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Quantum gravity correction to the thermodynamic quantities of the charged dRGT black hole

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Abstract: In this study, in the framework of the Hamilton approach, the quantum gravity effect on Hawking temperature, on the specific heat capacity, and on the stability conditions of the charged de Rham, Gabadadze, and Tolley (dRGT) black hole are investigated by using the tunneling processes of both charged massive scalar and Dirac particles. It is shown that quantum corrected Hawking temperature depends not only on the black hole properties but also on the properties of the tunnelling particles. It is also observed that quantum corrected Hawking temperature is lower than that of the standard Hawking temperature. Also, the specific heat capacity and the local stability conditions of the black hole are discussed in the context of the tunneling processes of both charged massive scalar and Dirac particles in the presence of quantum gravity effect. It is observed that the black hole might undergo second-type phase transitions to become stable during the tunneling processes of both scalar and Dirac particles. However, in the absence of the quantum gravity effect, the black hole might undergo the both first-type and second-type phase transitions to become stable.

Key words: Hawking radiation, particle tunneling, quantum gravity, black hole stability

1. Introduction

Black hole thermodynamics is one of the interesting topics of modern cosmology, which provides important clues to quantum gravity and is widely studied in the literature. Thanks to the pioneering studies of Bekenstein, Hawking and Bardeen [1–6], it was realized that a black hole can be understood by using thermodynamic parameters such as temperature, entropy, enthalpy, free energy and heat capacity just like a thermodynamic system. Several methods have been developed in the literature to derive Hawking temperature of a black hole after Hawking has proven that a black hole emits radiation at certain temperature $T_H = \frac{\kappa}{2\pi}$ by utilizing the quantum mechanical formulation in curved space-time [4–6]. The Hamilton-Jacobi approach based on the tunneling method is the most well known method in the literature. In the context of Hamilton–Jacobi approach, Hawking temperature of various black holes was recovered by using the tunneling process of massive point-like particles, such as scalar, Dirac and vector boson particles [7–31]. However, in all these studies, Hawking temperature is independent of the properties of a tunneling particle. Nowadays, the generalized uncertainty principle (GUP), the standard Heisenberg uncertainty principle that includes the gravity effect, has gained popularity [32–46]. Despite of the significant attempts to build a complete and self-consistent theory of quantum gravity, there is no satisfactory answer yet. One of the most interesting results of these attempts is the existence of a minimum measurable length identified by the Planck scale. This minimum length can be

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realized by the simple form of the GUP relation [38, 40, 50, 51]:

$$\Delta x \Delta p \geq \frac{h}{2}\left[1+\alpha(\Delta p)^2\right], \tag{1}$$

where $\alpha{=}\alpha_0/M_p^2$, M_p is the Planck mass and α_0 is a dimensionless parameter ranging from $\alpha_0 < 10^{21}$ to $\alpha_0 < 10^{50}$ [46–49]. Eq. (1) can be derived by using the modified commutation relation given as [38, 40, 50, 51]

$$[x_i, p_j] = i\hbar\delta_{ij}\left[1+\alpha p_{0i}^2\right]. \tag{2}$$

Here, x_i and p_j are the modified position and momentum operators defined in terms of the standard position, x_{0i}, and momentum operators, p_{0j}, as follows:

$$\begin{aligned} x_i &= x_{0i}, \\ p_i &= p_{0i}(1+\alpha p_0^2). \end{aligned} \tag{3}$$

Black holes are laboratories where quantum gravitational effects cannot be neglected and GUP applications can be tested. In the context of this thought, it is reported that the thermodynamic parameters of a black hole depend not only on the black hole properties but also on the properties of the tunnelled particles [50–67]. In light of these discussions, we focus on the thermodynamic quantities of the charged dRGT black hole by taking into account the quantum gravity effects.

The dRGT massive gravity is a covariant, ghost free and nonlinear theory of gravity in which the graviton has a mass [68, 69]. This theory can describe the current accelerating expansion of the universe without introducing the cosmological constant [70, 71]. In this theory the graviton mass naturally generates the cosmological constant. The dRGT massive gravity does not only admit cosmological solutions, but also admit a spherically symmetric solutions. The charged dRGT black hole is one of these spherical symmetric solutions. The charged dRGT black hole metric is given by [70]

$$ds^2 = f dt^2 - \frac{1}{f}dr^2 - r^2 d\theta^2 - r^2\sin^2(\theta)d\phi^2, \tag{4}$$

where

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{\Lambda}{3}r^2 + \gamma r + \varepsilon. \tag{5}$$

Here, M and Q are the mass and the charge of the black hole. Moreover, the parameters γ, ε and the cosmological constant, Λ, can be expressed in terms of the graviton mass, m_g, and the dimensionless free parameters, α_1 and β_1, of the dRGT massive gravity theory as follows:

$$\Lambda = 3m_g^2(1+\alpha_1+\beta_1), \quad \gamma = -cm_g^2(1+2\alpha_1+3\beta_1), \quad \varepsilon = c^2 m_g^2(\alpha_1+3\beta_1). \tag{6}$$

