions are valid for the state X_2 , which can be seen in Figures 4 and 5.

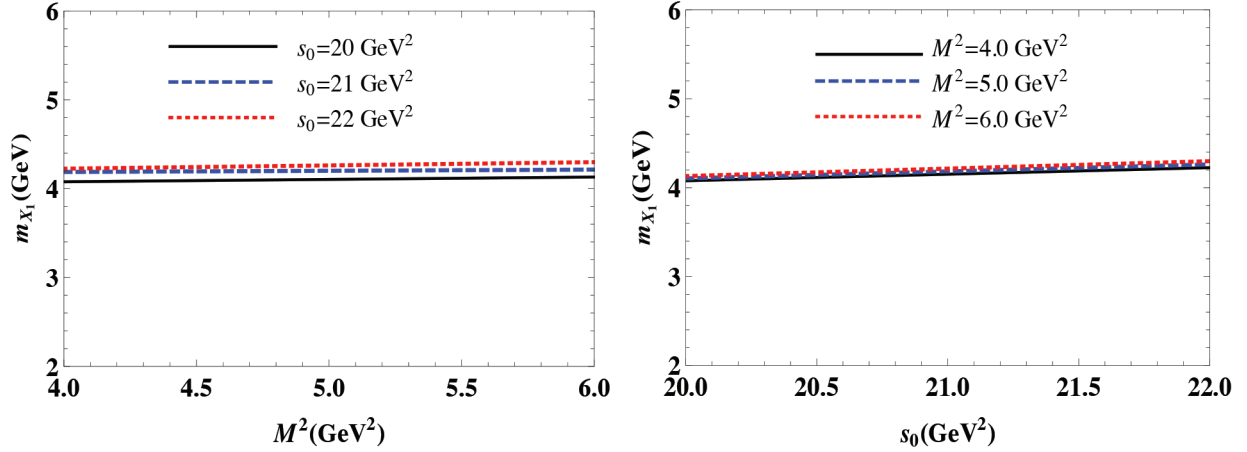


Figure 2. The mass of the X_1 state as a function of the Borel M^2 (left panel), and continuum threshold s_0 parameters (right panel).

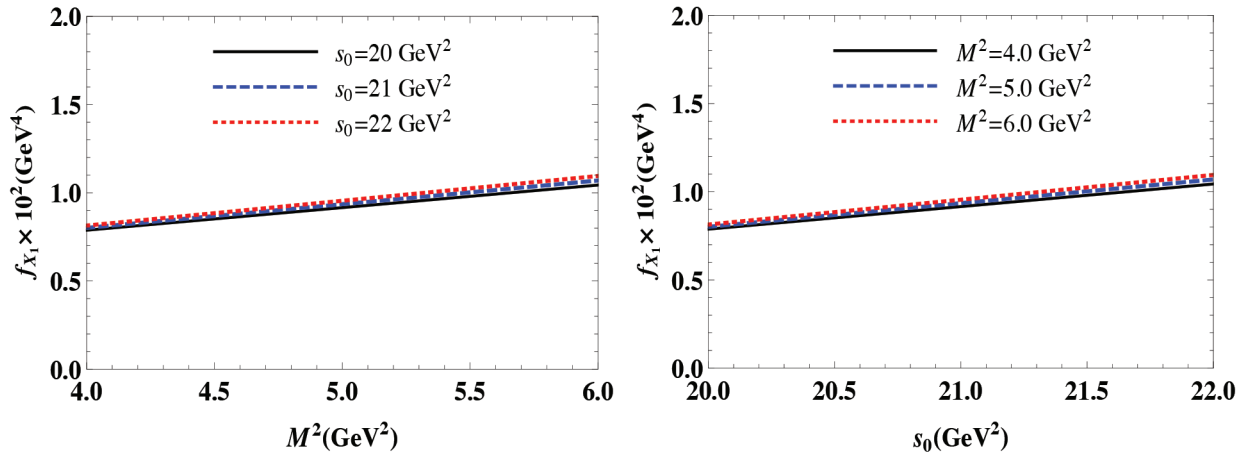


Figure 3. The dependence of the coupling f_X of the X_1 resonance on the Borel parameter at chosen s_0 (left panel), and on s_0 at fixed M^2 (right panel).

We see that our predictions for masses of the states X_1 and X_2 agree with the LHCb data. At this phase of studies one can conclude that the resonances X_1 and X_2 with the spin-parities $J^{PC} = 1^{++}$ enter to multiplets of tetraquarks composed of the color-triplet and -sextet diquarks, respectively.

4.2. Width of decays $X_1 \rightarrow J/\psi\phi$ and $X_2 \rightarrow J/\psi\phi$

Because X_1 and X_2 were discovered in the $J/\psi\phi$ invariant mass distribution, processes $X_1 \rightarrow J/\psi\phi$ and $X_2 \rightarrow J/\psi\phi$ are main decay modes of these resonances. In this subsection, we consider these two decays, and

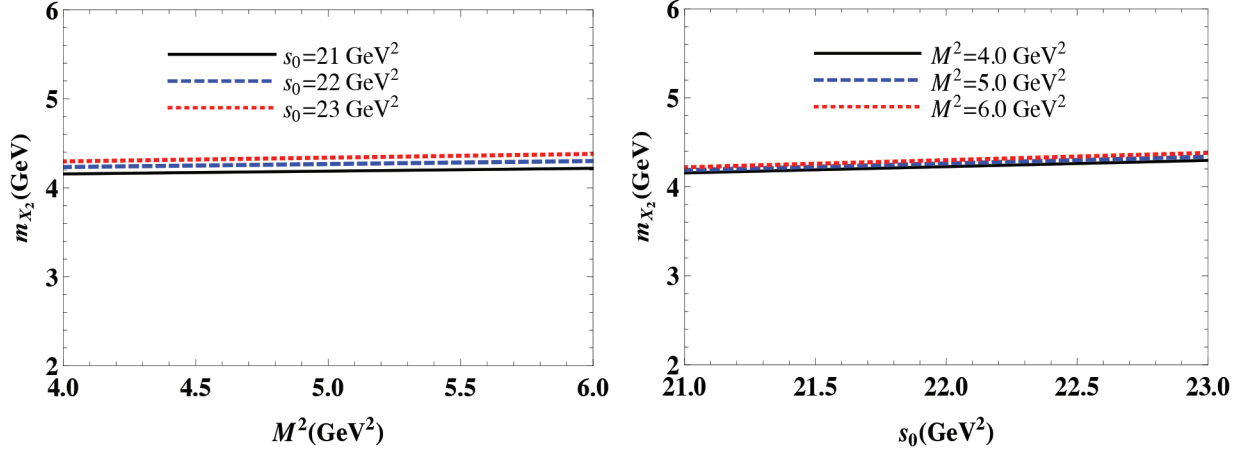


Figure 4. The mass of the X_2 resonance as a function of the Borel M^2 (left panel), and continuum threshold s_0 parameters (right panel).

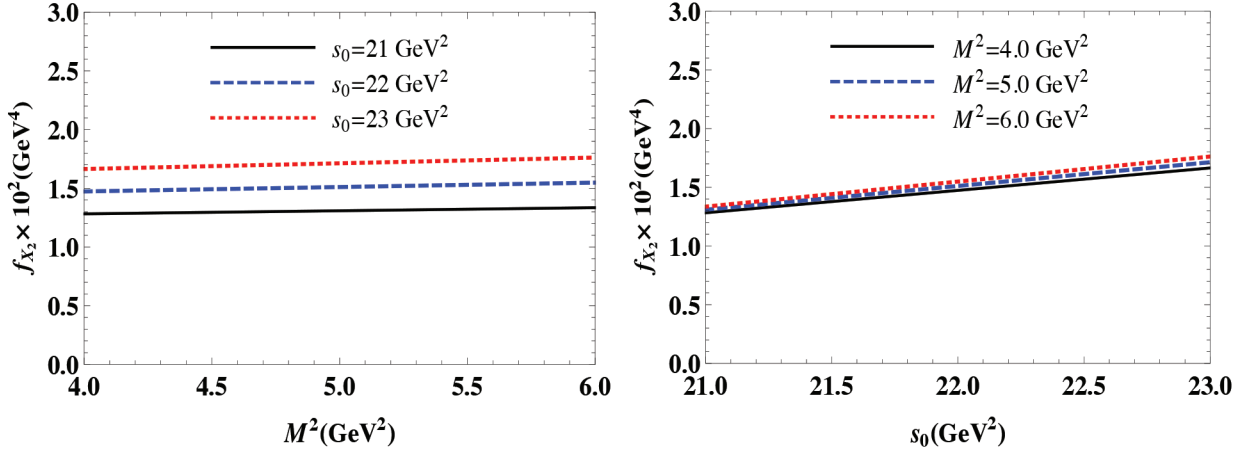


Figure 5. The coupling f_X of the resonance X_2 as a function of M^2 (left panel) and s_0 (right panel).

briefly explain operations required to explore the vertex $XJ/\psi\phi$, where X is one of states X_1 and X_2 . Below, we evaluate the strong coupling $g_{XJ/\psi\phi}$ and width of the corresponding process $X \rightarrow J/\psi\phi$.

The strong coupling $g_{XJ/\psi\phi}$ can be extracted from analysis of the correlation function

$$\Pi_{\mu\nu}(p, q) = i \int d^4x e^{ipx} \langle \phi(q) | \mathcal{T} \{ J_\mu^\psi(x) J_\nu^\dagger(0) \} | 0 \rangle, \quad (109)$$

with J_ν and J_μ^ψ being interpolating currents of the X state and J/ψ meson, respectively.

We calculate $\Pi_{\mu\nu}(p, q)$ using the LCSR method and the soft-meson approximation. For these purposes, at the first step of analysis, we express the function $\Pi_{\mu\nu}(p, q)$ in terms of the masses, decay constants (current couplings) of the particles X and J/ψ , and strong coupling $g_{XJ/\psi\phi}$.

For $\Pi_{\mu\nu}^{\text{Phys}}(p, q)$, we get

$$\Pi_{\mu\nu}^{\text{Phys}}(p, q) = \frac{\langle 0 | J_\mu^\psi | J/\psi(p) \rangle}{p^2 - m_{J/\psi}^2} \langle J/\psi(p) \phi(q) | X(p') \rangle \frac{\langle X(p') | J_\nu^\dagger | 0 \rangle}{p'^2 - m_X^2} + \dots \quad (110)$$

The matrix element of the J/ψ meson, necessary for our calculations, has been defined in Eq. (22), whereas for the vertex, we introduce the matrix element

$$\langle J/\psi(p) \phi(q) | X(p') \rangle = i g_{XJ/\psi\phi} \epsilon_{\alpha\beta\gamma\delta} \epsilon_\alpha^*(p) \epsilon_\beta(p') \epsilon_\gamma^*(q) p_\delta. \quad (111)$$

Here, $\epsilon_\gamma^*(q)$ is the polarization vector of the ϕ meson. Then the contribution to $\Pi_{\mu\nu}^{\text{Phys}}(p, q)$ of the ground-state particles is

$$\Pi_{\mu\nu}^{\text{Phys}}(p, q) = i \frac{f_{J/\psi} f_X m_{J/\psi} m_X g_{XJ/\psi\phi}}{(p'^2 - m_X^2)(p^2 - m_{J/\psi}^2)} \left(\epsilon_{\mu\nu\gamma\delta} \epsilon_\gamma^*(p) p_\delta - \frac{1}{m_X^2} \epsilon_{\mu\beta\gamma\delta} \epsilon_\gamma^*(p) p_\delta p'_\beta p'_\nu \right) + \dots \quad (112)$$

In the soft limit $p = p'$, and only the term $\sim i \epsilon_{\mu\nu\gamma\delta} \epsilon_\gamma^*(p) p_\delta$ survives in Eq. (112).

The correlation function $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$ for the current J_μ^1 is given by the expression

$$\begin{aligned} \Pi_{\mu\nu}^{\text{OPE}}(p, q) &= i \int d^4x e^{ipx} \epsilon^{ijk} \epsilon^{imn} \left\{ \left[\gamma_\nu \tilde{S}_c^{ak}(x) \gamma_\mu \tilde{S}_c^{na}(-x) \gamma_5 \right] - \left[\gamma_5 \tilde{S}_c^{ak}(x) \gamma_\mu \tilde{S}_c^{na}(-x) \gamma_\nu \right] \right\}_{\alpha\beta} \\ &\times \langle \phi(q) | \bar{s}_\alpha^j s_\beta^m | 0 \rangle. \end{aligned} \quad (113)$$

In the soft-meson approximation the matrix element

$$\langle 0 | \bar{s}(0) \gamma_\mu s(0) | \phi(p, \lambda) \rangle = f_\phi m_\phi \epsilon_\mu^{(\lambda)}, \quad (114)$$

of the ϕ meson contributes to the correlation function. Here, m_ϕ and f_ϕ are the mass and decay constant of the ϕ meson, respectively. The soft-meson limit reduces also possible Lorentz structures in $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$ to the term $\sim i \epsilon_{\mu\nu\gamma\delta} \epsilon_\gamma^*(p) p_\delta$, which should be equated to the same structure in $\Pi_{\mu\nu}^{\text{Phys}}(p, q = 0)$.

The invariant amplitude corresponding to this Lorentz structure in $\Pi_{\mu\nu}^{\text{OPE}}(p, q = 0)$ can be presented as a dispersion integral with the spectral density $\rho_c^{\text{OPE}}(s)$. We skip further details of calculations, and write down the final expression for $\rho_c^{\text{QCD}}(s)$, which reads

$$\rho_c^{\text{OPE}}(s) = \frac{f_\phi m_\phi m_c}{4} \left[\frac{\sqrt{s(s - 4m_c^2)}}{\pi^2 s} + F^{\text{n.-pert.}}(s) \right]. \quad (115)$$

The nonperturbative component of $\rho_c^{\text{QCD}}(s)$, i.e., $F^{\text{n.-pert.}}(s)$ is determined by the following formula

$$F^{\text{n.-pert.}}(s) = \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle \int_0^1 f_1(z, s) dz + \left\langle g_s^3 G^3 \right\rangle \int_0^1 f_2(z, s) dz + \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle^2 \int_0^1 f_3(z, s) dz. \quad (116)$$

The functions $f_1(z, s)$, $f_2(z, s)$ and $f_3(z, s)$ are given by the expressions

$$f_1(z, s) = \frac{1}{18r^2} \left\{ -(2 + 3r(3 + 2r)) \delta^{(1)}(s - \Phi) + (1 + 2r) [m_c^2 - sr] \delta^{(2)}(s - \Phi) \right\}, \quad (117)$$

$$f_2(z, s) = \frac{(1-2z)}{2^7 \cdot 9\pi^2 r^5} \left\{ 2r \left[3r(1+rR) \delta^{(2)}(s-\Phi) + [3sr^2(1+r) - 2m_c^2(1+rR)] \delta^{(3)}(s-\Phi) \right] \right. \\ \left. + [s^2 r^4 - 2sm_c^2 r^2(1+r) + m_c^4(1+rR)] \delta^{(4)}(s-\Phi) \right\}, \quad (118)$$

$$f_3(z, s) = \frac{m_c^2 \pi^2}{2^2 \cdot 3^4 r^2} \left[\delta^{(4)}(s-\Phi) - s \delta^{(5)}(s-\Phi) \right], \quad (119)$$

where the short hand notations

$$r = z(z-1), \quad R = 3+r, \quad \Phi = \frac{m_c^2}{z(1-z)}, \quad (120)$$

has been introduced. The function $\delta^{(n)}(s-\Phi)$ is defined as

$$\delta^{(n)}(s-\Phi) = \frac{d^n}{ds^n} \delta(s-\Phi). \quad (121)$$

For the interpolating current J_μ^2 we get

$$\Pi_{\mu\nu}^{\text{OPE}}(p, q) = i \int d^4x e^{ipx} \left\{ \left[\gamma_\nu \tilde{S}_c^{ib}(x) \gamma_\mu \tilde{S}_c^{ai}(-x) \gamma_5 - \gamma_5 \tilde{S}_c^{ib}(x) \gamma_\mu \tilde{S}_c^{ai}(-x) \gamma_\nu \right]_{\alpha\beta} \right. \\ \left. \times \langle \phi(q) | \bar{s}_\alpha^a s_\beta^b | 0 \rangle + \left[\gamma_\nu \tilde{S}_c^{ib}(x) \gamma_\mu \tilde{S}_c^{bi}(-x) \gamma_5 - \gamma_5 \tilde{S}_c^{ib}(x) \gamma_\mu \tilde{S}_c^{bi}(-x) \gamma_\nu \right]_{\alpha\beta} \langle \phi(q) | \bar{s}_\alpha^a s_\beta^a | 0 \rangle \right\}. \quad (122)$$

The corresponding spectral density is

$$\rho_c^{(2)\text{OPE}}(s) = 2\rho_c^{(1)\text{OPE}}(s), \quad (123)$$

where $\rho_c^{(1)\text{OPE}}(s)$ is given by Eq. (115).

The width of the decay $X \rightarrow J/\psi\phi$ can be found by means of the formula

$$\Gamma(X \rightarrow J/\psi\phi) = \frac{\lambda(m_X, m_{J/\psi}, m_\phi)}{48\pi m_X^4 m_\phi^2} g_{XJ/\psi\phi}^2 \left[(m_X^2 + m_\phi^2) m_{J/\psi}^4 + (m_X^2 - m_\phi^2)^2 \right. \\ \left. \times \left(m_X^2 + m_\phi^2 - 2m_{J/\psi}^2 \right) + 4m_X^2 m_{J/\psi}^2 m_\phi^2 \right], \quad (124)$$

where $\lambda(a, b, c)$ is the standard function (37).

In Table 7, we have collected our results for the couplings and decay widths. We also write down the regions for the parameters M^2 and s_0 used in numerical calculations to evaluate the couplings $g_{X_1 J/\psi\phi}$ and $g_{X_2 J/\psi\phi}$. In these regions computations meet all standard constraints of the sum rule analysis.

In Table 8, we have collected the LHCb data and our results for parameters of X_1 and X_2 . The states X_1 and X_2 were explored in numerous articles [113, 129–132]: some of their predictions are also shown. As is seen, our results for the masses of tetraquarks X_1 and X_2 , evaluated in the context of the QCD sum rule method, are in reasonable agreement with recent LHCb measurements [38]. We also see that width of the decay $X_1 \rightarrow J/\psi\phi$ is compatible with experimental data, but $\Gamma(X_2 \rightarrow J/\psi\phi)$ significantly overshoots and does not explain them.

Table 7. The strong coupling $g_{XJ/\psi\phi}$ and decay width $\Gamma(X \rightarrow J/\psi\phi)$.

X	$X(4140)$	$X(4274)$
M^2 (GeV ²)	5 – 7	5 – 7
s_0 (GeV ²)	20 – 22	21 – 23
$g_{XJ/\psi\phi}$	2.34 ± 0.89	3.41 ± 1.21
$\Gamma(X \rightarrow J/\psi\phi)$ (MeV)	80 ± 29	272 ± 81

Table 8. The LHCb data and theoretical predictions for the mass and width of the resonances X_1 and X_2 .

	m_{X_1} (MeV)	Γ_{X_1} (MeV)	m_{X_2} (MeV)	Γ_{X_2} (MeV)
LHCb	$4146 \pm 4.5^{+4.6}_{-2.8}$	$83 \pm 21^{+21}_{-14}$	$4273 \pm 8.3^{+17.2}_{-3.6}$	$56 \pm 11^{+8}_{-11}$
[134]	4183 ± 115	80 ± 29	4264 ± 117	272 ± 81
[113]	4140 ± 90	–	–	–
[129]	4070 ± 100	–	4220 ± 100	–
[131]	3950 ± 90 5000 ± 100	– –	– –	– –
[132]	–	–	4270 ± 90	1800

The resonance X_1 was considered in Ref. [113] as a molecule state $D_s^* \bar{D}_s^*$ with $J^{PC} = 0^{++}$. Mass of this molecule obtained by employing the QCD sum rule method correctly describes the experimental data. But the problem is that, LHCb ruled out interpretation of the resonance X_1 as a molecule-like state.

