A nonminimally coupled, conformally extended Einstein-Maxwell theory of pp-waves

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Abstract: A nonminimal coupling of Weyl curvatures to electromagnetic fields is considered in Brans-Dicke-Maxwell theory. The gravitational field equations are formulated in a Riemannian spacetime where the spacetime torsion is constrained to zero by the method of Lagrange multipliers in the language of exterior differential forms. The significance and ramifications of nonminimal couplings to gravity are examined in a pp-wave spacetime.

Key words: Classical general relativity, gravitational waves, modified theories of gravity

1. Introduction

Einstein's theory of general relativity is a theory of gravitation determined by the Riemannian geometry of a 4-dimensional spacetime. Brans-Dicke theory [1–3] on the other hand, provides a modification through the inclusion of a massless scalar field, in order to incorporate Mach's principle into Einstein's initial framework. The coupled field equations of the theory may be obtained by field variational principle from an action. It is well known that the second order principle where the metric variations of the Levi-Civita connections are taken into account; and the first order (Palatini) where independent variations of the action relative to the metric and connection yield, in these cases, the same set of field equations [4].The fact that the connection is Levi-Civita may be imposed by the method of Lagrange multipliers.

Electromagnetic fields can be coupled minimally to gravitation simply by incorporating the Maxwell Lagrangian density 4-form into the total action; and taking independent variations of the electromagnetic fields. Here we consider nonminimal couplings of gravity and electromagnetic fields in the Lagrangian density. Such nonminimal considerations for the Einstein-Maxwell couplings may be justified in extreme conditions where there are intense gravitational and electromagnetic fields at high temperatures, pressures and density [5–8].

We will use in what follows a first order Palatini type variational principle where the Levi-Civita connection is forced by zero-torsion constraint using the method of Lagrange multipliers [9, 10]. In a similar way, we may avoid the use of an electromagnetic potential 1-form A such that $F = dA$ by imposing the closedness of F, $dF = 0$, by the same method and varying F directly. We have previously discussed in the nonminimal coupling context, static, spherically symmetric field configurations where electromagnetic charges will be screened by gravitational fields[11]. We have also investigated pp-wave spacetimes in Brinkmann coordinates [12] for one simple case of RF^2 couplings which induced possible modifications to the metric functions[13]. Here we consider nonminimal couplings of RF^2 type to the Brans-Dicke scalar-tensor theory of gravity.

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The Brans-Dicke-Maxwell Lagrangian

$$\mathcal{L}_0 \;\; = \;\; \frac{\phi}{2}R_{ab} \wedge *(e^a \wedge e^b) - \frac{\omega}{2\phi}d\phi \wedge *d\phi - \frac{1}{2}F \wedge *F \tag{1}$$

is locally scale invariant under

$$g \to e^{2\sigma}, \quad \phi \to e^{-2\sigma}\phi, \quad F \to F, \tag{2}$$

in the special case $\omega = -\frac{3}{2}$. The nonminimal RF^2 coupling of curvature to electromagnetic fields that will be considered here is

$$\mathcal{L}_1 \;\; = \;\; \frac{\gamma}{2\phi}C_{ab} \wedge F^{ab} * F, \tag{3}$$

where

$$C_{ab} = R_{ab} - \frac{1}{2}(e_a \wedge R_b - e_b \wedge R_a) + \frac{R}{6}e_{ab} \tag{4}$$

are the Weyl conformal curvature 2-forms and γ is an arbitrary dimensionless coupling constant. $\mathcal{L}_1$ is also locally scale invariant.

2. Field equations

In a 4-dimensional spacetime manifold, the field equations will be derived by infinitesimal variations of the action $I[e^a, \omega^a{}_b, \phi, F, \lambda_a, \mu] = \int_M (\mathcal{L} + \mathcal{L}_\mathcal{C})$, where the Lagrangian density 4-form

$$\begin{aligned}
\mathcal{L} \;\; = \;\; & \frac{\phi}{2}R_{ab} \wedge *(e^a \wedge e^b) - \frac{\omega}{2\phi}d\phi \wedge *d\phi - \frac{1}{2}F \wedge *F \\
+ \;\; & \frac{c_1}{2\phi}R_{ab}F^{ab} \wedge *F + \frac{c_2}{2\phi}F_a \wedge R^a \wedge *F + \frac{c_3}{2\phi}RF \wedge *F.
\end{aligned} \tag{5}$$