The mass associated with the outer event horizon of the black hole is given as follows:

$$M = \frac{r_+}{2}\left[1+\frac{Q^2}{r_+^2}+\frac{\Lambda}{3}r_+^2+\gamma r_+ +\varepsilon\right]. \tag{7}$$

The black hole solution represented by the metric given in Eq. (4) can be reduced to black hole solutions well known in the literature found in the context of Einstein's theory of general relativity. If $m_g = 0$, it reduces to the Reissner–Nordström solution. Moreover, in the case of $c = 0$, Eq. (4) represents Reissner–Nordström-de Sitter type black hole for the condition $1 + \alpha_1 + \beta_1 < 0$ and Reissner–Nordström–anti-de Sitter type black hole for the condition $1 + \alpha_1 + \beta_1 > 0$.

The paper is organized as follows: In Section 2, we investigate the GUP effect on the tunnelling progress of the charged massive scalar particle from the charged dRGT black hole, and then calculate the modified Hawking temperature. In Sections 3, we repeat a similar procedure for the charged massive Dirac particle, as we performed for the scalar particle in Section 2. In Section 4, we discuss the local thermodynamic stability conditions and the possible phase transitions of the black hole by calculating the heat capacity within quantum gravity correction. The important outcomes of the paper are summarized in conclusion.

2. Corrected Hawking temperature derived by tunneling of charged massive scalar particle

In this section, we calculate the modified Hawking temperature of charged massive scalar particles which tunnel from the charged dRTG black hole. For a massive scalar particle, the modified Klein–Gordon equation is given by [51, 53, 64]

$$\hbar^2\partial_0\partial^0\widetilde{\Phi} + \hbar^2\partial_i\partial^i\widetilde{\Phi} + 2\alpha\hbar^4\partial_i\partial^i(\partial_i\partial^i\widetilde{\Phi}) + 2iq\hbar A^\mu(1-2\alpha M_0^2)\partial_\mu\widetilde{\Phi} + 4iq\alpha\hbar^3 A^\mu\partial_\mu(\partial_i\partial^i\widetilde{\Phi})$$
$$-2q^2\hbar^2\alpha A_\mu A^\mu\partial_i\partial^i\widetilde{\Phi} - (q^2A_\mu A^\mu - M_0^2)(1-2\alpha M_0^2)\widetilde{\Phi} = 0, \quad (8)$$

where $\widetilde{\Phi}$, M_0, q are the modified wave function, mass and charge of the scalar particle, respectively, and A_μ is the electromagnetic potential vector. The modified wave function can be defined by utilizing the Wentzel–Kramers–Brillouin (WKB) approximation as follows:

$$\widetilde{\Phi}(t,r,\theta,\phi) = A(t,r,\theta,\phi)\exp\left(\frac{i}{\hbar}S(t,r,\theta,\phi)\right), \quad (9)$$

where $S(t,r,\theta,\phi)$ is the classical action function. Therefore, in the presence of the Coulomb interaction, $A_0 = \frac{Q}{r}$, in the dRGT black hole background, the modified Klein–Gordon equation is reduced to the following modified Hamilton–Jacobi equation;

$$\left(\frac{\partial S}{\partial t}\right)^2 - f^2\left(\frac{\partial S}{\partial r}\right)^2 - \frac{f}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - \frac{f}{r^2\sin^2(\theta)}\left(\frac{\partial S}{\partial \phi}\right)^2 + \left(q^2A_0^2 - M_0^2 f\right) + 2qA_0\left(\frac{\partial S}{\partial t}\right)$$
$$+2\alpha\left[M_0^4 f + 2qA_0 f\left(\frac{\partial S}{\partial r}\right)^2\left(\frac{\partial S}{\partial t}\right) + \frac{2qA_0 f}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2\left(\frac{\partial S}{\partial t}\right) + \frac{q^2A_0^2}{r^2\sin^2(\theta)}\left(\frac{\partial S}{\partial \phi}\right)^2\right]$$
$$+2\alpha\left[\frac{q^2A_0^2}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - qA_0M_0^2\left(\frac{\partial S}{\partial t}\right) - \frac{f}{r^4\sin^4(\theta)}\left(\frac{\partial S}{\partial \phi}\right)^4 + q^2A_0^2 f\left(\frac{\partial S}{\partial r}\right)^2\right]$$
$$+2\alpha\left[\frac{2qA_0}{r^2\sin^2(\theta)}\left(\frac{\partial S}{\partial \phi}\right)^2\left(\frac{\partial S}{\partial t}\right) - q^2A_0^2M_0^2 - f^3\left(\frac{\partial S}{\partial r}\right)^4 - \frac{f}{r^4}\left(\frac{\partial S}{\partial \theta}\right)^4\right] = 0, \quad (10)$$