The parameters of X_1 and X_2 in the framework of the sum rule method were evaluated in Refs. [129, 130] as well. Results obtained there, are in accord with the LHCb data. Let us emphasize that the resonances X_1 and X_2 were considered in Refs. [129, 130] as the axial-vector states built of color-triplet and -sextet diquarks, respectively. The studies performed in Ref. [131] by means of the sum rule method and two interpolating currents, however excluded diquark-antidiquark interpretation for X_1 . The reason is that m_{X_1} evaluated using relevant sum rules is either below the LHCb data or exceeds them (see, Table 8).

The X_2 was investigated as a molecule-like color-octet state [132], and its mass m_{X_2} was found equal to

$$m_{X_2} = 4.27 \pm 0.09 \text{ GeV}. \quad (125)$$

But width of the decay $X_2 \rightarrow J/\psi\phi$

$$\Gamma(X_2 \rightarrow J/\psi\phi) = 1.8 \text{ GeV} \quad (126)$$

estimated using the QCD three-point sum rule method overshoots the LHCb value, and hence the author removed his assumption about the structure of the state X_2 from agenda.

In this section, we explored the resonances X_1 and X_2 . Our predictions for the mass and width of the resonance X_1 permit its interpretation as a serious candidate to a tetraquark with $J^{PC} = 1^{++}$ built of color-triplet diquark (antidiquark). But, in light of the LHCb data, consideration of X_2 as a tetraquark with only color-sextet diquark constituents seems is problematic. The reason is that LHCb specifies X_2 as a relatively narrow state, while our estimate for its width equals to a few hundred MeV. It is quite possible that X_2 is

an admixture of a tetraquark with color-sextet ingredients and an ordinary charmonium. But this and other assumptions on internal structure of the resonance X_2 require additional analyses.

5. The axial-vector resonance $a_1(1420)$

The resonance $a_1(1420)$ (or a_1 throughout this section) reported by COMPASS collaboration [142] enlarged a five-member family of axial-vector mesons with the spin-parities $J^{PC} = 1^{++}$. In order to find a partner of the isosinglet $f_1(1420)$ meson, COMPASS studied $J^{PC} = 1^{++}$ states in the diffractive process $\pi^- + p \rightarrow \pi^- \pi^- \pi^+ + p_{\text{recoil}}$. In the $f_0(980)\pi$ final state the collaboration discovered a resonance 1^{++} and identified it as a_1 meson with the mass and width

$$m = 1414_{-13}^{+15} \text{ MeV}, \quad \Gamma = 153_{-23}^{+8} \text{ MeV}. \quad (127)$$

Observation of the light axial-vector state a_1 that may be interpreted as isovector partner of $f_1(1420)$ meson, stimulated theoretical studies in the framework of numerous models and schemes. Goals of these investigations were to reveal structure of a_1 and compute its parameters. It is worth noting that by considering a_1 as an ordinary axial-vector meson COMPASS, at the same time, did not rule out its possible interpretation as an exotic state. The reason behind of this conclusion is discovery of only $a_1 \rightarrow f_0(980)\pi$ decay channel of the meson a_1 . Problems connected with identification of a_1 as a radially excited $a_1(1260)$ meson also feed ideas on its exotic nature.

The meson $f_0(980)$ that appears in the decay $a_1 \rightarrow f_0(980)\pi$ gives additional information on possible structure of a_1 . It is one of the first mesons that was considered as candidate to a light four-quark state. The meson $f_0(980)$ is a member of the first nonet of scalar particles, which already were analyzed as real candidates to four-quark $\bar{q}^2 q^2$ states [1]. Because, $f_0(980)$ has a considerable strange component it was considered also as a $K\bar{K}$ molecule [143]. Lattice simulations and various experiments seem confirm assumptions on four-quark structure of $f_0(980)$ and some other hadrons [57, 144–146]. On the basis of new theoretical analysis conclusions on a diquark-antidiquark structure of $f_0(980)$ and other light scalar mesons were also drawn in Refs. [147, 148].

Scalar mesons that form the first light nonet were investigated in the framework of the QCD sum rule method as well. These studies led to contradictory results about their internal organization [149–157]. In fact, some computations supported the diquark-antidiquark nature of these scalars [151–153], whereas the author of Ref. [154] could not find in the light mesons signs of diquark components. Different models were examined to explain properties of the light scalars and relevant experimental data. These models used various assumptions on their structure, including mixing of diquark-antidiquarks of different flavor structures [153], and admixtures of four- and two-quark components [155–157]. The modern theoretical studies and experimental data are in favor of the tetraquark picture for the light scalar mesons [2–5, 158].

It is evident that different theoretical models consider $f_0(980)$ mainly as a tetraquark state, or at least as a meson containing essential four-quark component. These features of $f_0(980)$ may provide useful information on an internal structure of the meson a_1 itself. Indeed, after discovery of the meson a_1 , in the literature appeared various models that considered it as an exotic state. It was modeled as an admixture of diquark-antidiquark and two-quark components, mass of which is in accord with the COMPASS data [159]. As a pure diquark-antidiquark compound a_1 was investigated in Refs. [160, 161], predictions of which also agree with the data.

Alternative confirmation of the four-quark structure of a_1 came from investigations performed in Ref. [162], where in the soft-wall AdS/QCD approach the authors derived and solved a Schrodinger-type equation for the tetraquark wave function. The result obtained there for the mass of the tetraquark with $J^{PC} = 1^{++}$ is in agreement with the data [142].

Explanations of a_1 as dynamical rescattering effects in $a_1(1260)$ meson's decays are presented in the literature by some articles [163–167]. Thus, a resonant structure in the $f_0(980)\pi$ mass distribution was considered in Ref. [163] as a triangle singularity in the relevant decay channel of the $a_1(1260)$ meson. The decay of the meson $a_1(1260)$ runs in accordance with the following scheme: at the first stage of transformations $a_1(1260)$ decays to $K^*\bar{K}$ -mesons, at the second phase K^* decays to K and π . Finally, K and \bar{K} combine to create the $f_0(980)$ meson. Investigation of these transformations and analysis of corresponding triangle diagram shows the existence of a singularity which may be considered as the resonance observed by COMPASS. The similar ideas were supported by Ref. [164], in which an anomalous triangle singularity were considered in various processes, including $a_1(1260) \rightarrow f_0(980)\pi$ decay.

Two-body strong decays of a_1 in the context of the covariant confined quark model were examined in Ref. [168]. The meson a_1 was modeled there as a four-quark state with diquark-antidiquark and molecule structures. In the analysis of the decay $a_1 \rightarrow f_0(980)\pi$ the meson $f_0(980)$ was also interpreted as a four-quark state with molecular or diquark organizations. Partial decay widths, and full width of the a_1 state found in this work allowed the authors to interpret a_1 as a four-quark state with a molecule-type structure.

It is seen that we can group theoretical studies of the axial-vector state a_1 into two almost equal classes: the first class contains articles, in which it is considered as a four-quark system with different structures, the second class encompasses works interpreting a_1 as dynamical effect observed in the process $a_1(1260) \rightarrow f_0(980)\pi$. In this section we present our investigation of a_1 and explain results obtained in Ref. [161].

5.1. Mass and current coupling of a_1

In the diquark picture the quark content of the neutral isovector state $I^G J^{PC} = 1^- 1^{++}$ has the form $([us][\bar{u}\bar{s}] - [ds][\bar{d}\bar{s}])/\sqrt{2}$. The isoscalar partner of a_1 , namely $f_1(1420)$ then should have the composition $([us][\bar{u}\bar{s}] + [ds][\bar{d}\bar{s}])/\sqrt{2}$. In the chiral limit particles a_1 and $f_1(1420)$ have equal masses.

A next problem connected with treatment of a_1 in the framework of the QCD sum rule method is a choice of the interpolating current. We choose the current $J_\mu(x)$ in the following form [160]

$$J_\mu(x) = \frac{1}{\sqrt{2}}[J_\mu^u(x) - J_\mu^d(x)]. \quad (128)$$

Here, $J_\mu^q(x)$ is given by the expression

$$\begin{aligned} J_\mu^q(x) = & q_a^T(x)C\gamma_5 s_b(x) [\bar{q}_a(x)\gamma_\mu C\bar{s}_b^T(x) - \bar{q}_b(x)\gamma_\mu C\bar{s}_a^T(x)] \\ & + q_a^T(x)C\gamma_\mu s_b(x) [\bar{q}_a(x)\gamma_5 C\bar{s}_b^T(x) - \bar{q}_b(x)\gamma_5 C\bar{s}_a^T(x)], \end{aligned} \quad (129)$$

with q being one of the light u , and d quarks.

After fixing $J_\mu(x)$, we should calculate the correlation function $\Pi_{\mu\nu}(p)$ given by Eq. (4), which allows us to evaluate the mass m_{a_1} and coupling f_{a_1} of the state a_1 . The remaining manipulations are standard ones, therefore we omit further details by emphasizing only that an invariant amplitude is calculated by including

into analysis vacuum condensates up to dimension 12. Let us note that contributions of terms up to dimension eight are found by using corresponding spectral density, effects of other terms are evaluated directly from their Borel transformations.

Sum rules depend on the auxiliary parameters M^2 and s_0 , the choice of which has to satisfy standard constraints. Our analyses allow us to find regions, where M^2 and s_0 can be varied:

$$M^2 \in [1.4, 1.8] \text{ GeV}^2, \quad s_0 \in [2.4, 3.1] \text{ GeV}^2. \quad (130)$$

Predictions for mass and coupling of the state a_1 extracted from the sum rules are plotted in Figures 6 and 7. In these figures they are shown as functions of the Borel and continuum threshold parameters. It is clear that m_{a_1} is rather stable against varying of M^2 and s_0 . The dependence of f_{a_1} on the Borel parameter is very weak, but its variations with s_0 are noticeable and generate essential part of theoretical ambiguities.

For m_{a_1} and f_{a_1} we find:

$$m_{a_1} = 1416_{-79}^{+81} \text{ MeV}, \quad f_{a_1} = (1.68_{-0.26}^{+0.25}) \times 10^{-3} \text{ GeV}^4. \quad (131)$$

The prediction for the mass of a_1 is in very nice agreement with data of the COMPASS collaboration. It is also in accord with the mass of the a_1 meson evaluated in Ref. [160] in the diquark-antidiquark model

$$m_{a_1} = (1440 \pm 80) \text{ MeV}, \quad f_{a_1} = (1.32 \pm 0.35) \times 10^{-3} \text{ GeV}^4. \quad (132)$$

Our result for f_{a_1} is compatible with prediction of Ref. [160] if one takes into account errors of computations: in fact, there is a large overlap region between (131) and (132). A discrepancy between two sets of parameters comes presumably from subleading terms in spectral density, which nevertheless do not change considerably the final predictions.

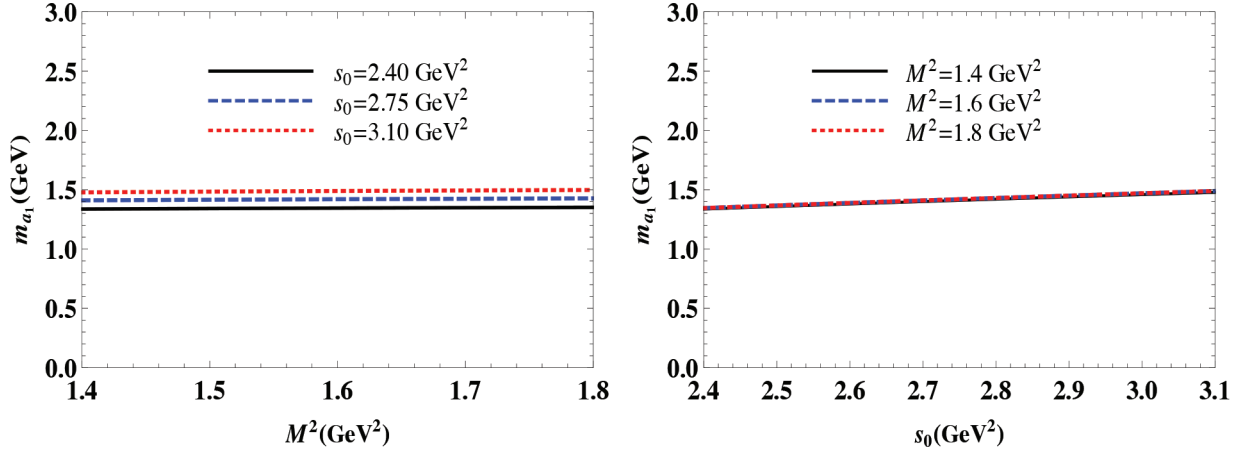


Figure 6. The dependence of m_{a_1} on M^2 (left panel), and on s_0 (right panel).

5.2. The decay channel $a_1 \rightarrow f_0(980)\pi^0$

The COMPASS observed the axial-vector state a_1 in the decay $a_1 \rightarrow f_0(980)\pi^0$. This process is P -wave decay for a_1 , and therefore is not its dominant decay mode. Nevertheless, it has to be analyzed in details because until now is a solely observed decay of the state a_1 .

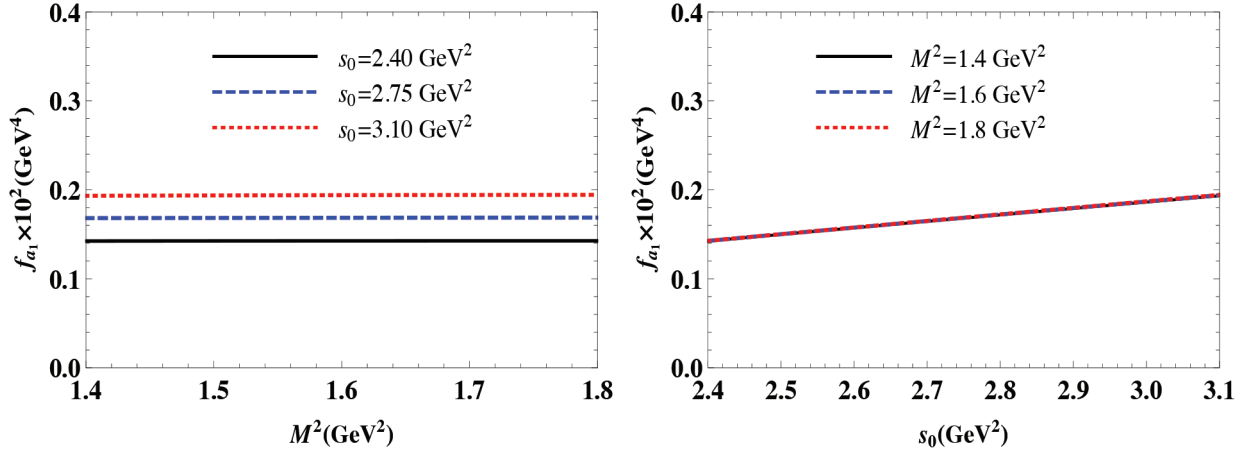


Figure 7. The coupling f_{a_1} of the a_1 state as a function of M^2 at fixed s_0 (left panel), and of s_0 at fixed M^2 (right panel).

In the framework of the LCSR method this decay can be studied starting from analysis of the correlation function

$$\Pi_\mu(p, q) = i \int d^4x e^{ipx} \langle \pi(q) | \mathcal{T} \{ J^f(x) J_\mu^\dagger(0) \} | 0 \rangle, \quad (133)$$

where $J^f(x)$ is the interpolating current of $f_0(980)$. We treat $f_0(980)$ as the scalar diquark-antidiquark state and fix its current $J^f(x)$ in the following form

$$J^f(x) = \frac{\epsilon^{dab} \epsilon^{dce}}{\sqrt{2}} \{ [u_a^T(x) C \gamma_5 s_b(x)] [\bar{u}_c(x) \gamma_5 C \bar{s}_e^T(x)] + [d_a^T(x) C \gamma_5 s_b(x)] [\bar{d}_c(x) \gamma_5 C \bar{s}_e^T(x)] \}. \quad (134)$$

After adopting the currents, we should analyze the strong vertex $a_1 f_0 \pi$ that contains two tetraquarks and an ordinary meson, and differs from vertices of a tetraquark and two conventional mesons. To find the sum rule for the coupling $g_{a_1 f_0 \pi}$, we perform well-known manipulations. Thus, at the first phase, we rewrite the correlation function using physical parameters of involved particles and get

$$\Pi_\mu^{\text{Phys}}(p, q) = \frac{\langle 0 | J^f | f_0(p) \rangle \langle f_0(p) \pi(q) | a_1(p') \rangle \langle a_1(p') | J_\mu^\dagger | 0 \rangle}{p^2 - m_{f_0}^2} + \dots \quad (135)$$

The representation for $\Pi_\mu^{\text{Phys}}(p, q)$ can be simplified by means of the matrix elements of the states a_1 , and $f_0(980)$, as well as by introducing the strong coupling $g_{a_1 f_0 \pi}$ to specify the vertex $a_1 f_0 \pi$

$$\langle f_0(p) \pi(q) | a_1(p') \rangle = g_{a_1 f_0 \pi} p \cdot \varepsilon'^*. \quad (136)$$

Here p' , p and q are four-momenta of a_1 , $f_0(980)$ and π , respectively. In Eq. (136) ε'_μ is the polarization vector of a_1 . The two-variable Borel transformations applied to $\Pi_\mu^{\text{Phys}}(p, q)$ yield

$$\mathcal{B} \Pi_\mu^{\text{Phys}}(p, q) = g_{a_1 f_0 \pi} m_{f_0} m_{a_1} f_{f_0} f_{a_1} e^{-m_{f_0}^2/M_1^2 - m_{a_1}^2/M_2^2} \left[\frac{1}{2} \left(-1 + \frac{m_{f_0}^2}{m_{a_1}^2} \right) p_\mu + \frac{1}{2} \left(1 + \frac{m_{f_0}^2}{m_{a_1}^2} \right) q_\mu \right], \quad (137)$$

where m_{f_0} and f_{f_0} are the mass and coupling of $f_0(980)$, and M_1^2 , M_2^2 are Borel parameters which correspond to variables p^2 and p'^2 , respectively. The $\Pi_\mu^{\text{Phys}}(p, q)$ and its Borel transformation contains structures proportional to p_μ and q_μ . In our studies, we use the invariant amplitude that correspond to the structure $\sim p_\mu$

$$\Pi^{\text{Phys}}(M_1^2, M_2^2) = g_{a_1 f_0 \pi} m_{f_0} m_{a_1} f_{f_0} f_{a_1} \frac{1}{2} e^{-m_{f_0}^2/M_1^2 - m_{a_1}^2/M_2^2} \left(-1 + \frac{m_{f_0}^2}{m_{a_1}^2} \right). \quad (138)$$

The sum rule can be derived after calculation of its second component. This means that the correlation function $\Pi_\mu(p, q)$ should be expressed in terms of quark propagators and of the pion's distribution amplitudes. After inserting currents into Eq. (133) and contracting quark fields, we get

$$\begin{aligned} \Pi_\mu^{\text{OPE}}(p, q) = & i \int d^4x \epsilon \tilde{\epsilon} \epsilon' \tilde{\epsilon}' e^{ipx} \left\{ \text{Tr} \left[\gamma_\mu \tilde{S}_u^{a'a}(x) \gamma_5 \tilde{S}_s^{b'b}(x) \right] \left[\gamma_5 \tilde{S}_s^{ee'}(-x) \gamma_5 \right]_{\alpha\beta} \langle \pi(q) | \bar{u}_\alpha^c(x) u_\beta^c(0) | 0 \rangle \right. \\ & - \text{Tr} \left[\gamma_5 \tilde{S}_s^{ee'}(-x) \gamma_5 \tilde{S}_u^{cc'}(-x) \right] \left[\gamma_\mu \tilde{S}_s^{b'b}(x) \gamma_5 \right]_{\alpha\beta} \langle \pi(q) | \bar{u}_\alpha^a(x) u_\beta^{a'}(0) | 0 \rangle + \text{Tr} \left[\gamma_5 \tilde{S}_u^{a'a}(x) \gamma_5 \tilde{S}_s^{b'b}(x) \right] \\ & \times \left[\gamma_5 \tilde{S}_s^{ee'}(-x) \gamma_\mu \right]_{\alpha\beta} \langle \pi(q) | \bar{u}_\alpha^c(x) u_\beta^c(0) | 0 \rangle + \text{Tr} \left[\gamma_5 \tilde{S}_s^{ee'}(-x) \gamma_\mu \tilde{S}_u^{cc'}(-x) \right] \left[\gamma_5 \tilde{S}_s^{b'b}(x) \gamma_5 \right]_{\alpha\beta} \\ & \left. \times \langle \pi(q) | \bar{u}_\alpha^a(x) u_\beta^{a'}(0) | 0 \rangle, \right. \end{aligned} \quad (139)$$

where $\epsilon \tilde{\epsilon} \epsilon' \tilde{\epsilon}' = \epsilon^{dab} \epsilon^{dce} \epsilon^{d'a'b'} \epsilon^{d'c'e'}$.