$\{e^a\}$'s are the co-frame 1-forms in terms of which the space-time metric $g = \eta_{ab}e^a \otimes e^b$ with $\eta_{ab} = diag(-+++)$. $*$ denotes the Hodge map. The orientation is given by $*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3$. We denote the Brans-Dicke scalar field by ϕ with its parameter ω and the electromagnetic field 2-form by F. The nonminimal dimensionless coupling constants are c_1, c_2 and c_3. $\{\omega^a{}_b\}$ are the connection 1-forms that satisfy the Cartan structure equations

$$de^a + \omega^a{}_b \wedge e^b = T^a \tag{6}$$

with the torsion 2-forms T^a and

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_b \tag{7}$$

with the curvature 2-forms $R^a{}_b$ of spacetime. The field equations are retrieved by independent variations of the action with respect to e^a, $\omega^a{}_b$, ϕ and F. In order to implement the method of Lagrange multipliers, we add to the Lagrangian density above the constraint terms

$$\mathcal{L}_\mathcal{C} = (de^a + \omega^a{}_b \wedge e^b) \wedge \lambda_a + dF \wedge \mu. \tag{8}$$

We impose the connection to be, through constrained variations, the unique metric compatible, torsion-free Levi-Civita connection. Constrained by the Lagrange multiplier 2-form μ, the homogeneous Maxwell equations $dF = 0$ are satisfied by the electromagnetic 2-form $F = \frac{1}{2}F_{ab}e^a \wedge e^b$. The infinitesimal variations of the total Lagrangian density with respect to the co-frames e^a, the scalar field ϕ, the electromagnetic field F and the connection 1-forms $\omega^a{}_b$ give, respectively, the following equations

$$\frac{\phi}{2}R^{bc} \wedge *e_{abc} + \frac{1}{2}(\iota_a F \wedge *F - F \wedge \iota_a * F) + \frac{\omega}{2\phi}(\iota_a d\phi \wedge *d\phi + d\phi \wedge \iota_a * d\phi) + D\lambda_a$$

$$-\frac{c_1}{4\phi}\big[-4F_{ac}F_b \wedge *R^{cb} + \iota_a R_{bc} \wedge F^{bc} * F - R_{bc}F^{bc} \wedge \iota_a * F + F^{bc}F_a \wedge *R_{bc} - F^{bc}F \wedge \iota_a * R_{bc}\big]$$

$$+\frac{c_2}{4\phi}\big[-F_{ac}R^c \wedge *F + 2RF_a \wedge *F - \iota_a R^c F_c \wedge *F + 2\iota_a R^{cb}F_{cb} \wedge *F - 2\iota_a R^{cb} \wedge F_c \wedge \iota_b * F$$

$$-F_a \wedge *(F_c \wedge R^c) - 2F_a \wedge R^c \wedge \iota_c * F + F_c \wedge R^c \wedge \iota_a * F + F \wedge \iota_a * (F_c \wedge R^c)\big]$$

$$+\frac{c_3}{2\phi}\big[-2\iota_a R^b F_b \wedge *F - 2\iota_a R^b F \wedge \iota_b * F - RF_a \wedge *F + RF \wedge \iota_a * F\big] = 0, \tag{9}$$

$$\frac{1}{2}R_{ab} \wedge *e^{ab} - \frac{1}{2\phi^2}(c_1 R_{ab}F^{ab} \wedge *F + c_2 F_a \wedge R^a \wedge *F + c_3 RF \wedge *F)$$

$$+\frac{\omega}{2\phi^2}d\phi \wedge *\mathrm{d}\phi + \omega d(*\frac{\mathrm{d}\phi}{\phi}) = 0, \tag{10}$$

$$\frac{1}{\phi}\big[c_1 F^{ab} * R_{ab} + \frac{c_2}{2}(R^a \wedge \iota_a * F - R * F + *(F_a \wedge R^a)) + c_3 R * F\big] - *F - d\mu = 0, \tag{11}$$

$$\frac{1}{2}D\big[\phi * e_{ab} + c_1(\frac{F_{ab} * F}{\phi}) + c_2(-\frac{F_{ab} * F}{\phi} + \frac{1}{2\phi}F_a \wedge \iota_b * F - \frac{1}{2\phi}F_b \wedge \iota_a * F)$$

$$+\frac{c_3}{\phi}\iota_b\iota_a(F \wedge *F)\big] + \frac{1}{2}(e_b \wedge \lambda_a - e_a \wedge \lambda_b) = 0. \tag{12}$$

D denotes the covariant exterior derivative with respect to the Levi-Civita connection and ι_a are the interior products that satisfies $\iota_a e^b = \delta_a^b$. Variations of the Lagrange multiplier 2-forms λ_a impose the zero-torsion condition $T^a = 0$.