where the higher order terms of $\hbar$ and α are neglected. Using separation of variables method, the classical trajectory function, $S(t,r,\theta,\phi)$, can be separated in terms of the quantum numbers of the tunneling scalar particle as $S(t,r,\theta,\phi) = -Et + j\phi + R(r) + W(\theta) + C$. Here, E is the energy of the emitted scalar particle, j is the angular momentum relating to the azimuth angle ϕ and C is a complex constant. Substituting this definition in Eq. (10), the radial trajectory, $R_{\pm}(r)$, of the tunneling charged massive scalar particle is obtained as follows:

$$R_{\pm}(r) = \pm \int \frac{\sqrt{(E-qA_0)^2 - f\left(M_0^2 + \frac{j^2}{r^2\sin^2(\theta)} + \frac{1}{r^2}(\frac{dW}{d\theta})^2\right)}}{f}\left[1+\alpha\frac{\Upsilon}{\Omega}\right]dr, \tag{11}$$

where $R_+(r)$ and $R_-(r)$ correspond to the outgoing and incoming particle trajectories, respectively, and the abbreviations Υ and Ω are

$$\begin{aligned}\Upsilon &= E^2M_0^2f(2E-qA_0) + (E-qA_0)^2\left[\frac{2f}{r^2}\left(\frac{dW}{d\theta}\right)^2 - E^2\right] - \frac{2f}{r^2}\left(\frac{dW}{d\theta}\right)^2\left[\frac{1}{r^2}\left(\frac{dW}{d\theta}\right)^2 + M_0^2\right] \\ &\quad - \frac{2f^2j^4}{r^4\sin^4(\theta)} - \frac{2fj^2}{r^2\sin^2(\theta)}\left[f\left(M_0^2 + \frac{1}{r^2}\left(\frac{dW}{d\theta}\right)^2\right) - (E-qA_0)^2\right],\end{aligned}$$

$$\Omega = f\left[(E-qA_0)^2 - f\left(M_0^2 + \frac{j^2}{r^2\sin^2(\theta)} + \frac{1}{r^2}\left(\frac{dW}{d\theta}\right)^2\right)\right]. \tag{12}$$

Since $f(r_+)=0$, there is a simple pole at the outer horizon, $r=r_+$. In this context, Eq. (11) can be calculated by using the residue theorem for semicircle. As $f \approx (r-r_+)f'(r_+)$ near the outer horizon, $R_{\pm}(r)$ are calculated as

$$R_{\pm}(r_+) = \pm i\pi\frac{\left(E - \frac{qQ}{r_+}\right)}{f'(r_+)}\left[1+\alpha\Sigma\right], \tag{13}$$

with the abbreviation Σ is

$$\Sigma = \frac{f'(r_+)\left(\frac{3E}{2} - \frac{qQ}{r_+}\right)\left[E\left(M_0^2 + \frac{1}{r_+^2}\left(\frac{dW}{d\theta}\right)^2\right) - \frac{2qQ}{r_+^3}\left(\frac{dW}{d\theta}\right)^2 + \frac{j^2f'(r_+)}{r_+^2\sin^2(\theta)}\left(E - \frac{2qQ}{r_+}\right)\right] - \frac{qQE^2}{r_+^2}\left(E - \frac{qQ}{r_+}\right)}{f'(r_+)\left(E - \frac{qQ}{r_+}\right)^2}. \tag{14}$$

The tunneling probabilities of particles crossing the outer horizon of a black hole are calculated by the following relations:

$$\begin{aligned}P_{out} &= \exp\left[-\frac{2}{\hbar}ImR_+(r_+)\right], \\ P_{in} &= \exp\left[-\frac{2}{\hbar}ImR_-(r_+)\right].\end{aligned} \tag{15}$$

Hence, the tunneling probability of a particle is

$$\Gamma = e^{-\frac{2}{\hbar}ImS} = \frac{P_{out}}{P_{in}} = e^{-\frac{E_{total}}{T_H}}, \tag{16}$$

where E_{total} and T_H are the total energy and Hawking temperature, respectively. Then, the modified Hawking temperature of the charged massive scalar particle, T_H^{KG}, is obtained as follows:

$$T_H^{KG} = \frac{T_H}{[1+\alpha\Sigma]}. \tag{17}$$

where T_H is the standard Hawking temperature of the black hole and its explicit expression is

$$T_H = \hbar\frac{f'(r_+)}{4\pi} = \frac{\hbar}{4\pi}\left[\frac{2M}{r_+^2} - \frac{2Q^2}{r_+^3} + \frac{2\Lambda r_+}{3} + \gamma\right]. \tag{18}$$

As seen in Eq. (17), if $\Sigma > 0$, the modified Hawking temperature of the charged massive scalar particle is lower than the standard Hawking temperature. On the other hand, in cases $\Sigma < 0$ and $\Sigma = 0$, the modified temperature will be higher than the standard temperature and equal to the standard temperature, respectively. Furthermore, since Σ is infinite in the case of $E = \frac{qQ}{r_+}$, the modified Hawking temperature becomes zero. Moreover, the modified Hawking temperature depends on not only the black hole but also on the mass, on the angular momentum, on the energy, and on the charge of the scalar particle. It is important to notice that the modified Hawking temperature of the charged massive scalar particle depends also on the polar angle θ and $\frac{dW}{d\theta}$. On the other hand, if $\alpha = 0$, the modified Hawking temperature of the scalar particle reduces to standard temperature [71–73].