Let us note that Eq. (139) is a full expression for $\Pi_\mu^{\text{OPE}}(p, q)$, that encompasses contributions due to both u and d components of the interpolating currents $J_\mu(x)$ and $J^J(x)$: this form of the correlation function is convenient for our analysis. Apart from propagators, the function $\Pi_\mu^{\text{OPE}}(p, q)$ contains also nonlocal quark operators sandwiched between the vacuum and pion states, which can be transformed in accordance with the prescription (26).

The matrix elements of operators $\bar{u}(x) \Gamma^j u(0)$ can be expanded over x^2 and written down using the pion's two-particle DAs of different twist [169, 170]. For example, in the case of $\Gamma = i\gamma_\mu \gamma_5$ and γ_5 one obtains

$$\begin{aligned} \sqrt{2} \langle \pi^0(q) | \bar{u}(x) i\gamma_\mu \gamma_5 u(0) | 0 \rangle = & f_\pi q_\mu \int_0^1 du e^{i\bar{u}qx} \left[\phi_\pi(u) + \frac{m_\pi^2 x^2}{16} \mathbb{A}_4(u) \right] \\ & + \frac{f_\pi m_\pi^2}{2} \frac{x_\mu}{qx} \int_0^1 du e^{i\bar{u}qx} \mathbb{B}_4(u), \end{aligned} \quad (140)$$

and

$$\sqrt{2} \langle \pi^0(q) | \bar{u}(x) i\gamma_5 u(0) | 0 \rangle = \frac{f_\pi m_\pi^2}{m_u + m_d} \int_0^1 du e^{iuqx} \phi_{3;\pi}^p(u). \quad (141)$$

Above, the twist-2 (or leading twist) DA of the pion is denoted by $\phi_\pi(u)$. The $\mathbb{A}_4(u)$ and $\mathbb{B}_4(u)$ are higher twist functions which can be rewritten in terms of the pion two-particle twist-4 distributions. The matrix element given by Eq. (141) is determined by one of two-particle twist-3 distribution amplitudes of the pion $\phi_{3;\pi}^p(u)$. Another two-particle twist-3 DA $\phi_{3;\pi}^\sigma(u)$ corresponds to matrix element (141) with $i\gamma_5 \rightarrow \sigma_{\mu\nu}$ replacement. The matrix elements which appear due to insertion into $\bar{u}(x) \Gamma^j u(0)$ of the gluon field strength tensor $G_{\mu\nu}(ux)$

can be expressed in terms of three-particle DAs of the pion. Their definitions and further details were presented in Refs. [169, 170].

Because the correlation function written down in terms pion's various DAs is rather cumbersome, we do not provide it here. The $\Pi_\mu^{\text{OPE}}(p, q)$ contains Lorentz structures proportional to p_μ and q_μ . We use the invariant amplitude $\Pi^{\text{OPE}}(p^2, p'^2)$ that corresponds to a structure $\sim p_\mu$ and equate it to similar amplitude from $\Pi_\mu^{\text{Phys}}(p, q)$. The Borel transform of the invariant amplitude $\Pi^{\text{OPE}}(p^2, p'^2)$ can be computed in a manner described in Ref. [51]. Afterwards, we carry out the continuum subtraction, which simplifies when two Borel parameters are equal to each other $M_1^2 = M_2^2$. In the case under discussion we assume that a choice $M_1^2 = M_2^2$ does not lead to essential ambiguities in sum rules, and introduce M^2 through

$$\frac{1}{M^2} = \frac{1}{M_1^2} + \frac{1}{M_2^2}. \quad (142)$$

Continuum subtraction is carried out by means of recipes explained in Ref. [51]. Some of formulas used during these manipulations were presented in Appendix B of Ref. [54].

Then, the strong coupling $g_{a_1 f_0 \pi}$ can be evaluated using the sum rule

$$g_{a_1 f_0 \pi} = \frac{2m_{a_1}^2}{m_{f_0}^2 - m_{a_1}^2} \frac{e^{(m_{f_0}^2 + m_{a_1}^2)/2M^2}}{m_{f_0} m_{a_1} f_{f_0} f_{a_1}} \Pi^{\text{OPE}}(M^2, s_0), \quad (143)$$

where $\Pi^{\text{OPE}}(M^2, s_0)$ is the invariant amplitude after Borel transformation and subtraction procedures. The partial width of decay $a_1 \rightarrow f_0(980)\pi^0$ is given by the formula

$$\Gamma(a_1 \rightarrow f_0 \pi^0) = g_{a_1 f_0 \pi}^2 \frac{\lambda^3(m_{a_1}, m_{f_0}, m_\pi)}{24\pi m_{a_1}^2}. \quad (144)$$

Important nonperturbative information in $\Pi^{\text{OPE}}(M^2, s_0)$ is included into DAs of the pion. A considerable part of $\Pi^{\text{OPE}}(M^2, s_0)$ is generated by two-particle DAs of the pion at $u_0 = 1/2$. The leading twist DA $\phi_\pi(u)$ contributes to $\Pi^{\text{OPE}}(M^2, s_0)$ not only directly, but also via higher-twist DAs with which it is connected by equations of motion. Therefore, $\phi_\pi(u)$ deserves a detailed analysis.

The DA $\phi_\pi(u)$ can be expanded over the Gegenbauer polynomials $C_{2n}^{3/2}(\zeta)$

$$\phi_\pi(u, \mu^2) = 6u\bar{u} \left[1 + \sum_{n=1,2,\dots} a_{2n}(\mu^2) C_{2n}^{3/2}(u - \bar{u}) \right], \quad (145)$$

where $\bar{u} = 1 - u$. It depends not only on a longitudinal momentum fraction u carried by a quark, but due to $a_{2n}(\mu^2)$ also on a scale μ . The Gegenbauer moments $a_{2n}(\mu_0^2)$ at a normalization scale $\mu = \mu_0$ fix an initial shape of the distribution amplitude, and should be determined by some nonperturbative method or extracted from experiment.

Here, we use two models for $\phi_\pi(u, \mu^2 = 1 \text{ GeV}^2)$. One of these models was obtained from LCSR analysis of the pion's electromagnetic transition form factor [171, 172]. The shape of this DA is fixed by the coefficients

$$a_2 = 0.1, \quad a_4 = 0.1, \quad a_6 = 0.1, \quad a_8 = 0.034. \quad (146)$$

At the middle point it equals to $\phi_\pi(1/2) \simeq 1.354$, which is not far from $\phi_{\text{asy}}(1/2) = 3/2$, where $\phi_{\text{asy}}(u) = 6u\bar{u}$ is the asymptotic DA. We use also the lattice model for $\phi_\pi(u)$ [173], which contains only one nonasymptotic term

$$\phi_\pi(u, \mu^2) = 6u\bar{u} \left[1 + a_2(\mu^2)C_2^{3/2}(u - \bar{u}) \right]. \quad (147)$$

The second Gegenbauer moment $a_2(\mu^2)$ of this DA at $\mu = 2$ GeV was estimated $a_2 = 0.1364 \pm 0.021$, and evolved to

$$a_2(1 \text{ GeV}^2) = 0.1836 \pm 0.0283 \quad (148)$$

at the scale $\mu = 1$ GeV.

The sum rule (143) contains the spectroscopic parameters of the particles a_1 and f_0 . The mass m_{a_1} and coupling f_{a_1} have been evaluated in the previous subsection. For m_{f_0} we use experimental data [96]

$$m_{f_0} = (990 \pm 20) \text{ MeV}, \quad (149)$$

whereas the coupling of the meson $f_0(980)$ is borrowed from Ref. [151]

$$f_{f_0} = (1.51 \pm 0.14) \times 10^{-3} \text{ GeV}^4. \quad (150)$$

In Ref. [151] f_{f_0} was extracted from the QCD sum rule analysis using the interpolating current (134), and hence is appropriate for our goals. Here, we take into account a difference between definitions of f_{f_0} employed in Ref. [151], and accepted in the present review.

Numerical computations are carried out by utilizing the following regions for the Borel and continuum threshold parameters

$$M^2 \in [1.5, 2.0] \text{ GeV}^2, \quad s_0 \in [2.4, 3.1] \text{ GeV}^2, \quad (151)$$

where all standard restrictions on M^2 , and s_0 imposed by the sum rules are satisfied.

For the pion DA (146) the strong coupling $g_{a_1 f_0 \pi}$ and width of the decay $a_1 \rightarrow f_0 \pi^0$ are equal to

$$g_{a_1 f_0 \pi} = 3.41 \pm 0.97, \quad \Gamma(a_1 \rightarrow f_0 \pi^0) = (3.14 \pm 0.96) \text{ MeV}, \quad (152)$$

respectively. For the DA from Eq. (148), we find

$$g_{a_1 f_0 \pi} = 3.38 \pm 0.93, \quad \Gamma(a_1 \rightarrow f_0 \pi^0) = (3.09 \pm 0.91) \text{ MeV}. \quad (153)$$

It is seen that an effect of different twist-2 DAs of the pion on final results is small.

5.3. The decay channels $a_1 \rightarrow K^{*\pm} K^\mp$, $K^{*0} \bar{K}^0$ and $\bar{K}^{*0} K^0$

Here, we consider S-wave decays of the exotic meson a_1 . For these purposes, we compute strong couplings $g_{a_1 K^* K^-}$ and $g_{a_1 K^* K^+}$ of the vertices $a_1 K^{*+} K^-$ and $a_1 K^{*-} K^+$, as well as find other two couplings corresponding to vertices $a_1 K^{*0} \bar{K}^0$ and $a_1 \bar{K}^{*0} K^0$. These vertices contain a tetraquark and two ordinary mesons. For their investigation, the LCSR method should be used in connection with the soft-meson approximation. In other words, to satisfy the four-momentum conservation at these vertices momentum of a final meson, for example, a momentum of K^- in $a_1 K^{*+} K^-$ has to be set $q = 0$.

We start from the decay channel $a_1 \rightarrow K^{*+}K^-$ which can be studied using of the correlation function

$$\Pi_{\mu\nu}(p, q) = i \int d^4x e^{ipx} \langle K^-(q) | \mathcal{T} \{ J_\mu^{K^{*+}}(x) J_\nu^\dagger(0) \} | 0 \rangle, \quad (154)$$

where $J_\mu^{K^{*+}}(x)$ is the interpolating current of the K^{*+} meson

$$J_\mu^{K^{*+}}(x) = \bar{s}(x) \gamma_\mu u(x). \quad (155)$$

Following standard prescriptions, we write $\Pi_{\mu\nu}(p, q)$ in terms of physical parameters of the particles a_1 , K^{*+} and K^-

$$\Pi_{\mu\nu}^{\text{Phys}}(p, q) = \frac{\langle 0 | J_\mu^{K^{*+}} | K^{*+}(p) \rangle \langle K^{*+}(p) K^-(q) | a_1(p') \rangle \langle a_1(p') | J_\nu^\dagger | 0 \rangle}{p^2 - m_{K^*}^2} + \dots, \quad (156)$$

where p' and p, q are momenta of the initial and final particles, respectively.

Further simplification of $\Pi_{\mu\nu}^{\text{Phys}}(p, q)$ is achieved by replacing the matrix elements with their explicit formulas

$$\begin{aligned} \langle 0 | J_\mu^{K^{*+}} | K^{*+}(p) \rangle &= f_{K^*} m_{K^*} \varepsilon_\mu, \\ \langle K^{*+}(p) K^-(q) | a_1(p') \rangle &= g_{a_1 K^* K^-} [(p \cdot p')(\varepsilon^* \cdot \varepsilon') - (p \cdot \varepsilon')(p' \cdot \varepsilon^*)]. \end{aligned} \quad (157)$$

First of them, i.e., $\langle 0 | J_\mu^{K^{*+}} | K^{*+}(p) \rangle$ is written in terms of the mass m_{K^*} , decay constant f_{K^*} and polarization vector ε_μ of the K^{*+} meson. The second matrix element is expressed by employing the strong coupling $g_{a_1 K^* K^-}$ that should be evaluated from a sum rule. In the soft approximation $q \rightarrow 0$ and $p' = p$: As a result, we should perform one-variable Borel transformation, which leads to

$$\mathcal{B}\Pi_{\mu\nu}^{\text{Phys}}(p) = g_{a_1 K^* K^-} m_{K^*} m_{a_1} f_{K^*} f_{a_1} \frac{e^{-m^2/M^2}}{M^2} (m^2 g_{\mu\nu} - p_\nu p'_\mu) + \dots, \quad (158)$$

where $m^2 = (m_{K^*}^2 + m_{a_1}^2)/2$

We preserve in Eq. (158) $p_\nu \neq p'_\mu$ to show explicitly the Lorentz structures of $\mathcal{B}\Pi_{\mu\nu}^{\text{Phys}}(p)$. It is known that in the soft limit there are contributions in Eq. (158) denoted by dots, which remain unsuppressed even after Borel transformation. They correspond to strong vertices of higher excitations of particles involved into a decay process. These terms appear as contaminations in the physical side of the sum rules and should be removed using well-known recipes [51].

In the soft-meson approximation the correlation function $\Pi_{\mu\nu}^{\text{OPE}}(p)$ is determined by the formula

$$\begin{aligned} \Pi_{\mu\nu}^{\text{OPE}}(p) &= i \int d^4x e^{ipx} \frac{\tilde{\epsilon}\tilde{\epsilon}}{\sqrt{2}} \left\{ \left[\gamma_5 \tilde{S}_u^{ic}(x) \gamma_\mu \tilde{S}_s^{bi}(-x) \gamma_\nu \right] + \left[\gamma_\nu \tilde{S}_u^{ic}(x) \gamma_\mu \tilde{S}_s^{bi}(-x) \gamma_5 \right] \right\}_{\alpha\beta} \\ &\times \langle K^-(q) | \bar{s}_\alpha^e(0) u_\beta^a(0) | 0 \rangle, \end{aligned} \quad (159)$$

where $\tilde{\epsilon}\tilde{\epsilon} = \epsilon^{dab} \epsilon^{dec}$.

It turns out that the matrix element of the K meson that contributes to this correlation function is

$$\langle 0 | \bar{u}(0) i \gamma_5 s(0) | K \rangle = f_K \mu_K, \quad (160)$$

where $\mu_K = m_K^2/(m_s + m_u)$. The function $\Pi_{\mu\nu}^{\text{OPE}}(p)$ contains the same Lorentz structures as its phenomenological counterpart (158). To derive the sum rule for $g_{a_1 K^* K^-}$, we choose the invariant amplitude proportional to $g_{\mu\nu}$. The Borel transform of this amplitude reads

$$\begin{aligned} \Pi^{\text{OPE}}(M^2) = & \int_{4m_s^2}^{\infty} ds \rho^{\text{pert.}}(s) e^{-s/M^2} + \frac{f_K \mu_K}{\sqrt{2}} \left[\frac{m_s}{6} (2\langle \bar{u}u \rangle - \langle \bar{s}s \rangle) + \frac{1}{72} \langle \frac{\alpha_s G^2}{\pi} \rangle \right. \\ & \left. + \frac{m_s}{36M^2} \langle \bar{s}g_s \sigma Gs \rangle - \frac{g_s^2}{243M^2} (\langle \bar{s}s \rangle^2 + \langle \bar{u}u \rangle^2) \right], \end{aligned} \quad (161)$$

where

$$\rho^{\text{pert.}}(s) = \frac{f_K \mu_K}{12\sqrt{2}\pi^2} s.$$

The function $\Pi^{\text{OPE}}(M^2)$ contains nonperturbative terms up to dimension six, and has a simple form. It is worth emphasizing that the spectral density $\rho^{\text{pert.}}(s)$ in Eq. (161) is calculated as imaginary part of the correlation function, whereas Borel transforms of other terms are extracted directly from $\Pi^{\text{OPE}}(p^2)$.

The sum rule for the strong coupling $g_{a_1 K^* K^-}$ is given by the equality

$$g_{a_1 K^* K^-} m_{K^*} m_{a_1} f_{K^*} f_{a_1} m^2 \frac{e^{-m^2/M^2}}{M^2} + \dots = \Pi^{\text{OPE}}(M^2). \quad (162)$$

But, before to carry out the continuum subtraction, we need to exclude unsuppressed terms from the physical side of this expression. To this end, we act on both sides of Eq. (162) by the operator $\mathcal{P}(M^2, m^2)$, which singles out the ground-state term and cancel contaminations. Remaining contributions can be subtracted in a standard way, which requires replacing $\infty \rightarrow s_0$ in the first term of $\Pi^{\text{OPE}}(M^2)$ while leaving components $\sim (M^2)^0$ and $\sim 1/M^2$ in their original forms [51]. The width of the decay $a_1 \rightarrow K^{*+} K^-$ after replacements $g_{Z\psi\pi}, m_\psi, \lambda(m_Z, m_\psi, m_\pi) \rightarrow g_{a_1 K^* K^-}, m_{K^*}, \lambda(m_{a_1}, m_{K^*}, m_K)$ can be calculated using Eq. (36).

The sum rule for $g_{a_1 K^* K^-}$ can be easily used for numerical calculations. The regions for parameters M^2 and s_0 employed in the decay $a_1 \rightarrow f_0(980)\pi$ are suitable for the process $a_1 \rightarrow K^{*+} K^-$ as well. For the masses and decay constants of the mesons K^{*+} and K^- we use

$$m_{K^\pm} = (493.677 \pm 0.016) \text{ MeV}, \quad m_{K^{*\pm}} = (891.76 \pm 0.25) \text{ MeV}, \quad (163)$$

and

$$f_{K^\pm} = (155.72 \pm 0.51) \text{ MeV}, \quad f_{K^{*0(\pm)}} = 225 \text{ MeV}, \quad (164)$$

respectively.

Results of calculations are presented below

$$g_{a_1 K^* K^-} = (2.84 \pm 0.79) \text{ GeV}^{-1}, \quad \Gamma(a_1 \rightarrow K^{*+} K^-) = (37.84 \pm 10.97) \text{ MeV}. \quad (165)$$

The width of the decay $\Gamma(a_1 \rightarrow K^{*-} K^+)$ are also given by Eq. (165).

The analysis of the decays $a_1 \rightarrow K^{*0} \bar{K}^0$ ($\bar{K}^{*0} K^0$) does not differ from one presented above. Let us write down only masses of the $K^0(\bar{K}^0)$ and $K^{*0}(\bar{K}^{*0})$ mesons

$$m_{K^0} = (497.611 \pm 0.013) \text{ MeV}, \quad m_{K^{*0}} = (895.55 \pm 0.20) \text{ MeV}, \quad (166)$$

employed in numerical calculations. The decay constants of these pseudoscalar and vector mesons are presented in Eq. (164). We skip further details and write down final sum rule predictions for one of these channels

$$g_{a_1 K^{*0} \bar{K}^0} = (2.85 \pm 0.82) \text{ GeV}^{-1}, \quad \Gamma(a_1 \rightarrow K^{*0} \bar{K}^0) = (33.35 \pm 9.76) \text{ MeV}. \quad (167)$$

Parameters of the process $a_1 \rightarrow \bar{K}^{*0} K^0$ are identical to ones of the decay $a_1 \rightarrow K^{*0} \bar{K}^0$ presented in Eq. (167).

Predictions for decays of the state a_1 obtained in this section allow us to find its full width Γ

$$\Gamma = (145.52 \pm 20.79) \text{ MeV}, \quad (168)$$

which is compatible with the COMPASS data, if we take into account ambiguities of computations.