The field equation from the variation of the constraint 1-form μ gives the homogeneous Maxwell equations $dF = 0$. Equivalently by the Poincaré Lemma, $F = dA$ for some unique potential A. We read the electromagnetic field equation from (11) by taking its exterior derivative

$$d * F = d\big[\frac{1}{\phi}\big(c_1 F^{ab} * R_{ab} + \frac{c_2}{2}(R^a \wedge \iota_a * F - R * F + *(F_a \wedge R^a)) + c_3 R * F\big)\big]. \tag{13}$$

Multiplying equation (10) we obtained from the ϕ variations by 2ϕ and subtracting it from the trace of the co-frame e^a variation equation (9), we simplify the scalar field equation to the form

$$2\omega d(*d\phi) - e^a \wedge D\lambda_a = 0. \tag{14}$$

λ_a's are Lagrange multiplier 2-forms to be solved from the connection variation equations (12) that is written as

$$(e_a \wedge \lambda_b - e_b \wedge \lambda_a) = d\phi \wedge *e_{ab} + K_{ab}, \tag{15}$$

where

$$K_{ab} = D(\frac{1}{\phi}\Sigma_{ab}) \tag{16}$$

and

$$\Sigma_{ab} = c_1 F_{ab} * F + c_2(-F_{ab} * F + \frac{1}{2}F_a \wedge \iota_b * F - \frac{1}{2}F_b \wedge \iota_a * F) + c_3 \iota_b \iota_a (F \wedge *F). \tag{17}$$

Then we obtain the following expression for the Lagrange multipliers

$$\lambda_a = \iota_a(*d\phi) + \iota_b K^b_a - \frac{e_a}{4} \wedge \iota_c \iota_b K^{bc}. \tag{18}$$

The Einstein 3-forms are given by $G_a = -\frac{1}{2}R^{bc} \wedge *e_{abc}$, so that the Einstein field equations can be written as follows:

$$\phi G_a =$$

$$\frac{1}{2}(\iota_a F \wedge *F - F \wedge \iota_a * F) + \frac{\omega}{2\phi}(\iota_a d\phi \wedge *d\phi + d\phi \wedge \iota_a * d\phi) + D(\iota_a * d\phi)$$

$$-\frac{c_1}{4\phi}\Big[-4F_{ac}F_b \wedge *R^{cb} + \iota_a R_{bc} \wedge F^{bc} * F - R_{bc}F^{bc} \wedge \iota_a * F + F^{bc}F_a \wedge *R_{bc} - F^{bc}F \wedge \iota_a * R_{bc}\Big]$$

$$+\frac{c_2}{4\phi}\Big[-F_{ac}R^c \wedge *F + 2RF_a \wedge *F - \iota_a R^c F_c \wedge *F + 2\iota_a R^{cb}F_{cb} \wedge *F - 2\iota_a R^{cb} \wedge F_c \wedge \iota_b * F$$

$$-F_a \wedge *(F_c \wedge R^c) - 2F_a \wedge R^c \wedge \iota_c * F + F_c \wedge R^c \wedge \iota_a * F + F \wedge \iota_a * (F_c \wedge R^c)\Big]$$

$$+\frac{c_3}{2\phi}\Big[-2\iota_a R^b F_b \wedge *F - 2\iota_a R^b F \wedge \iota_b * F - RF_a \wedge *F + RF \wedge \iota_a * F\Big]$$

$$+D(\iota_b K^b_a) + \frac{e_a}{4} \wedge D(\iota_c \iota_b K^{bc}). \tag{19}$$

We have to solve them together with the scalar field equation

$$(2\omega + 3)d(*d\phi) = d(\iota_b K^b_a \wedge e^a), \tag{20}$$

and the inhomogeneous Maxwell equations

$$d * F = d\Big[\frac{1}{\phi}\big(c_1 F^{ab} * R_{ab} + \frac{c_2}{2}(R^a \wedge \iota_a * F - R * F + *(F_a \wedge R^a)) + c_3 R * F\big)\Big]. \tag{21}$$

3. PP-wave solutions

We wish to explore solutions that characterize plane fronted waves with parallel rays. In Ehlers-Kundt form, we write the metric

$$g = 2dudv + 2H(u, \zeta, \bar{\zeta})du^2 + 2d\zeta d\bar{\zeta}, \tag{22}$$

where

$$u = \frac{z-t}{\sqrt{2}}, \quad v = \frac{z+t}{\sqrt{2}}, \quad \zeta = \frac{x+iy}{\sqrt{2}} \tag{23}$$

in terms of the Cartesian coordinates (t, x, y, z). H is a smooth function which will be determined by Einstein field equations. We take next an electromagnetic potential 1-form given by $A = f(u, \zeta, \bar{\zeta})du$, and the scalar field $\phi = \phi(u)$ only.