3. Corrected Hawking temperature derived by tunneling of charged massive Dirac particle

In order to perform a discussion about Hawking temperature of the black hole in the presence of the GUP effect by considering tunneling process of a charged massive Dirac particle, we use the modified Dirac equation given in the following explicit form [51, 53, 64];

$$\begin{aligned} i\gamma^0(x)\partial_0\widetilde{\Psi} + i\gamma^j(x)\left(1-\alpha m_0^2\right)\partial_j\widetilde{\Psi} + i\alpha\hbar^2\gamma^j(x)\partial_j\left(\partial_i\partial^i\widetilde{\Psi}\right) - \frac{m_0}{\hbar}\left(1+\alpha\hbar^2\partial_i\partial^i - \alpha m_0^2\right)\widetilde{\Psi} \\ -i\gamma^\mu(x)\Gamma_\mu\left(1+\alpha\hbar^2\partial_i\partial^i - \alpha m_0^2\right)\widetilde{\Psi} - \gamma^\mu(x)\frac{q}{\hbar}A_\mu\left(1+\alpha\hbar^2\partial_i\partial^i - \alpha m_0^2\right)\widetilde{\Psi} = 0, \end{aligned} \tag{19}$$

where, q, $\widetilde{\Psi}$, m_0 and $\gamma^\mu(x)$ are the charge, the modified Dirac spinor, the mass of Dirac particle and the spacetime dependent Dirac matrices, respectively. $\Gamma_\mu(x)$ are the spin affine coefficients associated with spin-1/2 Dirac particle and are expressed in terms of metric tensor, $g_{\mu\nu}(x)$, Christoffel symbols, $\Gamma^\alpha_{\nu\mu}$, tetrads, $e^{(j)}_\nu(x)$, and spacetime-dependent Dirac matrices, $\gamma^\mu(x)$, as follows:

$$\Gamma_\mu(x) = \frac{1}{8}g_{\lambda\alpha}(e^i_{\nu,\mu}e^\alpha_i - \Gamma^\alpha_{\nu\mu})[\gamma^\lambda(x),\gamma^\nu(x)]. \tag{20}$$

Dirac matrices are:

$$\gamma^t = \frac{1}{\sqrt{f}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}, \gamma^r = i\sqrt{f}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}, \gamma^\theta = \frac{i}{r}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$

$$\gamma^{\phi}=\frac{i}{r\sin(\theta)}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}. \tag{21}$$

Considering the WKB approximation, we employ the ansatz for the spin-up Dirac spinor as follows:

$$\widetilde{\Psi}(x)=\exp\left(\frac{i}{\hbar}S\left(t,r,\theta,\phi\right)\right)\begin{pmatrix}A\left(t,r,\theta,\phi\right)\\0\\B\left(t,r,\theta\phi\right)\\0\end{pmatrix} \tag{22}$$

where the $A\left(t,r,\theta,\phi\right)$ and $B\left(t,r,\theta,\phi\right)$ are the functions of spacetime coordinates. Putting Eq. (22) into the Eq. (19), we can get Dirac equation as follows:

$$\begin{aligned}A\left[\alpha\frac{qA_0m_0^2}{\sqrt{f}}-m_0(1-\alpha m_0^2)-\frac{1}{\sqrt{f}}\left(\frac{\partial S}{\partial t}\right)-\alpha\sqrt{f}\left(qA_0+m_0\sqrt{f}\right)\left(\frac{\partial S}{\partial r}\right)^2-\frac{qA_0}{\sqrt{f}}\right]\\-A\left[\alpha\left(\frac{qA_0}{\sqrt{f}}+m_0\right)\left(\frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2+\frac{1}{r^2\sin^2(\theta)}\left(\frac{\partial S}{\partial\phi}\right)^2\right)\right]\\-B\left[i\sqrt{f}\left(1-\alpha m_0^2\right)\left(\frac{\partial S}{\partial r}\right)+i\alpha f^{3/2}\left(\frac{\partial S}{\partial r}\right)^3+\frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2\left(\frac{\partial S}{\partial r}\right)+\frac{1}{r^2\sin^2(\theta)}\left(\frac{\partial S}{\partial\phi}\right)^2\left(\frac{\partial S}{\partial r}\right)\right]=0,\end{aligned} \tag{23}$$

$$B\left[\frac{i}{r}\left(\frac{\partial S}{\partial\theta}\right)-\frac{1}{r\sin(\theta)}\left(\frac{\partial S}{\partial\phi}\right)\right]\left[1+\alpha\left[f\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2+\frac{1}{r^2\sin^2(\theta)}\left(\frac{\partial S}{\partial\phi}\right)^2-m_0^2\right]\right]=0, \tag{24}$$