We have treated the meson a_1 as a diquark-antidiquark state, and calculated its mass and widths of five decay modes [161]. Our prediction for the mass $m_{a_1} = 1416_{-23}^{+81} \text{ MeV}$ of the a_1 is in good agreement with the experimental result. Within small computational errors, it is also in accord with the result of Ref. [160]. The full width of the meson a_1 calculated utilizing five decay channels led to prediction $\Gamma = (145.52 \pm 20.79) \text{ MeV}$. By taking into account errors of theoretical calculations and experimental measurements, it is consistent with the COMPASS data $\Gamma = 153_{-23}^{+8} \text{ MeV}$ as well. Analysis performed in Ref. [161] proved that the axial-vector meson $a_1(1420)$ can be considered as a viable candidate to a diquark-antidiquark state.

6. The resonance $Y(4660)$

The resonances $Y(4660)$ (in a short form Y) and $Y(4360)$ were seen by the Belle collaboration through initial-state radiation (ISR) in the electron-positron annihilation $e^+e^- \rightarrow \gamma_{\text{ISR}} \psi' \pi^+ \pi^-$: they were fixed as resonant structures in the $\psi' \pi^+ \pi^-$ invariant mass distribution [43, 44]. The mass and full width of the resonance Y measured by Belle are [44]

$$m_Y = 4652 \pm 10 \pm 8 \text{ MeV}, \quad \Gamma_Y = 68 \pm 11 \pm 1 \text{ MeV}. \quad (169)$$

It is interesting that there are theoretical papers in the literature claiming to interpret Y and $Y(4360)$ in the contexts of different models and schemes of the high-energy physics. In fact, the resonance Y was considered as the excited charmonia 5^3S_1 and 6^3S_1 in Refs. [174] and [175], respectively. To account for collected experimental data, Y was analyzed as a bound state of the scalar $f_0(980)$ and vector ψ' mesons [176–178], or as a baryonium [179, 180]. The hadrocharmonium model for the resonances Y and $Y(4360)$ was proposed in Ref. [81].

The diquark-antidiquark picture is among widely used models of $Y(4360)$ and Y , in which one assumes they are composition of a diquark and an antidiquark with certain features. Thus, computations performed in the context of the relativistic diquark model allowed the authors of Ref. [68] to interpret the resonance $Y(4360)$ as an excited $1P$ tetraquark composed of an axial-vector diquark and antidiquark. In this picture the resonance Y is $2P$ excitation of a scalar diquark-antidiquark state. As a radial excitation of the tetraquark $Y(4008)$ the resonance $Y(4360)$ was examined in Ref. [72]. In the framework of the QCD sum rule method Y was analyzed as the P -wave tetraquark with $C\gamma_5 \otimes D_\mu \gamma_5 C$ type structure and $[cs][\bar{c}\bar{s}]$ content in Ref. [181]. It was also modeled in Ref. [182] as the tetraquark $[cs][\bar{c}\bar{s}]$ with the interpolating current $C\gamma_5 \otimes \gamma_5 \gamma_\mu C$. The mass of such compound computed using the sum rule method agrees with experimental data.

Strictly speaking, there are some options to construct a tetraquark with required P and C parities from the five independent diquark fields without derivatives, which bear spins 0 or 1 and have different P -parities [129]. This means that there are numerous tetraquarks with different diquark-antidiquark structures, but the same quantum numbers $J^{PC} = 1^{--}$. In the context of the QCD sum rule method such currents, excluding ones with derivatives, were employed in Ref. [129] to compute masses of tetraquarks with $J^{PC} = 1^{-+}, 1^{--}, 1^{++}, 1^{+-}$ and quark contents $[cs][\bar{c}\bar{s}]$ and $[cq][\bar{c}\bar{q}]$. All examined currents for the tetraquark $[cq][\bar{c}\bar{q}]$ with $J^{PC} = 1^{--}$ predicted $m \sim 4.6 - 4.7$ GeV, which implies a possible tetraquark nature of Y as well. But these results do not exclude interpretation of Y as a state $[cs][\bar{c}\bar{s}]$ with $J^{PC} = 1^{--}$ and structure $C\gamma^\nu \otimes \sigma_{\mu\nu}C - C\sigma_{\mu\nu} \otimes \gamma^\nu C$, because the mass of such state $m = 4.64 \pm 0.09$ GeV is also consistent with the mass of the Y resonance. The sum rule method was utilized in Refs. [183–185] as well, in which the resonance Y was modeled as a tetraquark with $[cq][\bar{c}\bar{q}]$ or $[cs][\bar{c}\bar{s}]$ contents, and $C\gamma_\mu \otimes \gamma_\nu C - C\gamma_\nu \otimes \gamma_\mu C$ and $C \otimes \gamma_\mu C$ type structures.

6.1. Mass and coupling of the vector tetraquark Y

We consider Y as the $[cs][\bar{c}\bar{s}]$ tetraquark made of a scalar diquark and vector antidiquark with the $C\gamma_5 \otimes \gamma_5 \gamma_\mu C$ type structure [186]. In our calculations, we take into account condensates up to dimension 10 by including into consideration the gluon condensate $\langle g_s^3 G^3 \rangle$ omitted in aforementioned works, and improve an accuracy of the predictions obtained there.

We start from consideration of the correlation function (4), where the interpolating current $J_\mu(x)$ is

$$J_\mu(x) = \epsilon \tilde{\epsilon} [s_b^T(x) C \gamma_5 c_c(x) \bar{s}_d(x) \gamma_5 \gamma_\mu C \bar{c}_e^T(x) + s_b^T(x) C \gamma_\mu \gamma_5 c_c(x) \bar{s}_d(x) \gamma_5 C \bar{c}_e^T(x)]. \quad (170)$$

Remaining operations are standard, and the invariant amplitude proportional to a structure $g_{\mu\nu}$ in the physical side of the sum rule is equal to

$$\Pi^{\text{Phys}}(p^2) = \frac{m_Y^2 f_Y^2}{m_Y^2 - p^2} \quad (171)$$

The QCD side of the sum rule $\Pi_{\mu\nu}^{\text{OPE}}(p)$ should be expressed in terms of the quark propagators, and has the form

$$\begin{aligned} \Pi_{\mu\nu}^{\text{OPE}}(p) = & i \int d^4x e^{ipx} \epsilon \tilde{\epsilon} \epsilon' \tilde{\epsilon}' \left\{ \text{Tr} \left[\gamma_5 \tilde{S}_s^{bb'}(x) \gamma_5 S_c^{cc'}(x) \right] \text{Tr} \left[\gamma_5 \gamma_\mu \tilde{S}_c^{e'e}(-x) \gamma_\nu \gamma_5 S_s^{d'd}(-x) \right] \right. \\ & + \text{Tr} \left[\gamma_5 \gamma_\mu \tilde{S}_c^{e'e}(-x) \gamma_5 S_s^{d'd}(-x) \right] \text{Tr} \left[\gamma_5 \gamma_\nu \tilde{S}_s^{bb'}(x) \gamma_5 S_c^{cc'}(x) \right] + \text{Tr} \left[\gamma_5 \tilde{S}_c^{e'e}(-x) \gamma_\nu \gamma_5 S_s^{d'd}(-x) \right] \\ & \left. \times \text{Tr} \left[\gamma_5 \tilde{S}_s^{bb'}(x) \gamma_\mu \gamma_5 S_c^{cc'}(x) \right] + \text{Tr} \left[\gamma_5 \gamma_\nu \tilde{S}_s^{bb'}(x) \gamma_\mu \gamma_5 S_c^{cc'}(x) \right] \text{Tr} \left[\gamma_5 \tilde{S}_c^{e'e}(-x) \gamma_5 S_s^{d'd}(-x) \right] \right\}. \quad (172) \end{aligned}$$

The analysis performed by taking into account all usual restrictions of sum rule computations permits us to find

$$M^2 \in [4.9, 6.8] \text{ GeV}^2, \quad s_0 \in [23.2, 25.2] \text{ GeV}^2, \quad (173)$$

as working windows for M^2 and s_0 . Really, at $M^2 = 4.9 \text{ GeV}^2$ the convergence of the OPE is satisfied with high accuracy and $R(4.8 \text{ GeV}^2) = 0.017$ [R is evaluated using $\text{DimN} \equiv \text{Dim}(8 + 9 + 10)$ in Eq. (64)]. At $M^2 = 6.8 \text{ GeV}^2$ the pole contribution is $\text{PC} = 0.16$, and at $M^2 = 4.9 \text{ GeV}^2$ reaches its maximum $\text{PC} = 0.78$.

Moreover, at minimum of M^2 the perturbative contribution constitutes more than 74% of the result and significantly exceeds nonperturbative effects.

In Figures 8 and 9 we depict m_Y and f_Y as functions of the parameters M^2 and s_0 . It is evident that variations of the mass and coupling on the Borel parameter are very weak: predictions for m_Y and f_Y are stable against changes of M^2 within limits of the working region. But, m_Y and f_Y demonstrate a sensitivity to the continuum threshold parameter s_0 : this dependence forms an essential part of ambiguities in obtained predictions, which, however are within limits traditional for sum rule calculations. Then for the mass and coupling of the resonance $Y(4660)$, we get

$$m_Y = 4677_{-63}^{+71} \text{ MeV}, f_Y = (0.99 \pm 0.16) \times 10^{-2} \text{ GeV}^4. \quad (174)$$

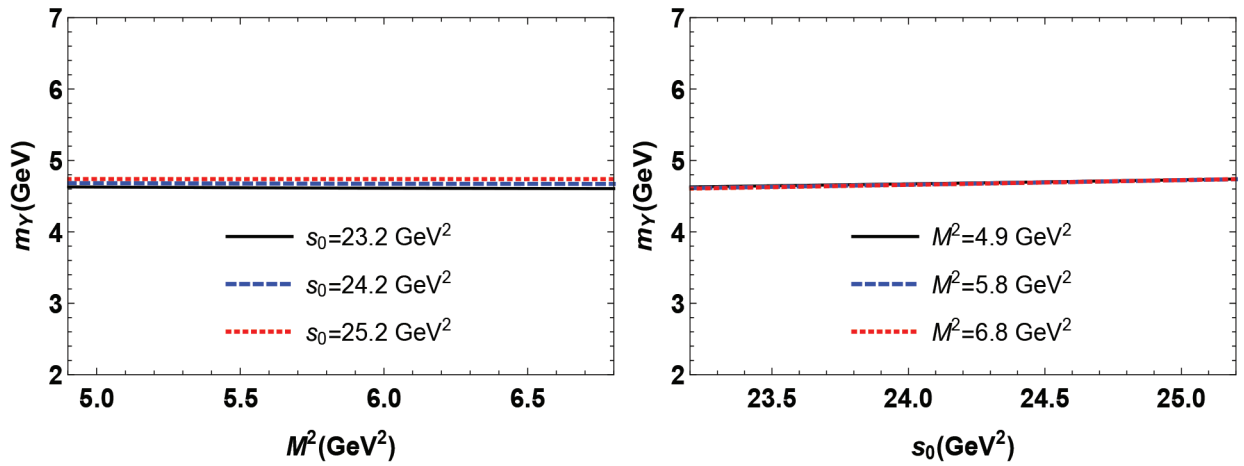


Figure 8. The dependence of the $Y(4660)$ resonance's mass on the Borel (left) and continuum threshold (right) parameters.

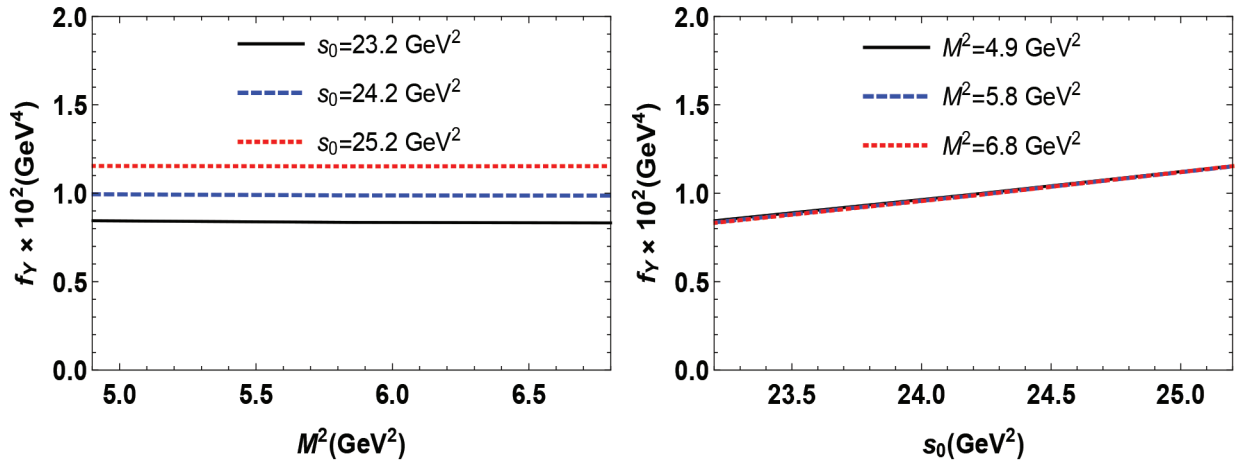


Figure 9. The same as in figure 8 but for the coupling f_Y .

The result for m_Y is compatible with experimental data [44]. It is interesting to confront m_Y with results of other theoretical studies. We have noted above, that the mass of the resonance Y was estimated

using QCD sum rule method in various publications. Thus, in Ref. [181], in which the authors examined Y as P -wave excitation of the scalar tetraquark $[cs][\bar{c}\bar{s}]$, its mass was found equal to $m_Y = (4.69 \pm 0.36)$ GeV. As the vector tetraquark $[cs][\bar{c}\bar{s}]$ the resonance Y was considered also in Ref. [182], with the prediction

$$m_Y = (4.65 \pm 0.10) \text{ GeV}. \quad (175)$$

These results agree with experimental data, and, by taking into account errors of calculations, are in accord also with our prediction.

Vector tetraquarks with $[cq][\bar{c}\bar{q}]$ or $[cs][\bar{c}\bar{s}]$ contents and charge conjugation parities $C = \pm$ were studied in Ref. [129] as well. In this article, sum rules for the mass were calculated by including into analysis vacuum condensates up to dimension 8. For the tetraquark $[cs][\bar{c}\bar{s}]$ built of the scalar diquark and vector antidiquark, the authors employed two currents denoted there by $J_{1\mu}$ and $J_{3\mu}$, respectively. The prediction obtained using the first current exceeds the mass of the resonance Y

$$m_{J_1} = (4.92 \pm 0.10) \text{ GeV}, \quad (176)$$

whereas the second one underestimates it, and leads to

$$m_{J_3} = (4.52 \pm 0.10) \text{ GeV}. \quad (177)$$

These results do not coincide with data, and agree neither with our result nor with prediction of Ref. [182] made by employing the current Eq. (170).

The Y was assigned in Ref. [185] to be the $C \otimes \gamma_\mu C$ -type vector tetraquark with the mass $m_Y = (4.66 \pm 0.09)$ GeV and pole residue $\lambda_Y = (6.74 \pm 0.88) \times 10^{-2}$ GeV⁵, which for the coupling f_Y leads to $f_Y = (1.45 \pm 0.19) \times 10^{-2}$ GeV⁴. The difference between this result and our prediction (174) for f_Y can be explained by assumptions on the internal structure of the vector resonance Y . In fact, we treat Y a state built of a scalar diquark and vector antidiquark, whereas in Ref. [185] it was considered as a bound state of a pseudoscalar diquark and axial-vector antidiquark.

It is evident that one can interpret the resonance Y as vector tetraquarks with the same content $[cs][\bar{c}\bar{s}]$, but distinct internal organizations and interpolating currents. Therefore, there is a necessity to analyze decay channels of the state Y to make a choice between existing models. In the next subsection we investigate strong decay modes of Y , where m_Y and f_Y appear as input information.

6.2. Strong decays of the tetraquark Y

The strong decays of the tetraquark Y can be determined by employing a kinematical constraint which is evident from Eq. (174). We consider S -wave decays of Y , therefore the spin and parity in these processes are conserved. Performed analysis allows us to see that processes $Y \rightarrow J/\psi f_0(980)$, $\psi' f_0(980)$, $J/\psi f_0(500)$, and $\psi' f_0(500)$ are among important decay modes of Y .

These decays in the final state contain scalar mesons $f_0(980)$ and $f_0(500)$, which will be considered as diquark-antidiquark states. A model proposed in Ref. [2] treats $f_0(980)$ and $f_0(500)$ as superpositions of the basic states $\mathbf{L} = [ud][\bar{u}\bar{d}]$ and $\mathbf{H} = ([su][\bar{s}\bar{u}] + [ds][\bar{d}\bar{s}])/\sqrt{2}$. Calculations carried out by employing this model predicted the mass and full width of mesons $f_0(980)$ and $f_0(500)$ [3, 4], which are in reasonable agreement with existing experimental data.

We consider here in a detailed form decays of the tetraquark Y to mesons $J/\psi f_0(980)$ and $\psi' f_0(980)$, and compute the strong couplings $g_{Y\psi f_0(980)}$ and $g_{Y\psi' f_0(980)}$ corresponding to the vertices $YJ/\psi f_0(980)$ and $Y\psi' f_0(980)$, respectively. To this end, we use the LCSR method and analyze the correlation function

$$\Pi_{\mu\nu}(p, q) = i \int d^4x e^{ipx} \langle f_0(q) | \mathcal{T} \{ J_\mu^\psi(x) J_\nu^\dagger(0) \} | 0 \rangle, \quad (178)$$

where $J_\mu^\psi(x)$ is the interpolating currents to J/ψ , and ψ' .

To extract from Eq. (178) the sum rules for $g_{Y\psi f_0(980)}$ and $g_{Y\psi' f_0(980)}$, we first find $\Pi_{\mu\nu}(p, q)$ in terms of the physical parameters of involved particles. After standard manipulations discussed in this review, we get

$$\begin{aligned} \Pi_{\mu\nu}^{\text{Phys}}(p, q) &= \frac{g_{Y\psi f_0(980)} f_\psi m_\psi f_Y m_Y}{(p^2 - m_Y^2)(p^2 - m_\psi^2)} \left(-p'_\mu p_\nu + \frac{m_Y^2 + m_\psi^2}{2} g_{\mu\nu} \right) \\ &+ \frac{g_{Y\psi' f_0(980)} f_{\psi'} m_{\psi'} f_Y m_Y}{(p^2 - m_Y^2)(p^2 - m_{\psi'}^2)} \left(-p'_\mu p_\nu + \frac{m_Y^2 + m_{\psi'}^2}{2} g_{\mu\nu} \right) + \dots, \end{aligned} \quad (179)$$

where m_ψ , and $m_{\psi'}$ are the mass of the mesons J/ψ and ψ' , respectively. The decay constants of these mesons are denoted by f_ψ and $f_{\psi'}$. In order to derive sum rules for couplings $g_{Y\psi f_0(980)}$ and $g_{Y\psi' f_0(980)}$, we use structures proportional $g_{\mu\nu}$ and corresponding invariant amplitudes.