Introducing null tetrads to the formalism developed in Ref[12] we have

$$l = du, \quad n = dv + Hdu, \quad m = d\zeta. \tag{24}$$

The following notation used for the connection and curvature forms have been previously introduced in [14, 15]; then using the Cartan structure equations, we determine the connection 1-forms

$$\omega_+ = 0, \quad \omega_- = iH_\zeta l, \quad \omega_0 = 0, \tag{25}$$

the curvature 2-forms

$$R_+ = 0, \quad R_- = -iH_{\zeta\zeta}\, l \wedge m - iH_{\bar{\zeta}\bar{\zeta}}\, l \wedge \bar{m}, \quad R_0 = 0, \tag{26}$$

and the Einstein 3-forms

$$\frac{G_3 + G_0}{\sqrt{2}} = 2iH_{\zeta\bar{\zeta}}\, l \wedge m \wedge \bar{m}, \quad \frac{G_3 - G_0}{\sqrt{2}} = 0, \quad \frac{G_1 + iG_2}{\sqrt{2}} = 0. \tag{27}$$

Furthermore the electromagnetic field 2-form becomes

$$F = dA = -f_\zeta\, l \wedge m - f_{\bar{\zeta}}\, l \wedge \bar{m}. \tag{28}$$

Regardless of the choice of the nonminimal coupling constants c_1, c_2 and c_3, in this particular geometry the right hand sides of both (20) and (21) vanish. The scalar field equation is then identically satisfied $d(*d\phi) = 0$.

The Einstein equations after a long calculation reduce to

$$\begin{aligned}
H_{\zeta\bar{\zeta}} &= -\frac{f_\zeta f_{\bar{\zeta}}}{\phi} - \omega\frac{\phi_u^2}{2\phi^2} - \frac{\phi_{uu}}{2\phi} + (c_1 - c_2)\frac{f_{\zeta\zeta}f_{\bar{\zeta}\bar{\zeta}}}{\phi^2} + \frac{c_3}{\phi^2}f_{\zeta\bar{\zeta}}^2 \\
&\quad + \frac{(c_2 - 2c_3)}{\phi^2}(f_{\zeta\bar{\zeta}\bar{\zeta}}f_\zeta + 2f_{\zeta\bar{\zeta}}^2 + f_{\zeta\zeta\bar{\zeta}}f_{\bar{\zeta}}).
\end{aligned} \tag{29}$$

From the Maxwell equations $dF = 0$ and $d*F = 0$ we get

$$f_{\zeta\bar{\zeta}} = 0. \tag{30}$$

Consequently the field equations reduce to

$$H_{\zeta\bar{\zeta}} = -\frac{f_\zeta f_{\bar{\zeta}}}{\phi} - (\omega\frac{\phi_u^2}{2\phi^2} + \frac{\phi_{uu}}{2\phi}) + (c_1 - c_2)\frac{f_{\zeta\zeta}f_{\bar{\zeta}\bar{\zeta}}}{\phi^2}. \tag{31}$$

An exact solution of the of the above equation is

$$H(u, \zeta, \bar{\zeta}) = H_0 - \frac{1}{2\phi}f^2 - (\omega\frac{\phi_u^2}{2\phi^2} + \frac{\phi_{uu}}{2\phi})\zeta\bar{\zeta} + (c_1 - c_2)\frac{f_\zeta f_{\bar{\zeta}}}{\phi^2}, \tag{32}$$

where $(H_0)_{\zeta\bar{\zeta}} = 0$.

4. Conclusion

It is remarkable that for an arbitrary choice of the coupling constants c_1, c_2 and c_3, the right hand side of both the Maxwell and scalar field equations identically vanish in the pp-wave geometry. Then the scalar and vector fields ϕ and F configurations of the nonminimal Brans-Dicke equations are still solutions. However, the Einstein field equations (29) are modified on its right hand side by nonminimal couplings.

The previous case studied in [13] corresponds to the choice $c_1 \neq 0$ and $c_2 = c_3 = 0$ which give a nontrivial contribution in the metric function when integrated. We note the emergence of the scalar field coupling to the electromagnetic potential on the right hand side of the Einstein equations, which in turn leads to an exact solution that differs due to the conformal coupling. It may be interesting to include in future works a cosmological constant with a generalization of pp-waves in AdS or dS.

Here with the choice $c_1 = c_2 = \gamma$ and $c_3 = \frac{\gamma}{3}$ we see that all nonminimal contributions drop out on-shell, i.e. provided the Maxwell equation $f_{\zeta\bar{\zeta}} = 0$ is satisfied. This is a manifestation of the fact that the nonminimal coupling we study here is locally scale invariant. The non-minimal couplings that we have considered here polarize spacetime and the presence of a dark matter distribution around a black hole will be affected for $c_1 \neq c_2$, which could provide dominant signs of the peculiar quality of dark matter[16]. Further studies[17] shows that the effects of cold dark matter on primordial gravitational waves; a frequency dependent modification, detectable but small, on the propagation speed. Consequently nonminimal couplings of RF^2 type may be of essence also in understanding the effects of magnetization and polarization on a binary inspiral. Remarkably, it has been pointed out recently that dark matter overdensities around black holes inescapably reshape the motion of binary mergers[18].

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