$$\begin{aligned}B\left[-\alpha\frac{qA_0m_0^2}{\sqrt{f}}-m_0(1-\alpha m_0^2)+\frac{1}{\sqrt{f}}\left(\frac{\partial S}{\partial t}\right)+\alpha\sqrt{f}\left(qA_0-m_0\sqrt{f}\right)\left(\frac{\partial S}{\partial r}\right)^2+\frac{qA_0}{\sqrt{f}}\right]\\-B\left[\alpha\left(\frac{qA_0}{\sqrt{f}}+m_0\right)\left(\frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2+\frac{1}{r^2\sin^2(\theta)}\left(\frac{\partial S}{\partial\phi}\right)^2\right)\right]\\-A\left[i\sqrt{f}\left(1-\alpha m_0^2\right)\left(\frac{\partial S}{\partial r}\right)+i\alpha f^{3/2}\left(\frac{\partial S}{\partial r}\right)^3+\frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2\left(\frac{\partial S}{\partial r}\right)+\frac{1}{r^2\sin^2(\theta)}\left(\frac{\partial S}{\partial\phi}\right)^2\left(\frac{\partial S}{\partial r}\right)\right]=0,\end{aligned} \tag{25}$$

$$A\left[\frac{i}{r}\left(\frac{\partial S}{\partial\theta}\right)-\frac{1}{r\sin(\theta)}\left(\frac{\partial S}{\partial\phi}\right)\right]\left[1+\alpha\left[f\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2+\frac{1}{r^2\sin^2(\theta)}\left(\frac{\partial S}{\partial\phi}\right)^2-m_0^2\right]\right]=0, \tag{26}$$

where we consider terms only up to the first order in $\hbar$. From Eqs. (24) and (26), one can get the following relation;

$$\frac{i}{r}\left(\frac{\partial S}{\partial\theta}\right)-\frac{1}{r\sin(\theta)}\left(\frac{\partial S}{\partial\phi}\right)=0, \tag{27}$$

which implies

$$\frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2+\frac{1}{r^2\sin^2(\theta)}\left(\frac{\partial S}{\partial \phi}\right)^2=0. \tag{28}$$

The nontrivial solution of Eq. (27) is a complex function. This means that the angular part of the action does not contribute to the tunneling probability of Dirac particle. After putting Eq. (28) into Eqs. (23) and (25), the modified Hamilton–Jacobi equation for the charged massive Dirac particle is obtained from the matrix of coefficients of A and B:

$$\frac{1}{F}\left[\left(\frac{\partial S}{\partial t}\right)+qA_0\right]^2-m_0^2-f\left(\frac{\partial S}{\partial r}\right)^2+2\alpha\left[m_0^4-q^2A_0^2\left(\frac{\partial S}{\partial r}\right)^2+qA_0\left(\frac{\partial S}{\partial t}\right)\left(\frac{\partial S}{\partial r}\right)^2\right]$$
$$-2\alpha\left[\frac{q^2A_0^2m_0^2}{f}+\frac{qA_0m_0^2}{f}\left(\frac{\partial S}{\partial t}\right)+f^2\left(\frac{\partial S}{\partial r}\right)^4\right]=0. \tag{29}$$

Putting $S\left(t,r,\theta,\phi\right)=-Et+j\phi+R(r)+W(\theta)+C$ into Eq. (29) and solving it for the radial trajectory, $R_{\pm}(r)$, yields

$$R_{\pm}(r)=\pm\int\frac{\sqrt{(E-qA_0)^2-fm_0^2}}{f}\left[1+\alpha\frac{E(E-qA_0)\left[2fm_0^2-(E-qA_0)^2\right]}{f\left[(E-qA_0)^2-fm_0^2\right]}\right]dr, \tag{30}$$

where $R_{+}(r)$ and $R_{-}(r)$ represent the the trajectories of the outgoing and incoming Dirac particles, respectively. Integrating Eq. (30) near the event horizon by using the residue theorem for semi circle, we acquire

$$R_{\pm}(r_+)=\pm i\pi\frac{\left(E-\frac{qQ}{r_+}\right)}{f'(r_+)}\left[1+\alpha\Delta\right], \tag{31}$$

with the abbreviation Δ is

$$\Delta=\frac{E\left[\frac{3}{2}f'(r_+)m_0^2r_+^2-2qQ(E-\frac{qQ}{r_+})\right]}{f'(r_+)r_+^2\left(E-\frac{qQ}{r_+}\right)}. \tag{32}$$

Thus, the modified Hawking temperature of the charged massive Dirac particle, T_H^D, is calculated by inserting Eq. (31) into Eq. (16):

$$T_H^D=\frac{T_H}{\left[1+\alpha\Delta\right]} \tag{33}$$

in term of the standard Hawking temperature, T_H, given in Eq. (18). According to Eq. (33), since Δ is infinite in the case of $E=\frac{qQ}{r_+}$, the modified Hawking temperature of the charged massive Dirac particle becomes zero. Moreover, in the cases $\Delta>0$, $\Delta<0$ and $\Delta=0$, the modified Hawking temperature of the charged massive Dirac particle is lower than, higher than and equal to the standard temperature, respectively. We also see that the modified Hawking temperature of the black hole depends on the mass, charge and energy of Dirac particle. However, in contrast to the scalar particle case, the modified Hawking temperature of Dirac particle is independent of the angular momentum of Dirac particle. On the other hand, in the case of $\alpha=0$, the modified Hawking temperature of Dirac particle reduces to standard temperature [71–73].