At the next stage of calculations, we express the correlation function using the quark propagators, and obtain

$$\begin{aligned} \Pi_{\mu\nu}^{\text{OPE}}(p, q) &= \int d^4x e^{ipx} \bar{\epsilon} \epsilon \left[\gamma_5 \tilde{S}_c^{ic}(x) \gamma_\mu \tilde{S}_c^{ei}(-x) \gamma_\nu \gamma_5 - \gamma_\nu \gamma_5 \tilde{S}_c^{ic}(x) \gamma_\mu \tilde{S}_c^{ei}(-x) \gamma_5 \right]_{\alpha\beta} \\ &\times \langle f_0(q) | \bar{s}_\alpha^b(0) s_\beta^d(0) | 0 \rangle, \end{aligned} \quad (180)$$

Our computations for the Borel transformed correlation function $\Pi^{\text{OPE}}(M^2)$ give

$$\Pi^{\text{OPE}}(M^2) = \frac{\lambda_{f'}}{24\pi^2} \int_{4m_c^2}^{\infty} \frac{ds}{s} \sqrt{s(s-4m_c^2)}(s+8m_c^2) + \lambda_{f'} \int_0^1 d\alpha e^{-m_c^2/M^2 Z} F(\alpha, M^2). \quad (181)$$

Here, $\lambda_{f'}$ is the matrix element

$$\langle f_0(980)(q) | \bar{s}(0) s(0) | 0 \rangle = \lambda_{f'}, \quad (182)$$

which has been calculated by employing the QCD two-point sum rule method in Ref. [186]. In Eq. (181) $F(\alpha, M^2)$ is a function that contains all nonperturbative contributions to $\Pi^{\text{OPE}}(M^2)$ up to dimension 8

$$\begin{aligned} F(\alpha, M^2) &= -\frac{\langle \alpha_s G^2 / \pi \rangle m_c^2}{72M^4} \frac{1}{Z} [m_c^2(1-2Z) - M^2 Z(3-7Z)] \\ &+ \frac{\langle g_s^3 G^3 \rangle}{45 \cdot 2^9 \pi^2 M^8 Z^5} \{ m_c^6(1-2\alpha)^2(9-11Z) + 2m_c^2 M^4 Z^2 [-42 + Z(122+9Z)] \\ &- 2M^6 Z^3 [6 - Z(22-9Z)] + m_c^4 M^2 Z(-11 + 119Z - 190Z^2) \} \\ &+ \frac{\langle \alpha_s G^2 / \pi \rangle^2 m_c^2 \pi^2}{648M^{10} Z^3} [m_c^4 - m_c^2 M^2(1+4Z) + 2M^4 Z(2-Z)], \end{aligned} \quad (183)$$

where $Z = \alpha(1 - \alpha)$.

After equating the Borel transform of the invariant amplitudes $\Pi^{\text{Phys}}(p^2)$ and $\Pi^{\text{OPE}}(M^2)$, and carrying out continuum subtraction, we obtain an expression that contains two unknown variables $g_{YJf_0(980)}$ and $g_{Y\Psi f_0(980)}$. It is worth noting that continuum subtraction in the perturbative part has to be done by $\infty \rightarrow s_0$ replacement. As is seen, all terms in Eq. (183) are proportional to inverse powers of the parameter M^2 , therefore the nonperturbative contributions should be left in an unsubtracted form [51]. The second equality can be found by applying the operator $d/d(-1/M^2)$ to the first expression. By solving these equalities it is possible to extract sum rules for $g_{YJf_0(980)}$ and $g_{Y\Psi f_0(980)}$.

The similar analysis is valid also for decays of Y to $J/\psi f_0(500)$, and $\psi' f_0(500)$. The width of the decays under analysis can be evaluated by means of the formula (36), where instead of parameters $g_{Z\psi\pi}$, m_ψ , and $\lambda(m_Z, m_\psi, m_\pi)$ one has to use $g_{Y\psi f_0}$, m_ψ , and $\lambda(m_Y, m_\psi, m_{f_0})$: here, ψ and f_0 are one of the mesons J/ψ , ψ' and $f_0(500)$, $f_0(980)$, respectively.

The numerical computations are fulfilled by employing the vacuum condensates given in Eq. (18), and using the mass and decay constant of the mesons J/ψ and ψ' (see, Table 2). The parameters of the resonance Y have been found in the previous subsection, and for the mass of the $f_0(980)$ meson we use its experimental value $m_{f_0} = (990 \pm 20)$ MeV.

The parameters M^2 and s_0 are changed in the regions $M^2 \in [4.9, 6.8]$ GeV² and $s_0 \in [23.2, 25.2]$ GeV². The results obtained for the strong couplings read

$$|g_{YJf_0(980)}| = (0.22 \pm 0.07) \text{ GeV}^{-1}, \quad g_{Y\Psi f_0(980)} = (1.22 \pm 0.33) \text{ GeV}^{-1}. \quad (184)$$

Then partial widths of the decay modes under analysis become equal to (in units of MeV):

$$\Gamma(Y \rightarrow J/\psi f_0(980)) = 18.8 \pm 5.4, \quad \Gamma(Y \rightarrow \psi' f_0(980)) = 30.2 \pm 8.5. \quad (185)$$

Exploration of the next decays does not differ from previous analysis and gives the following predictions

$$g_{YJf_0(500)} = (0.07 \pm 0.02) \text{ GeV}^{-1}, \quad |g_{Y\Psi f_0(500)}| = (0.18 \pm 0.05) \text{ GeV}^{-1}, \quad (186)$$

and (in MeV)

$$\Gamma(Y \rightarrow J/\psi f_0(500)) = 2.7 \pm 0.7, \quad \Gamma(Y \rightarrow \psi' f_0(500)) = 13.1 \pm 3.7. \quad (187)$$

The full width of the resonance Y evaluated by taking into account these four strong decay modes

$$\Gamma_Y = (64.8 \pm 10.8) \text{ MeV} \quad (188)$$

agrees with the experimental result $(68 \pm 11 \pm 1)$ MeV from Eq. (169). The Particle Data Group for the full width of Y provides the world averaged estimate $\Gamma_Y = (72 \pm 11)$ MeV [96], which exceeds (169). But, the result Eq. (188) within errors of computations and experimental measurements is consistent also with the world average. We also take into account that in the diquark-antidiquark model there are other S -wave decay channels $Y \rightarrow D_s^\pm D_{s1}^\mp(2460)$ and $Y \rightarrow D_s^{*\pm} D_{s0}^\mp(2317)$ of the resonance Y which contribute to Γ_Y and may improve this agreement.

We have calculated the full width of the resonance Y by taking into account the strong decays $Y \rightarrow J/\psi f_0(500)$, $\psi' f_0(500)$, $J/\psi f_0(980)$ and $\psi' f_0(980)$. However, only the process $Y \rightarrow \psi' \pi^+ \pi^-$ was observed

experimentally. It is known that the dominant decays of the $f_0(500)$ and $f_0(980)$ mesons are processes $f_0 \rightarrow \pi^+\pi^-$ and $f_0 \rightarrow \pi^0\pi^0$. Therefore, the chains $Y \rightarrow \psi' f_0(980) \rightarrow \psi' \pi^+\pi^-$ and $Y \rightarrow \psi' f_0(500) \rightarrow \psi' \pi^+\pi^-$ explain a dominance of the observed $\psi' \pi^+\pi^-$ final state in the decay of the resonance Y . In the diquark-antidiquark model the width of the mode $Y \rightarrow J/\psi f_0(980)$ is considerable. Besides, decays to mesons $\psi' \pi^0\pi^0$ and $J/\psi \pi^0\pi^0$ have to be detected as well. But neither $J/\psi \pi^+\pi^-$ nor $\pi^0\pi^0$ were seen in decays of Y . It is interesting that aforementioned final-state particles were observed in decays of the vector resonance $Y(4260)$: its decays to $J/\psi \pi^+\pi^-$ and $J/\psi \pi^0\pi^0$ as well as to $J/\psi K^+K^-$ were discovered in experiments. Therefore, more accurate measurements may fix these modes in decays of the resonance Y as well.

7. The light resonances $X(2100)$ and $X(2239)$

In previous sections we have explored heavy resonances which are candidates to exotic four-quark mesons. They are heavy particles and contain a pair of $c\bar{c}$ quarks. Only small number of resonances observed experimentally may be interpreted as multi-quark mesons composed exclusively of light quarks. The famous resonance $Y(2175)$ seen by BaBar in the process $e^+e^- \rightarrow \gamma_{\text{ISR}} \phi f_0(980)$ [187] is one of such states. It was fixed as a resonant structure in the $\phi f_0(980)$ invariant mass spectrum. The BESII, Belle, and BESIII collaborations observed this state as well [188–190]. The mass and width of the resonance $Y(2175)$ with spin-parities $J^{PC} = 1^{--}$ are $m = (2175 \pm 10 \pm 15)$ MeV and $\Gamma = (58 \pm 16 \pm 20)$ MeV, respectively.

Some other light resonances that can be considered as four-quark states were discovered recently by BESIII. One of them, i.e., $X(2239)$ was fixed in the cross section's lineshape of the process $e^+e^- \rightarrow K^+K^-$ [191]. The mass and width of this resonant structure are equal to $m = (2239.2 \pm 7.1 \pm 11.3)$ MeV and $\Gamma = (139.8 \pm 12.3 \pm 20.6)$ MeV, respectively. The $X(2100)$ was discovered in the process $J/\psi \rightarrow \phi \eta \eta'$ as a resonance in the $\phi \eta'$ mass spectrum [192]. The BESIII explored angular distribution of $J/\psi \rightarrow X(2100)\eta$, but because of limited statistics could not distinguish 1^+ and 1^- options for the spin-parity J^P of the $X(2100)$. Therefore, the mass and width of this state were extracted by employing both of these options. For $J^P = 1^-$ the mass and width of the $X(2100)$ were extracted to be $m = (2002.1 \pm 27.5 \pm 21.4)$ MeV and $\Gamma = (129 \pm 17 \pm 9)$ MeV. In the case $J^P = 1^+$ BESIII found $m = (2062.8 \pm 13.1 \pm 7.2)$ MeV and $\Gamma = (177 \pm 36 \pm 35)$ MeV.

Almost all models and methods of the high energy physics were used to understand structures of these light resonances. Because $Y(2175)$ was observed for the first time more than ten years ago, there are various articles in literature, in which it was investigated thoroughly. The $Y(2175)$ was interpreted as 2^3D_1 excited state of the ordinary meson $\bar{s}s$ [193, 194]. It was considered as a dynamically generated $\phi K\bar{K}$ system [195], or a system appeared due to self-interaction between ϕ and $f_0(980)$ mesons [196]. Other models suggest to explain the resonance $Y(2175)$ as a hybrid meson $\bar{s}sg$, or a baryon-antibaryon $qq\bar{s}\bar{q}\bar{s}$ state that couples strongly to the $\Lambda\bar{\Lambda}$ channel (see Ref. [191] for other models).

As a vector tetraquark with $s\bar{s}s\bar{s}$ or $ss\bar{s}\bar{s}$ contents $Y(2175)$ was examined in Refs. [197] and [198, 199], respectively. An alternative suggestion on nature of this state was made in Ref. [200], where it was interpreted as a vector diquark-antidiquark system with the content $sq\bar{s}\bar{q}$. The resonances $X(2100)$ and $X(2239)$ (hereafter X_1 and X_2 , respectively) were explored as vector or axial-vector tetraquarks as well. Indeed, in Ref. [201] the $ss\bar{s}\bar{s}$ four-quark compounds were studied within the relativized quark model. The authors made a conclusion that the resonance X_2 can be considered as a P -wave $ss\bar{s}\bar{s}$ tetraquark. The X_1 was investigated within framework the QCD sum rule method in Refs. [202, 203]. Results of these analyses can be explained by

interpreting X_1 as the axial-vector $ss\bar{s}\bar{s}$ tetraquark with $J^{PC} = 1^{+-}$. An alternative explanation of X_1 as the second radial excitation of the meson $h_1(1380)$ was suggested in Ref. [204].

In our article [205], we explored the light axial-vector T_{AV} and vector T_V tetraquarks $ss\bar{s}\bar{s}$ and calculated their spectroscopic parameters. It appears, the resonance X_1 may be considered as a axial-vector tetraquark: we identified X_1 with T_{AV} . Among the vector particles $Y(2175)$ and X_2 , which we treated as different resonances, parameters of the latter are closer to our predictions. Therefore, we interpreted the resonance X_2 as the tetraquark T_V . We evaluated also width of the decays $X_1 \rightarrow \phi\eta'$ and $X_1 \rightarrow \phi\eta$ which are essential for our interpretation of X_1 . Presentation in this section is based on Ref. [205].

7.1. Mass and coupling of the axial-vector tetraquark $ss\bar{s}\bar{s}$

In this subsection, we calculate the spectroscopic parameters of the axial-vector tetraquark $T_{AV} = ss\bar{s}\bar{s}$ using the QCD sum rule method. We consider the two-point correlation function $\Pi_{\mu\nu}(p)$ given by Eq. (4), with $J_\mu(x)$ being the interpolating current for the axial-vector tetraquark T_{AV} . The choice of $J_\mu(x)$ is a main decision in our analysis, because T_{AV} with spin-parities $J^{PC} = 1^{+-}$ can be modeled by employing various currents. The current that leads to credible results for parameters of T_{AV} is given by the expression [202]

$$J_\mu(x) = [s_a^T(x)C\gamma^\nu s_b(x)] [\bar{s}_a(x)\sigma_{\mu\nu}\gamma_5 C\bar{s}_b^T(x)] - [s_a^T(x)C\sigma_{\mu\nu}\gamma_5 s_b(x)] [\bar{s}_a(x)\gamma^\nu C\bar{s}_b^T(x)]. \quad (189)$$

The sum rules for the mass m and coupling f of the tetraquark T_{AV} can be obtained in accordance with prescriptions of the method. First, we should rewrite the correlation function $\Pi_{\mu\nu}(p)$ by utilizing the physical parameters of T_{AV} . After some operations for the physical side of the sum rules, we obtain

$$\Pi_{\mu\nu}^{\text{Phys}}(p) = \frac{m^2 f^2}{m^2 - p^2} \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \right) + \dots \quad (190)$$

The correlation function $\Pi_{\mu\nu}(p)$ calculated using the quark propagators forms the QCD side of the sum rules. It is determined by the formula

$$\begin{aligned} \Pi_{\mu\nu}^{\text{OPE}}(p) = & \frac{i}{4} \int d^4x e^{ipx} \left\{ \text{Tr} \left[\gamma^\alpha \tilde{S}^{a'b}(-x) \gamma^\beta S^{b'a}(-x) \right] \text{Tr} \left[S^{ab'}(x) \gamma_\nu \gamma_\beta \gamma_5 \tilde{S}^{ba'}(x) \gamma_5 \gamma_\mu \gamma_\alpha \right] \right. \\ & \left. - \text{Tr} \left[\gamma^\alpha \tilde{S}^{bb'}(-x) \gamma^\beta S^{a'a}(-x) \right] \text{Tr} \left[S^{ab'}(x) \gamma_\nu \gamma_\beta \gamma_5 \tilde{S}^{ba'}(x) \gamma_5 \gamma_\mu \gamma_\alpha \right] + 62 \text{ similar terms} \right\}. \quad (191) \end{aligned}$$

In these computations, we employ the x -space light-quark propagator

$$S^{ab}(x) \Rightarrow S^{ab}(x) + \frac{m_s g_s}{32\pi^2} G_{ab}^{\mu\nu} \sigma_{\mu\nu} \left[\ln \left(\frac{-x^2 \Lambda^2}{4} \right) + 2\gamma_E \right] + \dots, \quad (192)$$

where $\gamma_E \simeq 0.577$ is the Euler constant, and Λ is the QCD scale parameter. The scale parameter Λ can be fixed inside of the region $[0.5, 1]$ GeV; we use the central value $\Lambda = 0.75$ GeV. We introduce also the notation $G_{ab}^{\mu\nu} \equiv G_A^{\mu\nu} t_{ab}^A$, $A = 1, 2, \dots, 8$, and $t^A = \lambda^A/2$, with λ^A being the Gell-Mann matrices.

At the next phase, we compute four- x Fourier integrals appeared in $\Pi_{\mu\nu}^{\text{OPE}}(p)$ due to propagators. The correlation function $\Pi_{\mu\nu}^{\text{OPE}}(p)$ found by this manner contains two Lorentz structures. To extract the sum rules, we work with terms proportional to $g_{\mu\nu}$, because they do not receive contributions from scalar particles. By

equating the relevant invariant amplitudes $\Pi^{\text{Phys}}(p^2)$ and $\Pi^{\text{OPE}}(p^2)$, we get an expression in momentum space, which after applying the Borel transformation and subtracting continuum effects becomes the first sum rule equality. An expression obtained after these operations depends on the Borel M^2 and continuum subtraction s_0 parameters.

A second equality which is necessary to get sum rules for m and f , can be derived by applying the operator $d/d(-1/M^2)$ to the first expression. These two equalities are enough to fix the sum rules for m and f . Obtained expressions for the mass and coupling of T_{AV} contains perturbative and nonperturbative parts. In numerical computations, we take into account nonperturbative terms up to dimension-20, and bear in mind that higher dimensional contributions appear due to the factorization hypothesis as product of basic condensates, and do not embrace all dimension-20 terms.

Traditionally an important question is a choice of regions for the Borel M^2 and continuum threshold s_0 parameters. These parameters should meet some known requirements. Our investigations prove that the regions

$$M^2 \in [1.4, 2] \text{ GeV}^2, \quad s_0 \in [6, 7] \text{ GeV}^2, \quad (193)$$

obey required constraints. Indeed, at $M^2 = 2 \text{ GeV}^2$ the pole contribution is equal to 0.39, and reaches $\text{PC} = 0.68$ at the minimum of $M^2 = 1.4 \text{ GeV}^2$. In Figure 10 we plot the pole contribution as functions of M^2 and s_0 , where these effects are seen explicitly. Convergence of the sum rules is also satisfied $R(1.4 \text{ GeV}^2) < 0.01$. The predictions for the mass and coupling of the tetraquark T_{AV} are

$$m = (2067 \pm 84) \text{ MeV}, \quad f = (0.89 \pm 0.11) \times 10^{-2} \text{ GeV}^4. \quad (194)$$

Obtained results should not depend on the parameter M^2 . But m and f evaluated from relevant sum rules are sensitive to a choice of M^2 . Theoretical errors in computations appear namely due to choices of M^2 and s_0 . Errors generated by ambiguities of m_s and vacuum condensates are not considerable. Varying of m_s , for example, inside boundaries $88 \text{ MeV} \leq m_s \leq 104 \text{ MeV}$ generates corrections $({}_{-1}^{+2}) \text{ MeV}$ to m and $({}_{-0.0001}^{+0.0002}) \text{ GeV}^4$ to f . Of course, these errors and others connected with condensates are included into Eq. (194). The mass m is depicted in Figure 11, where one sees its weak dependence on M^2 and s_0 .

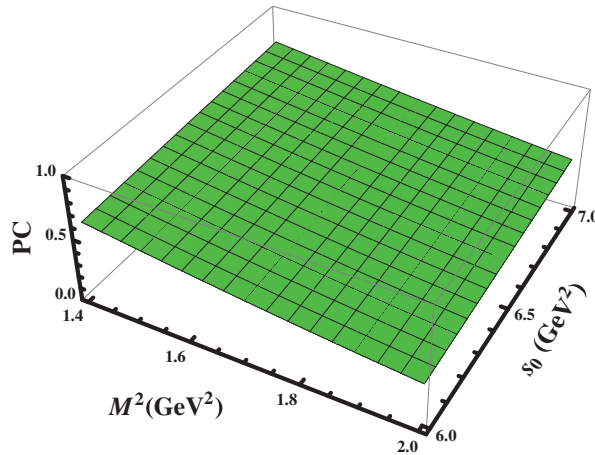


Figure 10. The pole contribution vs M^2 and s_0 .

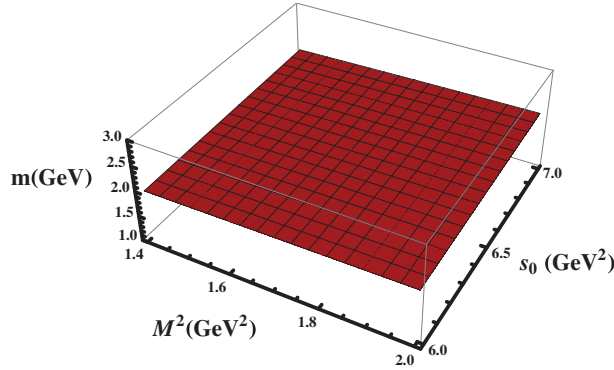


Figure 11. The dependence of the mass m on M^2 and s_0 .

The prediction for the mass of T_{AV} agrees with the mass of the resonance X_1 measured by BESIII. Therefore, it is reasonable to identify T_{AV} with the resonance X_1 . This is in line with existing theoretical predictions for m extracted from the QCD sum rule computations. In fact, the mass of X_1 was found in Refs. [202, 203] equal to

$$m = 2000_{-90}^{+100} \text{ MeV}, \quad m = (2080 \pm 120) \text{ MeV}, \quad (195)$$

where computations were performed by taking into account condensates until dimensions 12 and 10, respectively. There are differences between results (194) and (195), but all of them support a suggestion about a diquark-antidiquark structure of the axial-vector resonance X_1 . But to unveil a whole picture, we should explore decay channels $X_1 \rightarrow \phi\eta'$ and $X_1 \rightarrow \phi\eta$ to find width of X_1 and compare it with experimental information: only after successful outcome, we will be able to make a firm decision about the structure of X_1 . In this section, we are going to analyze this problem as well.

7.2. Spectroscopic parameters of the vector tetraquark $ss\bar{s}\bar{s}$

We have investigated the axial-vector tetraquark T_{AV} and classified it as a candidate to the resonance X_1 . But there are light resonances $Y(2175)$ and X_2 which may be interpreted as four-quark states. Here, we study the vector tetraquark $T_V = ss\bar{s}\bar{s}$ with spin-parities $J^{PC} = 1^{--}$ and compute its mass. After confronting our result with the experimental data of the BaBar and BESIII collaborations, we can identify $Y(2175)$ or X_2 as the state T_V .

The mass \tilde{m} and coupling \tilde{f} of the tetraquark T_V can be found using standard tools of the sum rule method. This analysis does not differ significantly from operations performed above. A difference in the case under consideration stems from a choice of the interpolating current $\tilde{J}_\mu(x)$, which for the vector tetraquark is given by the formula [199]

$$\tilde{J}_\mu(x) = [s_a^T(x)C\gamma_5 s_b(x)] [\bar{s}_a(x)\gamma_\mu\gamma_5 C\bar{s}_b^T(x)] - [s_a^T(x)C\gamma_\mu\gamma_5 s_b(x)] [\bar{s}_a(x)\gamma_5 C\bar{s}_b^T(x)]. \quad (196)$$

We should determine both sides of the sum rule equalities. The physical side of the sum rule is fixed by Eq.

(190) with evident replacements. The QCD side of the sum rule $\tilde{\Pi}_{\mu\nu}^{\text{OPE}}(p)$ has the form

$$\begin{aligned} \tilde{\Pi}_{\mu\nu}^{\text{OPE}}(p) = i \int d^4x e^{ipx} \left\{ \text{Tr} \left[\gamma_5 \tilde{S}^{b'b}(-x) \gamma_\nu \gamma_5 S^{a'a}(-x) \right] \text{Tr} \left[S^{a'a}(x) \gamma_5 \tilde{S}^{bb'}(x) \gamma_5 \gamma_\mu \right] \right. \\ \left. - \text{Tr} \left[\gamma_5 \tilde{S}^{a'b}(-x) \gamma_\nu \gamma_5 S^{b'a}(-x) \right] \text{Tr} \left[S^{a'a}(x) \gamma_5 \tilde{S}^{bb'}(x) \gamma_5 \gamma_\mu \right] + 14 \text{ similar terms} \right\}. \end{aligned} \quad (197)$$

The regions for the Borel and continuum threshold parameters M^2 and s_0 are given by the intervals

$$M^2 \in [1.4, 2] \text{ GeV}^2, \quad s_0 \in [7, 8] \text{ GeV}^2. \quad (198)$$

These regions differ from working windows of the axial-vector state (193) by a small shift in s_0 . The regions (198) meet all restrictions on the PC and convergence of OPE imposed by the sum rule method. In fact, at $M^2 = 1.4 \text{ GeV}^2$ the PC amounts to 0.6, and at $M^2 = 2 \text{ GeV}^2$ is equal to 0.3. Convergence of OPE is also fulfilled. The spectroscopic parameters of the vector tetraquark T_V are

$$\tilde{m} = (2283 \pm 114) \text{ MeV}, \quad \tilde{f} = (0.57 \pm 0.10) \times 10^{-2} \text{ GeV}^4. \quad (199)$$

In figure 12 we depict \tilde{m} and \tilde{f} as functions of M^2 and s_0 .

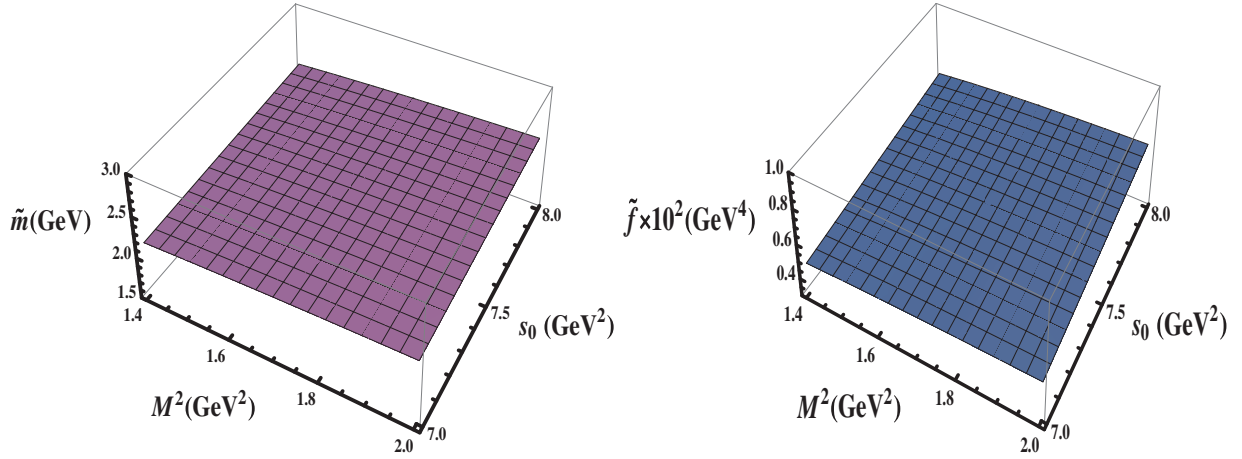


Figure 12. The \tilde{m} (left panel) and \tilde{f} (right panel) as functions of the Borel and continuum threshold parameters.

Confronting now \tilde{m} with collected data on the resonances $Y(2175)$ and X_2 , we see that T_V can be interpreted as the resonance X_2 . Indeed, a mass gap $T_V - X_2$ is approximately 60 MeV smaller than mass splitting of T_V and $Y(2175)$. The mass $m_{X_2} = 2227 \text{ MeV}$ of the vector tetraquark $ss\bar{s}\bar{s}$ found in Ref. [201] also agrees with BESIII data for X_2 . This fact forced the authors to draw the similar conclusion about internal organization of X_2 .

Parameters of the vector tetraquark $ss\bar{s}\bar{s}$ were also calculated in the context the sum rule method in Refs. [203] and [199]. The result for the mass of this exotic meson $\tilde{m} = (3080 \pm 110) \text{ MeV}$ obtained in Ref. [203] disfavors interpreting it as the resonance $Y(2175)$. Confronting this prediction with the BESIII data, we see that it is also difficult to classify this structure as the resonance X_2 . In Ref. [199] the authors employed two interpolating currents to study the vector tetraquark $ss\bar{s}\bar{s}$. For \tilde{m} these currents led to different values

$$\tilde{m}_1 = (2410 \pm 250) \text{ MeV}, \quad \tilde{m}_2 = (2340 \pm 170) \text{ MeV}. \quad (200)$$

The first tetraquark was interpreted as a structure at around 2.4 GeV in the invariant mass spectrum $\phi f_0(980)$ [199]. The second structure was identified with the resonance $Y(2175)$. We think that it is closer to X_2 , which was discovered later and the authors did not know about existence of this resonance.

7.3. Decays $X_1 \rightarrow \phi\eta'$ and $X_1 \rightarrow \phi\eta$

In this subsection we investigate decays $X_1 \rightarrow \phi\eta'$ and $X_1 \rightarrow \phi\eta$ of the resonance X_1 . We concentrate on the process $X_1 \rightarrow \phi\eta'$ and present our studies in a detailed form. The second mode $X_1 \rightarrow \phi\eta$ can be considered in the same way.

We begin from analysis of the correlation function

$$\widehat{\Pi}_{\mu\nu}(p, q) = i \int d^4x e^{ipx} \langle \eta'(q) | \mathcal{T} \{ J_\mu^\phi(x) J_\nu^\dagger(0) \} | 0 \rangle, \quad (201)$$

which is necessary to study the decay $X_1 \rightarrow \phi\eta'$. In Eq. (201) $J_\mu^\phi(x)$ is the interpolating current of the ϕ meson

$$J_\mu^\phi(x) = \bar{s}_i(x) \gamma_\mu s_i(x). \quad (202)$$

Following the standard recipes, we write down $\widehat{\Pi}_{\mu\nu}(p, q)$ in terms of the physical parameters of the particles X_1 , ϕ and η'

$$\widehat{\Pi}_{\mu\nu}^{\text{Phys}}(p, q) = \frac{\langle 0 | J_\mu^\phi(x) | \phi(p) \rangle \langle \phi(p) \eta'(q) | X_1(p') \rangle \langle X_1(p') | J_\nu^\dagger | 0 \rangle}{p^2 - m_\phi^2} + \dots, \quad (203)$$

where the momenta of the initial and final particles are denoted by p' and p, q , respectively. By utilizing the matrix elements

$$\langle 0 | J_\mu^\phi(x) | \phi(p) \rangle = f_\phi m_\phi \varepsilon_\mu, \quad \langle \phi(p) \eta'(q) | X_1(p') \rangle = g_{X_1 \phi \eta'} [(p \cdot p')(\varepsilon^* \cdot \varepsilon') - (p \cdot \varepsilon')(p' \cdot \varepsilon^*)], \quad (204)$$

one can simplify $\widehat{\Pi}_{\mu\nu}^{\text{Phys}}(p, q)$. The matrix element $\langle 0 | J_\mu^\phi(x) | \phi(p) \rangle$ is determined by the mass m_ϕ , decay constant f_ϕ and polarization vector ε_μ of ϕ meson. The vertex $X_1 \phi \eta'$ is modeled using the strong coupling $g_{X_1 \phi \eta'}$, that should be extracted from the sum rule. In the soft-meson approximation $q \rightarrow 0$ and $p' = p$, we have to perform one-variable Borel transformation which gives

$$\mathcal{B} \widehat{\Pi}_{\mu\nu}^{\text{Phys}}(p) = g_{X_1 \phi \eta'} m_\phi m f_\phi f \frac{e^{-\bar{m}^2/M^2}}{M^2} (\bar{m}^2 g_{\mu\nu} - p_\nu p'_\mu) + \dots, \quad (205)$$

where $\bar{m}^2 = (m_\phi^2 + m^2)/2$. To make explicit Lorentz structures of $\mathcal{B} \widehat{\Pi}_{\mu\nu}^{\text{Phys}}(p)$, we keep in Eq. (205) $p_\nu \neq p'_\mu$. The sum rule for $g_{X_1 \phi \eta'}$ will be derived by using a structure proportional to $g_{\mu\nu}$. In the soft limit we also act on both the physical and QCD sides of the sum rule by the operator $\mathcal{P}(M^2, \bar{m}^2)$, that cancels unsuppressed terms in $\mathcal{B} \widehat{\Pi}_{\mu\nu}^{\text{Phys}}(p)$.

In the soft limit $\widehat{\Pi}_{\mu\nu}^{\text{OPE}}(p)$ is given by the formula

$$\begin{aligned} \widehat{\Pi}_{\mu\nu}^{\text{OPE}}(p) = & 2i \int d^4x e^{ipx} \left\{ \left[\sigma_{\mu\rho} \gamma_5 \widetilde{S}^{ib}(x) \gamma_\nu \widetilde{S}^{bi}(-x) \gamma^\rho - \gamma^\rho \widetilde{S}^{ib}(x) \gamma_\nu \widetilde{S}^{bi}(-x) \gamma_5 \sigma_{\mu\rho} \right]_{\alpha\beta} \langle \eta'(q) | \bar{s}_\alpha^a(0) s_\beta^a(0) | 0 \rangle \right. \\ & \left. + \left[\gamma^\rho \widetilde{S}^{ia}(x) \gamma_\nu \widetilde{S}^{bi}(-x) \gamma_5 \sigma_{\mu\rho} - \gamma_5 \sigma_{\mu\rho} \widetilde{S}^{ia}(x) \gamma^\rho \widetilde{S}^{bi}(-x) \gamma_\nu \right]_{\alpha\beta} \langle \eta'(q) | \bar{s}_\alpha^b(0) s_\beta^a(0) | 0 \rangle \right\}. \quad (206) \end{aligned}$$

The local matrix element of the η' meson $\langle \eta'(q) | \bar{s}_\alpha^b s_\beta^a | 0 \rangle$ can be transformed in accordance with Eq. (26). After this transformation operators $\bar{s}\Gamma^j s$, and ones generated due to $G_{\mu\nu}$ insertions from quark propagators, form local matrix elements of the η' meson. Applying Eq. (26) to the correlation function, performing summation over color and calculating traces of over spinor indices, we determine local matrix elements of the η' meson that contribute to $\widehat{\Pi}_{\mu\nu}^{\text{OPE}}(p)$.

Our studies show that in the soft limit the twist-3 matrix element $\langle \eta' | \bar{s}i\gamma_5 s | 0 \rangle$ contributes to the correlation function $\widehat{\Pi}_{\mu\nu}^{\text{OPE}}(p)$. The matrix elements of the η and η' mesons have some peculiarities connected with mixing in the η - η' system. In fact, through the mixing both the η' and η mesons contains $\bar{s}s$ components. It is clear that in the η' meson dominant is a strange component, but it plays some role also in the η meson. Due to existence of strange components both the decays $X_1 \rightarrow \phi\eta'$ and $X_1 \rightarrow \phi\eta$ can be realized.

The η - η' mixing can be described using two basics. For our analysis a suitable is the quark-flavor basis, which was employed to investigate various exclusive processes with η' and η mesons [206–208]. In this basis the twist-3 matrix element $\langle \eta' | \bar{s}i\gamma_5 s | 0 \rangle$ is given by the formula

$$2m_s \langle \eta' | \bar{s}i\gamma_5 s | 0 \rangle = h_{\eta'}^s, \quad (207)$$

where the parameter $h_{\eta'}^s$ is defined by the expression

$$h_{\eta'}^s = m_{\eta'}^2 f_{\eta'}^s - A_{\eta'}, \quad A_{\eta'} = \langle 0 | \frac{\alpha_s}{4\pi} G_{\mu\nu}^a \tilde{G}^{a,\mu\nu} | \eta' \rangle. \quad (208)$$

In Eq. (208) $m_{\eta'}$ and $f_{\eta'}^s$ are the mass and s -component of the η' meson decay constant. Here, $A_{\eta'}$ is the matrix element which appear due to $U(1)$ axial-anomaly. The parameter $h_{\eta'}^s$ can be calculated using Eqs. (207) and (208). It is also possible to use the phenomenological value

$$h_{\eta'}^s = h_s \cos \varphi, \quad h_s = (0.087 \pm 0.006) \text{ GeV}^3, \quad (209)$$

where $\varphi = 39^\circ.3 \pm 1^\circ.0$ is the mixing angle in the quark-flavor basis.

The Borel transform of the invariant function $\widehat{\Pi}^{\text{OPE}}(p^2)$ which is related to a structure $\sim g_{\mu\nu}$ reads

$$\widehat{\Pi}^{\text{OPE}}(M^2) = \int_{16m_s^2}^{\infty} ds \rho^{\text{pert.}}(s) e^{-s/M^2} - h_{\eta'}^s \langle \bar{s}s \rangle - \left\langle \frac{\alpha_s G^2}{\pi} \right\rangle \frac{h_{\eta'}^s}{8m_s} - \frac{h_{\eta'}^s}{6M^2} \langle \bar{s}g_s \sigma G s \rangle + \frac{2g_s^2 h_{\eta'}^s}{81m_s M^2} \langle \bar{s}s \rangle^2, \quad (210)$$

where the perturbative contribution is given in terms of the spectral density

$$\rho^{\text{pert.}}(s) = -\frac{h_{\eta'}^s}{4m_s \pi^2} (s + 3m_s^2). \quad (211)$$

Other components of $\widehat{\Pi}^{\text{OPE}}(M^2)$ are nonperturbative contributions calculated by including terms up to dimension 6. To carry out the continuum subtraction, we need to apply the operator $\mathcal{P}(M^2, \bar{m}^2)$ to $\widehat{\Pi}^{\text{OPE}}(M^2)$. Afterwards, we should replace in the first term ∞ by s_0 , and leave in original forms contributions $\sim (M^2)^0$ and $\sim 1/M^2$. The width of the decay $X_1 \rightarrow \phi\eta'$ is determined by the formula (36), where substitutions $g_{Z\psi\pi}$, m_ψ , $\lambda(m_Z, m_\psi, m_\pi) \rightarrow g_{X_1\phi\eta'}$, m_ϕ , $\lambda(m, m_\phi, m_{\eta'})$ must be done.

The parameters M^2 and s_0 in numerical analysis are varied inside limits

$$M^2 \in [1.4, 2] \text{ GeV}^2, \quad s_0 \in [6.2, 7.2] \text{ GeV}^2. \quad (212)$$

The masses of the mesons ϕ and η' are borrowed from Ref. [96]

$$m_\phi = (1019.461 \pm 0.019) \text{ MeV}, \quad m_{\eta'} = (957.78 \pm 0.06) \text{ MeV}, \quad f_\phi = (215 \pm 5) \text{ MeV}. \quad (213)$$

We get the following results

$$g_{X_1\phi\eta'} = (2.82 \pm 0.54) \text{ GeV}^{-1}, \quad \Gamma(X_1 \rightarrow \phi\eta') = (105.3 \pm 28.6) \text{ MeV}. \quad (214)$$

The $X_1 \rightarrow \phi\eta'$ is the dominant decay mode of the tetraquark X_1 . The width of the decay $X_1 \rightarrow \phi\eta$ can be computed using formulas derived in the present subsection. The distinctions between two decays of X_1 are connected with the twist-3 matrix element

$$2m_s \langle \eta | \bar{s} i \gamma_5 s | 0 \rangle = -h_s \sin \varphi, \quad (215)$$

and the mass of the η meson $m_\eta = (547.862 \pm 0.018) \text{ MeV}$. Computations lead to results

$$|g_{X_1\phi\eta}| = (0.85 \pm 0.22) \text{ GeV}^{-1}, \quad \Gamma(X_1 \rightarrow \phi\eta) = (24.9 \pm 9.5) \text{ MeV}. \quad (216)$$

It is worth emphasizing that $|g_{X_1\phi\eta}|$ has been evaluated using the region $s_0 \in [5.8, 6.8] \text{ GeV}^2$.

The full width of the resonance X_1 saturated by two decays is equal to

$$\Gamma = (130.2 \pm 30.1) \text{ MeV}. \quad (217)$$

This result for Γ is comparable with experimentally measured width of the resonance X_1 .

In this section we have investigated the axial-vector and vector states $ss\bar{s}\bar{s}$. The mass $m = (2067 \pm 84) \text{ MeV}$ of T_{AV} evaluated here agrees with results of BESIII. The width of T_{AV} which has been found equal to $\Gamma = (130.2 \pm 30.1) \text{ MeV}$ is consistent with these data as well. This information has allowed us to interpret the resonance X_1 as an axial-vector exotic meson $ss\bar{s}\bar{s}$.

Another conclusion that can be made is that the vector tetraquark $ss\bar{s}\bar{s}$ may be considered as the structure X_2 rather than the resonance $Y(2175)$. Let us note that we have treated the resonances X_2 and $Y(2175)$ as different particles, though their masses are close to each other. This picture is typical for a family of heavy vector resonances Y as well [186]. Some of these states may be treated as the same resonances, but even in this situation the mass range 4 – 5 GeV is overpopulated by $J^{PC} = 1^{--}$ mesons. A similar picture seems persists also in a light sector of $J^{PC} = 1^{--}$ particles. Hence, more detailed experimental analyses are necessary to differentiate these resonances, and determine reliably their parameters.