4. Quantum gravity effect on the local stability of the black hole

At this stage of the study, under the quantum gravity effect, we would like to investigate the local stability criteria and the possible phase transition of the charged dRGT black hole in the framework of the tunneling of the both scalar and Dirac particles. It is well known that the information regarding the local stability/instability conditions and the associated phase transitions of a black hole can be extracted from its heat capacity. The heat capacity of a black hole can be computed in terms of the mass, M, and Hawking temperature, T_H, of the black hole by the following relation:

$$C_Q = \left(\frac{\partial M}{\partial T_H}\right)_Q . \tag{34}$$

The sign of the heat capacity determines whether a black hole is locally stable. If the heat capacity is positive, $C > 0$, the black hole is stable, otherwise (i.e. $C < 0$) it is unstable.

In order to determine the stability condition of the black hole under the tunneling process of the scalar particle, we visualize the behavior of the modified heat capacity, C_Q^{KG}, by putting Eqs. (7) and (17) into Eq. (34). As can be seen from Figure 1, the modified heat capacity is negative in the regions $0 < r_+ < 0.154526$ and $0.6 < r_+ < 0.855543$. Therefore, both very small and intermediate black holes are thermally unstable. On the other hand, in the regions $0.154526 < r_+ < 0.6$ and $0.855543 < r_+$, the modified heat capacity is positive, which implies that both small and large black holes are locally stable. Moreover, the modified heat capacity diverges at points $r_+ = 0.154526$, $r_+ = 0.6$ and $r_+ = 0.855543$. This situation indicates that the black hole may undergo only second-type phase transition [74–76] to becomes thermally stable. Consequently, very small unstable black hole changes to small stable black hole at point $r_+ = 0.154526$, the small stable black hole changes to intermediate unstable one at point $r_+ = 0.6$, and the intermediate unstable black hole evolves into a large stable black hole at point $r_+ = 0.855543$.

During the tunneling of the charged massive Dirac particle, the behavior of the modified heat capacity, $C_Q^{D\,rac}$, is demonstrated in Figure 2. The modified heat capacity calculated by using the modified Hawking temperature of the charged massive Dirac particle is negative in the region $0 < r_+ < 0.361054$. Therefore, the black hole is locally unstable in this region. On the other hand, in the region $0.361054 < r_+$, the modified heat capacity is positive, so that the black hole is locally stable in this region. Furthermore, the modified heat capacity diverges at point $r_+ = 0.361054$. Therefore, during the tunneling process of the charged massive Dirac particle, the black hole may undergo only a second-type phase transition to become thermally stable. Consequently, a small unstable black hole evolves into a large stable black hole at point $r_+ = 0.361054$.

In the absence of the quantum gravity, i.e. $\alpha = 0$, the standard heat capacity is figure out in Figure 3. As displayed in the Figure 3, while the standard heat capacity is negative in the regions $0 < r_+ < 0.156393$ and $0.231062 < r_+ < 0.583566$, it is positive in the regions $0.156393 < r_+ < 0.231062$ and $0.583566 < r_+$. On the other hand, while the standard heat capacity is zero at the point $r_+ = 0.156393$, it diverges an the points $r_+ = 0.231062$ and $r_+ = 0.583566$. Therefore, in the absence of the quantum gravity effect, the charged dRGT black hole may undergo both first-type and second-type phase transitions [74–76] to become thermally local stable. Consequently, very small unstable black hole might undergo a first-type phase transition to evolves to small stable black hole. Also, intermediate unstable black hole might undergo a second-type phase transition to evolves to large stable black hole.