8. The resonance $Y(2175)$

As we have noted above, the vector resonance $Y(2175)$ (in this section, \tilde{Y}) is one of a few light particles which can be considered as a serious candidate to an exotic meson. Because it was observed in $\phi f_0(980)$ invariant mass distribution, usually was treated as a state containing exclusively strange quarks and antiquarks $ss\bar{s}\bar{s}$. The reason for such interpretation of \tilde{Y} is quite natural. Indeed, in the conventional model both ϕ and

$f_0(980)$ are mesons with $\bar{s}s$ structure, a difference being only in their quantum numbers: While ϕ is the vector particle, $f_0(980)$ is the scalar meson. But, the light meson $f_0(980)$, as a member of the first scalar nonet, can also be treated as a four-quark state. In this picture $f_0(980)$ is a superposition of diquark-antidiquark states $\mathbf{L} = [ud][\bar{u}\bar{d}]$ and $\mathbf{H} = ([su][\bar{s}\bar{u}] + [ds][\bar{d}\bar{s}])/\sqrt{2}$. Then it appears that \tilde{Y} can be interpreted as a tetraquark with $s\bar{q}\bar{s}q$ content. In this section, we provide results of our analysis, obtained in Ref. [200] by treating \tilde{Y} as a vector tetraquark $[su][\bar{s}\bar{u}]$.

8.1. Spectroscopic parameters of the tetraquark \tilde{Y} : the mass m_Y and current coupling f_Y

To evaluate the mass m_Y and coupling f_Y of the vector tetraquark \tilde{Y} , we use the QCD two-point sum rule method and start our calculations from analysis of the correlation function (4), where we use the current

$$J_\mu^Y(x) = [u_a^T(x)C\gamma_5 s_b(x)][\bar{u}_a(x)\gamma_\mu\gamma_5 C\bar{s}_b^T(x)] - [u_a^T(x)C\gamma_\mu\gamma_5 s_b(x)][\bar{u}_a(x)\gamma_5 C\bar{s}_b^T(x)]. \quad (218)$$

The current J_μ^Y consists of two pieces and each of them describes a vector $J^P = 1^-$ tetraquark. This is evident from quantum numbers of the diquark-antidiquark fields: the first term is built of the scalar diquark $u^T C\gamma_5 s$ and vector antidiquark $\bar{u}\gamma_\mu\gamma_5 C\bar{s}^T$, whereas in the second term the diquark and antidiquark are vector and scalar states, respectively. The J_μ^Y corresponds to the vector tetraquark with a definite charge-conjugation parity $J^{PC} = 1^{--}$. Indeed, because the charge-conjugation transforms diquarks to antidiquarks (and antidiquarks to diquarks) the minus sign between two currents in J_μ^Y implies $C = -1$.

The analysis of the phenomenological side of the sum rules $\Pi_{\mu\nu}^{\text{Phys}}(p)$ does not differ from similar expression (190), where now one should use the mass m_Y and coupling f_Y of the state \tilde{Y} . Because a part of $\Pi_{\mu\nu}^{\text{Phys}}(p)$ proportional to $g_{\mu\nu}$ is formed due to contributions of vector states, we work with this term and corresponding invariant amplitude $\Pi^{\text{Phys}}(p^2)$.

To get the sum rules' QCD side, we compute $\Pi_{\mu\nu}(p)$ using quark-gluon degrees of freedom, and find

$$\begin{aligned} \Pi_{\mu\nu}^{\text{OPE}}(p) = & i \int d^4x e^{ipx} \left\{ \text{Tr} \left[\gamma_5 \tilde{S}_s^{b'b}(-x) \gamma_5 \gamma_\nu S_u^{a'a}(-x) \right] \text{Tr} \left[S_u^{aa'}(x) \gamma_5 \tilde{S}_s^{bb'}(x) \gamma_5 \gamma_\mu \right] \right. \\ & + \text{Tr} \left[\gamma_\mu \gamma_5 \tilde{S}_s^{b'b}(-x) \gamma_5 S_u^{a'a}(-x) \right] \text{Tr} \left[S_u^{aa'}(x) \gamma_\nu \gamma_5 \tilde{S}_s^{bb'}(x) \gamma_5 \right] + \text{Tr} \left[S_u^{aa'}(x) \gamma_5 \tilde{S}_s^{bb'}(x) \gamma_5 \right] \\ & \left. \times \text{Tr} \left[\gamma_\mu \gamma_5 \tilde{S}_s^{b'b}(-x) \gamma_5 \gamma_\nu S_u^{a'a}(-x) \right] + \text{Tr} \left[\gamma_5 \tilde{S}_s^{b'b}(-x) \gamma_5 S_u^{a'a}(-x) \right] \text{Tr} \left[S_u^{aa'}(x) \gamma_\nu \gamma_5 \tilde{S}_s^{bb'}(x) \gamma_5 \gamma_\mu \right] \right\}. \quad (219) \end{aligned}$$

The required sum rules for the mass and coupling of the tetraquark \tilde{Y} can be obtained by extracting the invariant amplitude $\Pi^{\text{OPE}}(p^2)$ related to a structure $g_{\mu\nu}$ in Eq. (219), and equating it to $\Pi^{\text{Phys}}(p^2)$. Afterwards, one should apply to this equality the Borel transformation and perform continuum subtraction. These operations generate a dependence of sum rules on the Borel M^2 and continuum threshold s_0 parameters. Next steps to get sum rules for m_Y and f_Y were described many times in this review, therefore we omit further details. Let us only note that calculation of $\Pi^{\text{OPE}}(p^2)$ is carried out by including into analysis nonperturbative terms up to dimension 15.

The quantities m_Y and f_Y should be stable against variations of the Borel parameter M^2 . But in actual computations one can minimize these effects by fixing a plateau where dependence of physical quantities

on M^2 is minimal. The continuum threshold parameter s_0 separates contributions of ground-state particles from ones due to higher resonances and continuum states. In other words, s_0 should be below the first excited state of the particle under discussion \tilde{Y} . In the case of ordinary hadrons, masses of excited states are known either from experimental measurements or from alternative theoretical studies. For exotic particles a situation is more complicated: there is not information on their radial and/or orbital excitations. For tetraquarks this problem was addressed only in a few publications [72–74]. Therefore, one chooses s_0 by demanding maximum for the pole contribution, and a stability of extracting physical quantity. In this situation a self-consistency of the prediction for m_Y , and s_0 used for its computation is very important: $\sqrt{s_0}$ may exceed m_Y approximately [0.3, 0.5] MeV, then a first excited state of \tilde{Y} is above $\sqrt{s_0}$.

Computations show that the regions

$$M^2 \in [1.2, 1.7] \text{ GeV}^2, \quad s_0 \in [6, 6.5] \text{ GeV}^2 \quad (220)$$

satisfy all restrictions imposed on M^2 and s_0 by the sum rule analysis. Predictions for m_Y and f_Y extracted from this analysis read

$$m_Y = (2173 \pm 85) \text{ MeV}, \quad f_Y = (2.8 \pm 0.5) \times 10^{-3} \text{ GeV}^4. \quad (221)$$

Comparing m_Y with $\sqrt{s_0}$, we see that $\sqrt{s_0} - m_Y = [0.28, 0.38] \text{ MeV}$ is a reasonable mass gap to separate \tilde{Y} from its excitations.

Our result for m_Y is in good agreement with the BaBar datum $(2175 \pm 10 \pm 15) \text{ MeV}$, but is below new result of BESIII $(2200 \pm 6 \pm 5) \text{ MeV}$. Nevertheless, if one takes into account theoretical errors of computations, and errors of the experiment m_Y is consistent with BESIII data as well. In this situation, when there are different models for \tilde{Y} , a prediction for full width of this tetraquark and its confrontation with data can shed light on internal structure of \tilde{Y} .

8.2. The decay $\tilde{Y} \rightarrow \phi f_0(980)$

The process $\tilde{Y} \rightarrow \phi f_0(980)$ is an important decay channel of the tetraquark \tilde{Y} . The partial width of this mode can be expressed in term of the strong coupling $G_{Y\phi f}$ describing the vertex $\tilde{Y}\phi f_0(980)$. In its turn, the coupling $G_{Y\phi f}$ can be evaluated in the context of the LCSR method and expressed using various vacuum condensates and distribution amplitudes of the ϕ meson.

We extract the sum rule for $G_{Y\phi f}$ by computing the correlation function

$$\Pi_\mu(p, q) = i \int d^4x e^{ipx} \langle \phi(q) | \mathcal{T} \{ J^f(x) J_\mu^{Y\dagger}(0) \} | 0 \rangle. \quad (222)$$

We treat the scalar meson $f_0(980)$ [hereafter in expressions $f = f_0(980)$] as a pure \mathbf{H} state, interpolating current of which has been presented in Eq. (134). The phenomenological side of the sum rule is equal to the expression

$$\Pi_\mu^{\text{Phys}}(p, q) = G_{Y\phi f} \frac{m_Y f_Y m_f F_f}{2(p'^2 - m_Y^2)(p^2 - m_f^2)} \left[(m_f^2 - m_Y^2 - m_\phi^2) \varepsilon_\mu^* + \frac{m_Y^2 + m_f^2 - m_\phi^2}{m_Y^2} p \cdot \varepsilon^* q_\mu \right]. \quad (223)$$

The function $\Pi_\mu^{\text{Phys}}(p, q)$ is a sum of two terms with different Lorentz structures. We choose a structure $\sim \varepsilon_\mu^*$ to extract the sum rule necessary for our purposes.

The QCD side of the sum rule $\Pi_\mu^{\text{OPE}}(p, q)$ is derived by inserting interpolating currents into Eq. (222). After contracting quark fields, and rewriting an obtained expression using the quarks' light-cone propagators $\mathcal{S}_q(x)$, we see that the matrix element in Eq. (222) is a sum of terms

$$[A(x)]_{\alpha\beta}^{ab} \langle \phi(q) | \bar{s}_\alpha^a(x) s_\beta^b(0) | 0 \rangle, \quad [B(x)]_{\alpha\beta}^{ab} \langle \phi(q) | \bar{s}_\alpha^a(0) s_\beta^b(x) | 0 \rangle. \quad (224)$$

Here $A(x)$ and $B(x)$ are composed of the propagators $\mathcal{S}_q(\pm x)$, $\tilde{\mathcal{S}}_q(\pm x) = C\mathcal{S}_q^T(\pm x)C$, and $\gamma_{5(\sigma)}$ matrices. Explicit expression of $\mathcal{S}_q(x)$ is moved to Appendix.

Besides propagators, the function $\Pi_\mu^{\text{OPE}}(p, q)$ depends on nonlocal matrix elements of operator $\bar{s}s$ placed between the vacuum and ϕ meson. These matrix elements, after using Eq. (26), can be expressed via the ϕ meson's distribution amplitudes. In fact, after performed operations $A(x)$ and $B(x)$ depend on colorless operators $\bar{s}(x)\Gamma^j s(0)$ which can be expanded over x^2 and expressed in terms of the ϕ meson's DAs of different twist. For $\Gamma^j = \mathbf{1}$ and $i\gamma_\mu\gamma_5$ we employ the following formulas

$$\langle 0 | \bar{s}(x) s(0) | \phi(q) \rangle = -if_\phi^\perp \varepsilon \cdot x m_\phi^2 \int_0^1 du e^{i\bar{u}qx} \psi_3^\parallel(u), \quad (225)$$

and

$$\langle 0 | \bar{s}(x) \gamma_\mu \gamma_5 s(0) | \phi(q) \rangle = \frac{1}{2} f_\phi^\parallel m_\phi \varepsilon_{\mu\nu\alpha\beta} \varepsilon^\nu q^\alpha x^\beta \int_0^1 du e^{i\bar{u}qx} \psi_3^\perp(u). \quad (226)$$

For the structures $\Gamma^j = \gamma_\mu$ and $\sigma_{\mu\nu}$ we have

$$\begin{aligned} \langle 0 | \bar{s}(x) \gamma_\mu s(0) | \phi(q) \rangle &= f_\phi^\parallel m_\phi \left\{ \frac{\varepsilon \cdot x}{q \cdot x} q_\mu \int_0^1 du e^{i\bar{u}qx} \left[\phi_2^\parallel(u) + \frac{m_\phi^2 x^2}{4} \phi_4^\parallel(u) \right] \right. \\ &\left. + \left(\varepsilon_\mu - q_\mu \frac{\varepsilon \cdot x}{q \cdot x} \right) \int_0^1 du e^{i\bar{u}qx} \phi_3^\perp(u) - \frac{1}{2} x_\mu \frac{\varepsilon \cdot x}{(q \cdot x)^2} m_\phi^2 \int_0^1 du e^{i\bar{u}qx} C(u) + \dots \right\}, \end{aligned} \quad (227)$$

and

$$\begin{aligned} \langle 0 | \bar{s}(x) \sigma_{\mu\nu} s(0) | \phi(q) \rangle &= if_\phi^\perp \left\{ (\varepsilon_\mu q_\nu - \varepsilon_\nu q_\mu) \int_0^1 du e^{i\bar{u}qx} \left[\phi_2^\perp(u) + \frac{m_\phi^2 x^2}{4} \phi_4^\perp(u) \right] \right. \\ &+ \frac{1}{2} (\varepsilon_\mu x_\nu - \varepsilon_\nu x_\mu) \frac{m_\phi^2}{q \cdot x} \int_0^1 du e^{i\bar{u}qx} [\psi_4^\perp(u) - \phi_2^\perp(u)] + (q_\mu x_\nu - q_\nu x_\mu) \\ &\left. \times \frac{\varepsilon \cdot x}{(q \cdot x)^2} m_\phi^2 \int_0^1 du e^{i\bar{u}qx} D(u) + \dots \right\}. \end{aligned} \quad (228)$$

In equations above u is a longitudinal momentum fraction carried by a quark, and $\bar{u} = 1 - u$ is a momentum of an antiquark. The mass and polarization vector of the ϕ meson are denoted respectively by m_ϕ and ε_μ . Combinations of two-particle DAs $C(u)$ and $D(u)$ are given by the following expressions

$$C(u) = \psi_4^\parallel(u) + \phi_2^\parallel(u) - 2\phi_3^\perp(u), \quad D(u) = \phi_3^\parallel(u) - \frac{1}{2}\phi_2^\perp(u) - \frac{1}{2}\psi_4^\perp(u), \quad (229)$$

where subscripts in functions denote their twists. Expressions of matrix elements $\langle 0 | \bar{s}(x) \Gamma^J G_{\mu\nu}(vx) s(0) | \phi(q) \rangle$ in terms of the ϕ meson higher twist DAs, and detailed information on features of these DAs themselves can be found in Refs. [98, 99, 209–211].

The main contribution to $\Pi_\mu^{\text{OPE}}(p, q)$ comes from the terms (224), where only perturbative components of the propagators are used (see, figure 13). Contribution of this diagram can be evaluated by employing the ϕ meson's two-particle distribution amplitudes. The one-gluon exchange diagrams shown in figure 14 are corrections, which can be expressed and calculated using relevant three-particle DAs. An analytic expression of the $\Pi_\mu^{\text{OPE}}(p, q)$ in terms of the ϕ meson's DAs is lengthy enough, hence we refrain from providing it here.

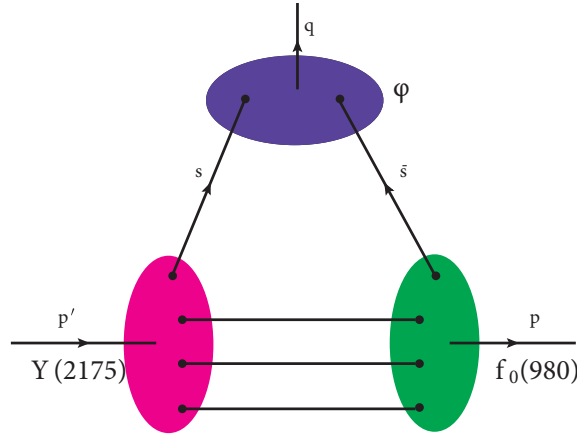


Figure 13. The leading order diagram contributing to $\Pi_\mu^{\text{OPE}}(p, q)$.

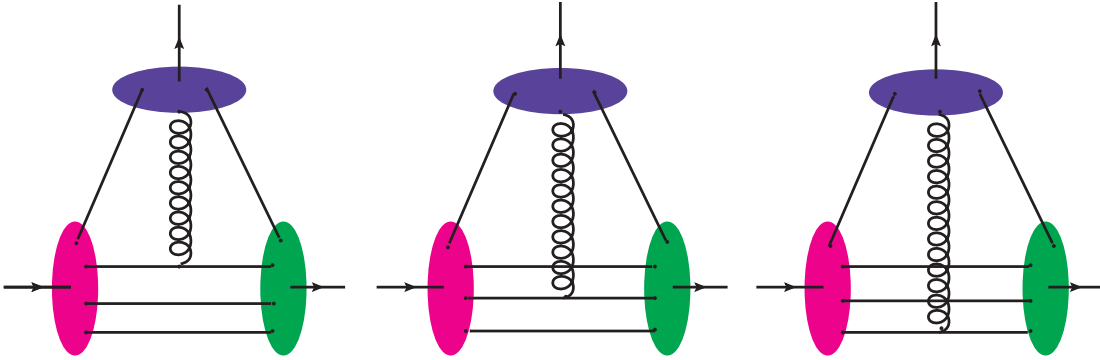


Figure 14. The one-gluon exchange diagrams connected to three-particle DAs of ϕ meson.

In calculations, we use the amplitude $\Pi^{\text{OPE}}(p'^2, p^2)$ extracted from a term proportional to ε_μ^* , and equate it to relevant amplitude from $\Pi_\mu^{\text{Phys}}(p, q)$. The invariant amplitudes depend on p'^2 and p^2 , therefore one has to perform double Borel transformation over p'^2 and p^2

$$\Pi^{\text{OPE}}(M_1^2, M_2^2) = \mathcal{B}_{p'^2}^{M_1^2} \mathcal{B}_{p^2}^{M_2^2} \Pi^{\text{OPE}}(p'^2, p^2). \quad (230)$$

The Borel transformed amplitude $\Pi^{\text{OPE}}(M_1^2, M_2^2)$ can be computed using recipes of Ref. [54], and written down

in form of a double dispersion integral. To simplify following operations, it is convenient to relate parameters M_1^2 and M_2^2 to each other by employing the relation $M_1^2/M_2^2 = m_Y^2/m_f^2$ and introducing the common parameter M^2 through the relation

$$\frac{1}{M^2} = \frac{1}{M_1^2} + \frac{1}{M_2^2}. \quad (231)$$

This implies replacements

$$M_1^2 = \frac{m_f^2 + m_Y^2}{m_f^2} M^2, \quad M_2^2 = \frac{m_f^2 + m_Y^2}{m_Y^2} M^2, \quad (232)$$

which allows us to carry out an integration over one of variables in the double dispersion integral. The expression obtained in this phase of computations depends also on the parameter u_0

$$u_0 = \frac{M_1^2}{M_1^2 + M_2^2} = \frac{m_Y^2}{m_f^2 + m_Y^2}. \quad (233)$$

As a result, we get a single integral representation for $\Pi^{\text{OPE}}(M^2)$ which simplifies the continuum subtraction procedure: Formulas required to fulfil the subtraction are collected in Appendix B of Ref. [54].

Distribution amplitudes of the ϕ meson contain a lot of parameters. Thus, the leading twist DAs of the longitudinally and transversely polarized ϕ meson have the forms

$$\phi_2^{\parallel(\perp)}(u) = 6u\bar{u} \left[1 + \sum_{n=1,2,\dots}^{\infty} a_{2n}^{\parallel(\perp)} C_{2n}^{3/2}(2u-1) \right], \quad (234)$$

which are general expressions for $\phi_2^{\parallel}(u)$ and $\phi_2^{\perp}(u)$. In computations we use DAs with only one nonzero coefficients $a_2^{\parallel(\perp)} \neq 0$. Analytic forms of higher twist DAs of the ϕ meson are borrowed from Refs. [99, 211], where one can find also parameters of these functions (see Tables 1 and 2 in Ref. [99]).