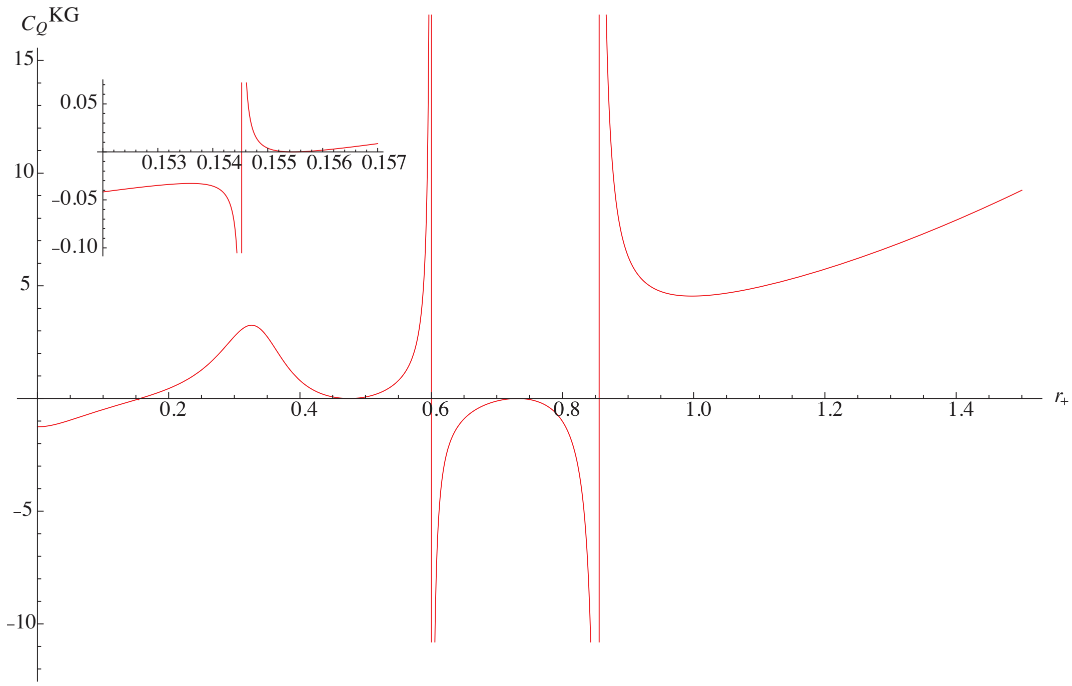

Figure 1. Modified heat capacity (C_Q^{KG}) versus event horizon (r_+). We have taken $E = j = \hbar = M_0 = 1, q = 3, Q = 0.2, \theta = \theta_0 = \frac{\pi}{2}, \epsilon = 1.6, \Lambda = 6.6, \gamma = -3.6, \alpha = 0.1$.

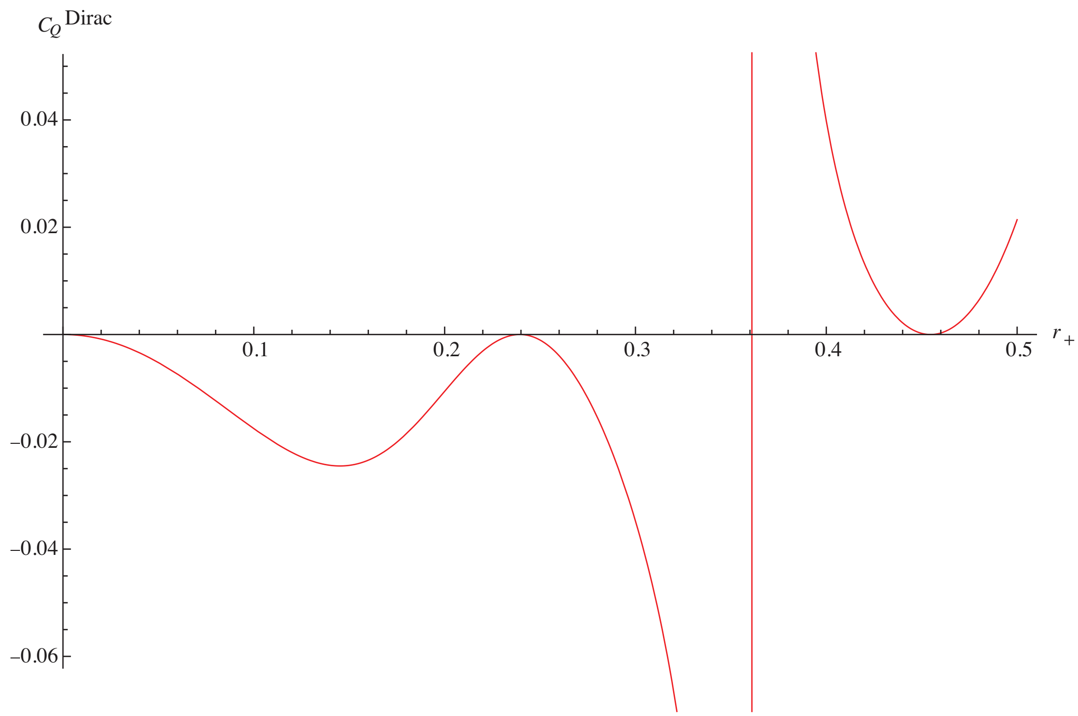

Figure 2. Modified heat capacity (C_Q^{Dirac}) versus event horizon (r_+). We have taken $E = \hbar = m_0 = 1, q = 3, Q = 0.2, \epsilon = 1.6, \Lambda = 6.6, \gamma = -3.6, \alpha = 0.1$.

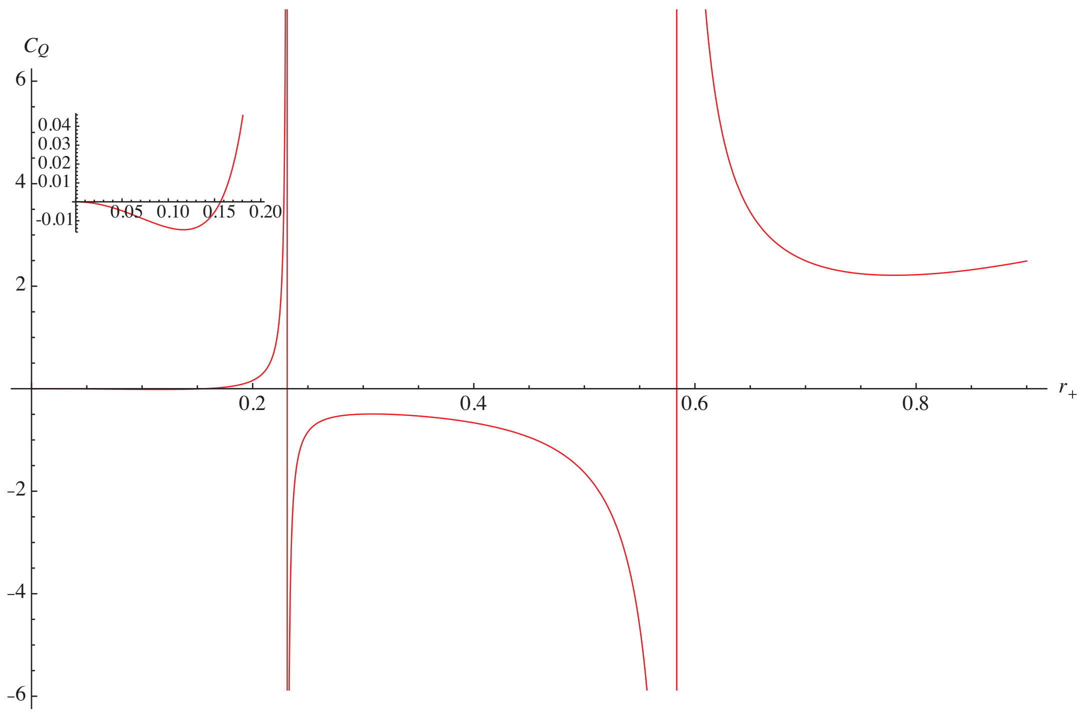

Figure 3. Standard heat capacity (C_Q) versus event horizon (r_+). We have taken $\hbar = 1, Q = 0.2, \epsilon = 1.6, \Lambda = 6.6, \gamma = -3.6$.

5. Conclusion

In this paper, we have investigated the quantum gravity effect on the thermodynamic quantities of the (3+1) dimensional charged dRGT black hole by applying the particle tunneling process. Firstly, we have considered the modified Klein–Gordon and Dirac equations to discuss the tunneling process of the scalar and Dirac particles from the black hole, respectively. The tunneling probability and corresponding modified Hawking temperatures of each particle were derived. To discuss the quantum gravity effect on the local stability of the black hole, we also calculated its modified heat capacity. The important results of the study are as follows:

- In the absence of the quantum gravity effect, i.e. $\alpha = 0$, the standard Hawking temperature of the (3+1) dimensional charged dRGT black hole is recovered.
- In the presence of the quantum gravity effect, quantum numbers of the scalar particle such as mass, energy, charge, and the angular momentum affect the modified Hawking temperature of the charged dRGT black hole. As can be seen from Eq. (17), the modified Hawking temperature also varies with the polar angle θ and $\frac{dW}{d\theta}$. Furthermore, in the cases $\Sigma > 0$, $\Sigma < 0$ and $\Sigma = 0$, it is lower than, higher than and equal to the standard Hawking temperature, respectively. Moreover, in the case of $E = \frac{qQ}{r_+}$, the modified Hawking temperature becomes zero.
- In the cases $\Delta > 0$, $\Delta < 0$ and $\Delta = 0$, the modified Hawking temperature of the charged massive Dirac particle will be lower than, higher than and equal to the standard temperature, respectively [see Eq. (33)]. Additionally, it will be vanished for the case $E = \frac{qQ}{r_+}$. The corrected Hawking temperature depends on the mass, charge and event horizon radius of the charged dRGT black hole, and also on the

mass, charge and energy of the tunneling Dirac particle. However, it is independent of the angular parts. In the absence of the quantum gravity effect, i.e $\alpha = 0$, the modified Hawking temperature reduces to the standard Hawking temperature.

- During the tunneling processes of the both scalar and Dirac particles in the context of quantum gravity, the charged dRGT black hole might undergo only second-type phase transition to be thermodynamically local stabilized (see Figures 1 and 2). On the other hand, in the absence of the quantum gravity effect, i.e $\alpha = 0$, the charged dRGT black hole might undergo the both first-type and second-type phase transitions to become locally stable (see Figure 3).

Finally, in the context of quantum gravity, it is important to emphasize that the thermodynamic quantities of the charged dRGT black hole are affected by quantum numbers [energy, mass, total angular momentum (spin+orbital), charge] of the particles radiating from it. The present study, in the presence of the quantum gravity effect, includes the calculation of the Hawking temperature of the dRGT black hole using the Hamilton–Jacobi method and discussion of its local stability. We plan to carry on this study to investigate some other thermodynamics quantities such as entropy, Gibbs, and Helmholtz free energies of the charged dRGT black hole together with some other black holes. Such an attempt may reveal some new information about both local and global stabilities of black holes. This is going to be the subject of our next study in the near future.

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