The sum rule for $G_{Y\phi f}$ depends on various condensates, and on mass of s quark presented already in Eq. (18). The masses and decay constants (couplings) of the particles \tilde{Y} , ϕ , and $f_0(980)$ are input information of computations as well. The spectroscopic parameters of \tilde{Y} have been evaluated in the previous subsection. For mass and decay constant of the ϕ meson, we employ experimental data $m_\phi = (1019.461 \pm 0.019)$ MeV and $f_\phi = (215 \pm 5)$ MeV. The mass of the meson $f_0(980)$ is known from measurements, whereas the coupling F_f of the meson $f_0(980)$ is taken from Ref. [3]

$$F_f \equiv F_{\mathbf{H}} = (1.35 \pm 0.34) \times 10^{-3} \text{ GeV}^4. \quad (235)$$

Let us remind that in Ref. [3] the meson $f_0(980)$ was considered as a scalar diquark-antidiquark state. The sum rule depends also on the Borel and continuum threshold parameters M^2 and s_0 . We fix working windows for the M^2 and s_0

$$M^2 \in [2.4, 3.4] \text{ GeV}^2, \quad s_0 \in [6, 6.5] \text{ GeV}^2, \quad (236)$$

which satisfy constraints of sum rule computations.

For the strong coupling $G_{Y\phi f}$ our computations yield

$$G_{Y\phi f} = (1.62 \pm 0.41) \text{ GeV}^{-1}. \quad (237)$$

The width of the decay $\tilde{Y} \rightarrow \phi f_0(980)$ is found by employing Eq. (36), in which one has to make the substitutions $g_{Z\psi\pi}$, m_ψ , $\lambda(m_Z, m_\psi, m_\pi) \rightarrow G_{Y\phi f}$, m_ϕ , $\lambda(m_Y, m_\phi, m_f)$.

For the partial width of the decay $\tilde{Y} \rightarrow \phi f_0(980)$, we get

$$\Gamma(\tilde{Y} \rightarrow \phi f) = (49.2 \pm 17.6) \text{ MeV}. \quad (238)$$

The result for $\Gamma(\tilde{Y} \rightarrow \phi f)$ is the principal output of this subsection, and we are going to use it to evaluate the full width of the tetraquark \tilde{Y} .

8.3. The decays $\tilde{Y} \rightarrow \phi\eta$ and $\tilde{Y} \rightarrow \phi\eta'$

The decay modes $\tilde{Y} \rightarrow \phi\eta$ and $\tilde{Y} \rightarrow \phi\eta'$ are next two processes which will be analyzed in this section. We are going to concentrate here on the channel $\tilde{Y} \rightarrow \phi\eta$, and provide final predictions for the process $\tilde{Y} \rightarrow \phi\eta'$.

In the context of the LCSR method the vertex $\tilde{Y}\phi\eta$ can be examined using the correlation function

$$\Pi_{\mu\nu}(p, q) = i \int d^4x e^{ipx} \langle \eta(q) | \mathcal{T} \{ J_\mu^\phi(x) J_\nu^{Y\dagger}(0) \} | 0 \rangle, \quad (239)$$

where $J_\mu^\phi(x)$ is the interpolating current for the vector ϕ meson (202).

The phenomenological side of the required sum rule for the strong coupling $g_{Y\phi\eta}$ is equal to

$$\Pi_{\mu\nu}^{\text{Phys}}(p, q) = g_{Y\phi\eta} \frac{f_\phi m_\phi f_Y m_Y}{(p^2 - m_\phi^2)(p'^2 - m_Y^2)} \varepsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta + \dots \quad (240)$$

In deriving Eq. (240), we have introduced the vertex

$$\langle \phi(p) \eta(q) | Y(p') \rangle = g_{Y\phi\eta} \varepsilon_{\mu\nu\alpha\beta} p^\mu q^\nu \epsilon^{*\alpha} \epsilon'^\beta, \quad (241)$$

with ϵ'^β being the polarization vector of the tetraquark \tilde{Y} .

It is evident that the correlation function $\Pi_{\mu\nu}^{\text{Phys}}(p, q)$ has a simple Lorentz structure. The invariant amplitude $\Pi^{\text{Phys}}(p'^2, p^2)$, which is necessary to obtain the sum rule for $g_{Y\phi\eta}$, can be extracted from Eq. (240) by factoring out the structure $\varepsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta$.

We calculate the invariant amplitude $\Pi^{\text{OPE}}(p'^2, p^2)$ from the correlation function $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$. In our case $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$ is determined by the formula

$$\begin{aligned} \Pi_{\mu\nu}^{\text{OPE}}(p, q) &= -i \int d^4x e^{ipx} \left[\gamma_5 \tilde{S}_s^{ib}(x) \gamma_\mu \tilde{S}_s^{bi}(-x) \gamma_5 \gamma_\nu + \gamma_\nu \gamma_5 \tilde{S}_s^{ib}(x) \gamma_\mu \tilde{S}_s^{bi}(-x) \gamma_5 \right]_{\alpha\beta} \\ &\times \langle \eta(q) | \bar{u}_\alpha^a(0) u_\beta^a(0) | 0 \rangle. \end{aligned} \quad (242)$$

The correlation function $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$ is expressed using s quark propagators and local matrix elements of the η meson. The local matrix elements $\langle \eta(q) | \bar{u}_\alpha^a u_\beta^a | 0 \rangle$ should be transformed in accordance with Eq. (26). Our analysis proves that $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$ receives a contribution from the matrix element $\langle \eta(q) | \bar{u} \gamma_\mu \gamma_5 u | 0 \rangle$ of the η meson

$$\langle \eta(q) | \bar{u} \gamma_\mu \gamma_5 u | 0 \rangle = -i \frac{f_\eta^q}{\sqrt{2}} q_\mu, \quad (243)$$

where f_η^q is the decay constant of the η meson's q component. Here, some comments are in order concerning the matrix element (243). It differs from matrix elements of other pseudoscalar mesons, and this is related to the mixing in the $\eta - \eta'$ system. Relevant problems have been discussed in Sec. 7, where one can find further details.

Using Eqs. (242) and (243), we find the invariant amplitude $\Pi^{\text{OPE}}(p'^2, p^2)$. This amplitude has to be equated to $\Pi^{\text{Phys}}(p'^2, p^2)$ which allows us to extract the sum rule for the strong coupling $g_{Y\phi\eta}$.

Because the correlation function $\Pi_{\mu\nu}^{\text{OPE}}(p, q)$ depends on local matrix elements of the η meson, we apply technical tools of the soft-meson approximation. In the soft limit $p' = p$, and for the strong coupling $g_{Y\phi\eta}$ we get the sum rule

$$g_{Y\phi\eta} = \frac{1}{f_\phi m_\phi f_Y m_Y} \mathcal{P}(M^2, m^2) \Pi^{\text{OPE}}(M^2, s_0), \quad (244)$$

where $\Pi^{\text{OPE}}(M^2, s_0)$ is the invariant amplitude $\Pi^{\text{OPE}}(p^2)$ after the Borel transformation and subtraction procedures. The amplitude $\Pi^{\text{OPE}}(M^2, s_0)$ computed by including into analysis nonperturbative terms up to dimension-5 is

$$\Pi^{\text{OPE}}(M^2, s_0) = \frac{f_\eta^q m_s}{8\sqrt{2}\pi^2} \int_{4m_s^2}^{s_0} ds e^{-s/M^2} + \frac{f_\eta^q m_s^2}{6\sqrt{2}M^2} \langle \bar{s}s \rangle + \frac{f_\eta^q}{12\sqrt{2}M^2} \langle \bar{s}g_s \sigma Gs \rangle. \quad (245)$$

The width of the process $\tilde{Y} \rightarrow \phi\eta$ is determined by the formula

$$\Gamma(\tilde{Y} \rightarrow \phi\eta) = \frac{g_{Y\phi\eta}^2 \lambda^3(m_Y, m_\phi, m_\eta)}{12\pi}. \quad (246)$$

For the strong coupling $g_{Y\phi\eta}$ and width of the decay $\tilde{Y} \rightarrow \phi\eta$ numerical computations yield

$$g_{Y\phi\eta} = (1.85 \pm 0.38) \text{ GeV}^{-1}, \quad \Gamma(\tilde{Y} \rightarrow \phi\eta) = (35.8 \pm 10.4) \text{ MeV}. \quad (247)$$

Let us note that in calculations of $g_{Y\phi\eta}$, we have varied M^2 and s_0 within the intervals

$$M^2 \in [1.3, 1.8] \text{ GeV}^2, \quad s_0 \in [6, 6.5] \text{ GeV}^2. \quad (248)$$

The partial width of the decay $\tilde{Y} \rightarrow \phi\eta'$ can be found by using expressions obtained for the first process. To this end, we take into account the mass of the η' meson, the new coupling $f_{\eta'}^q$, and function $\lambda(m_Y, m_\phi, m_{\eta'})$

$$f_{\eta'}^q = f_q \sin \varphi, \quad \lambda \rightarrow \lambda(m_Y, m_\phi, m_{\eta'}), \quad (249)$$

which can be easily implemented into analysis. For the parameters of the second decay channel, we get

$$g_{Y\phi\eta'} = (1.59 \pm 0.31) \text{ GeV}^{-1}, \quad \Gamma(\tilde{Y} \rightarrow \phi\eta') = (6.1 \pm 1.7) \text{ MeV}. \quad (250)$$

Saturating the full width of the \tilde{Y} resonance by three decay channels analyzed in the present section, we find

$$\Gamma_{\text{full}} = (91.1 \pm 20.5) \text{ MeV}. \quad (251)$$

The result for the mass m_Y obtained in this section by treating \tilde{Y} as the vector tetraquark $\tilde{Y} = [su][\bar{s}\bar{u}]$ agrees with the BaBar data, but is consistent with BESIII measurements as well. The full width Γ_{full} has some overlapping region with $\Gamma = (58 \pm 16 \pm 20)$ MeV extracted in Ref. [187], but agreement with data of BESIII is considerably better.

Encouraging is also our prediction for the ratio

$$\frac{\Gamma(\tilde{Y} \rightarrow \phi\eta)}{\Gamma(\tilde{Y} \rightarrow \phi f)} \approx 0.73, \quad (252)$$

which is almost identical to its experimental value ≈ 0.74 . The latter has been obtained from analysis of experimental information on the ratios

$$\frac{\Gamma(\tilde{Y} \rightarrow \phi\eta) \times \Gamma(\tilde{Y} \rightarrow e^+e^-)}{\Gamma_{\text{total}}} = 1.7 \pm 0.7 \pm 1.3, \quad (253)$$

and

$$\frac{\Gamma(\tilde{Y} \rightarrow \phi f) \times \Gamma(\tilde{Y} \rightarrow e^+e^-)}{\Gamma_{\text{total}}} = 2.3 \pm 0.3 \pm 0.3, \quad (254)$$

from Ref. [96] [\tilde{Y} is denoted there $\phi(2170)$].

In calculations of Γ_{full} , we have included into analysis only three strong decays of the resonance \tilde{Y} . Decay channels $\tilde{Y} \rightarrow \phi\pi\pi$, $K^+K^-\pi^+\pi^-$, $K^*(892)^0\bar{K}^*(892)^0$ of \tilde{Y} (observed in experiments and/or theoretically allowed) and other possible modes have not been taken into account. Partial width of these decays may improve our prediction for Γ_{full} . Theoretical analyses of these channels, as well as their detailed experimental investigations can help to answer open questions about the structure of the resonance $Y(2175)$.

9. Concluding notes

In this article, we have reviewed our works devoted to investigations of resonances observed by different collaborations, and which are considered as candidates to exotic four-quark mesons. As usual, experimental measurements provide valuable information on masses, widths and quantum numbers of these states. Corresponding theoretical studies start from interpretations of observed resonances as ground-state or excited conventional mesons, from assumptions on their dynamical or exotic nature. These suggestions should be supported by successful confrontation of theoretical predictions for their spectroscopic parameters and decay widths with experimental data. It is worth noting that existing theoretical computations use all diversity of available methods and schemes.

In our articles, we studied the resonances $Z_c(3900)$, $Z_c(4430)$, $Z_c^-(4100)$, $X(4140)$, $X(4274)$, $a_1(1420)$, $Y(4660)$, $X(2100)$, $X(2239)$, and $Y(2175)$ by assuming that they are exotic four-quark mesons with diquark-antidiquark structures. We constructed relevant interpolating currents for these states and calculated their masses and couplings. All resonances considered here are strong-interaction unstable particles, and decay to a pair of ordinary mesons. We computed partial widths of their dominant decay modes. Obtained results allowed us to interpret these resonances as diquark-antidiquark states with different spin-parities and quark contents or pose additional questions on their nature.

Some of our predictions deserves to be mentioned here. Thus, we interpreted the resonance $Z_c(4430)$ as the radial excitation of $Z_c(3900)$, and computed the masses and couplings, as well as estimated full widths

of these states. It seems experimental data do not contradict to this assumption, and resonances $Z_c(3900)$ and $Z_c(4430)$ are the ground-state and radial excitation of the same tetraquark. Another interesting result is connected with an assumption about quark content of $Y(2175)$. In fact, despite widespread $4s$ picture of $Y(2175)$, we argued that this vector resonance may have a content $[sq][\bar{s}\bar{q}]$. Predictions for parameters of this tetraquark agree with available measurements. Interesting was also our suggestion about different internal color structures of the axial-vector resonances $X(4140)$ and $X(4274)$. We interpreted them as tetraquarks with identical quark contents and spin-parities, but built of color-triplet and -sextet diquarks, respectively. While results for masses of these states confirm our assignments, prediction for the full width of $X(4274)$ overshoots experimental data considerably. Alternative ideas on the structure $X(4274)$ seem may be helpful to solve this problem.

In our calculations, we used the QCD sum rules approach, which is a powerful tool to explore features not only of conventional, but also exotic hadrons. The spectral parameters of four-quark states were computed by employing two-point correlation functions, and sum rules extracted from their analysis. To explore numerous strong decays of particles under discussion, we applied either three-point or light-cone sum rule methods. From these sum rules it is possible to extract numerical values of strong couplings corresponding to vertices of involved particles. The three-point sum rule method is effective for computations in the case of heavy final-state mesons. The light-cone sum rules are applicable to situations when at least one of final mesons is a particle with well-known distribution amplitudes or local matrix elements. Let us note that tetraquark-tetraquark-meson vertices can be explored by means of standard methods of LCSR, whereas treatment of tetraquark-meson-meson vertices requires additionally a soft-meson technique.

There are a lot of results left beyond the scope of the present review. Thus, we did not consider our papers on the structure of the resonance $X(5568)$ and its charmed partner, which presumably are particles made of four quarks of different flavors [212–215]. Very interesting investigations of tetraquarks stable against strong and electromagnetic decays were not included into this review as well. Because stable tetraquarks can dissociate to conventional mesons only through weak transformations, lifetimes of these states are approximately $10^{-12} - 10^{-13}$ s and significantly longer than that of unstable tetraquarks. The famous member of this class is the axial-vector tetraquark $T_{bb;\bar{u}\bar{d}}^-$ which is one of candidates to stable heavy exotic mesons [216–218]. We calculated the full width of $T_{bb;\bar{u}\bar{d}}^-$ through its semileptonic decays [219]. It is remarkable that family of stable exotic mesons is wider than one might suppose: some of these states were studied in our articles [220–223]. Problems of the resonance $X(5568)$, in general tetraquarks built of four quarks of different flavors, and stable heavy tetraquarks deserve detailed investigations and separate reviews.

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Appendix

10. The quark propagators

The light and heavy quark propagators are necessary to find QCD side of the different correlation functions. In the present work, we use the light quark propagator $S_q^{ab}(x)$ which is given by the following formula

$$\begin{aligned}
 S_q^{ab}(x) &= i\delta_{ab} \frac{\not{x}}{2\pi^2 x^4} - \delta_{ab} \frac{m_q}{4\pi^2 x^2} - \delta_{ab} \frac{\langle \bar{q}q \rangle}{12} + i\delta_{ab} \frac{\not{x} m_q \langle \bar{q}q \rangle}{48} - \delta_{ab} \frac{x^2}{192} \langle \bar{q}g_s \sigma Gq \rangle \\
 &+ i\delta_{ab} \frac{x^2 \not{x} m_q}{1152} \langle \bar{q}g_s \sigma Gq \rangle - i \frac{g_s G_{ab}^{\alpha\beta}}{32\pi^2 x^2} [\not{x} \sigma_{\alpha\beta} + \sigma_{\alpha\beta} \not{x}] - i\delta_{ab} \frac{x^2 \not{x} g_s^2 \langle \bar{q}q \rangle^2}{7776} \\
 &- \delta_{ab} \frac{x^4 \langle \bar{q}q \rangle \langle g_s^2 G^2 \rangle}{27648} + \dots
 \end{aligned} \tag{J.255}$$

For the heavy quarks Q we utilize the propagator $S_Q^{ab}(x)$

$$\begin{aligned}
 S_Q^{ab}(x) &= i \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \left\{ \frac{\delta_{ab} (\not{k} + m_Q)}{k^2 - m_Q^2} - \frac{g_s G_{ab}^{\alpha\beta}}{4} \frac{\sigma_{\alpha\beta} (\not{k} + m_Q) + (\not{k} + m_Q) \sigma_{\alpha\beta}}{(k^2 - m_Q^2)^2} \right. \\
 &\left. + \frac{g_s^2 G^2}{12} \delta_{ab} m_Q \frac{k^2 + m_Q \not{k}}{(k^2 - m_Q^2)^4} + \frac{g_s^3 G^3}{48} \delta_{ab} \frac{(\not{k} + m_Q)}{(k^2 - m_Q^2)^6} [\not{k} (k^2 - 3m_Q^2) + 2m_Q (2k^2 - m_Q^2)] (\not{k} + m_Q) + \dots \right\}.
 \end{aligned} \tag{J.256}$$

The light-cone propagator of the light quark is given by the expression

$$\begin{aligned}
 S_q^{ab}(x) &= \frac{i\not{x}}{2\pi^2 x^4} \delta_{ab} - \frac{m_q}{4\pi^2 x^2} \delta_{ab} - \frac{\langle \bar{q}q \rangle}{12} \left(1 - i \frac{m_q}{4} \not{x} \right) \delta_{ab} - \frac{x^2}{192} \langle \bar{q}g_s \sigma Gq \rangle \left(1 - i \frac{m_q}{6} \not{x} \right) \delta_{ab} \\
 &- ig_s \int_0^1 du \left\{ \frac{\not{x}}{16\pi^2 x^2} G_{ab}^{\mu\nu}(ux) \sigma_{\mu\nu} - \frac{iux_\mu}{4\pi^2 x^2} G_{ab}^{\mu\nu}(ux) \gamma_\nu - \frac{im_q}{32\pi^2} G_{ab}^{\mu\nu}(ux) \sigma_{\mu\nu} \left[\ln \left(\frac{-x^2 \Lambda^2}{4} \right) + 2\gamma_E \right] \right\},
 \end{aligned} \tag{J.257}$$

For the heavy Q quark's light-cone propagator we have

$$\begin{aligned}
 S_Q^{ab}(x) &= \frac{m_Q^2}{4\pi^2} \frac{K_1(m_Q \sqrt{-x^2})}{\sqrt{-x^2}} \delta_{ab} + i \frac{m_Q^2}{4\pi^2} \frac{\not{x} K_2(m_Q \sqrt{-x^2})}{(\sqrt{-x^2})^2} \delta_{ab} \\
 &- \frac{g_s m_Q}{16\pi^2} \int_0^1 dv G_{ab}^{\mu\nu}(vx) \left[(\sigma_{\mu\nu} \not{x} + \not{x} \sigma_{\mu\nu}) \frac{K_1(m_Q \sqrt{-x^2})}{\sqrt{-x^2}} + 2\sigma^{\mu\nu} K_0(m_Q \sqrt{-x^2}) \right].
 \end{aligned} \tag{J.258}$$

In the expressions above

$$G_{ab}^{\alpha\beta} = G_A^{\alpha\beta} t_{ab}^A, \quad G^2 = G_{\alpha\beta}^A G_{\alpha\beta}^A, \quad G^3 = f^{ABC} G_{\mu\nu}^A G_{\nu\delta}^B G_{\delta\mu}^C, \tag{J.259}$$

where $a, b = 1, 2, 3$ are color indices and $A, B, C = 1, 2 \dots 8$. Here $t^A = \lambda^A/2$, where λ^A are the Gell-Mann matrices, and the gluon field strength tensor is fixed at $x = 0$, i.e., $G_{\alpha\beta}^A \equiv G_{\alpha\beta}^A(0)$. In Eq. (J.258) $K_\nu(z)$ are Bessel functions of the second kind.

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