Modern quantum mechanics, why we need it, how to use it?

Vahid KARIMIPOUR[*]

Department of Physics, Sharif University of Technology, Tehran, Iran

Abstract: A new field of research centered on quantum science and technology is emerging. This rapidly developing field deals with the study and control of single quantum entities, i.e. atoms, ions, photons, which are no longer isolated but are parts of a larger system. Manipulation of these quantum entities forms the basis of modern quantum technology, from quantum computation to quantum communication and quantum sensing. One, therefore, needs to reconsider the basic concepts of state, evolution, and measurement for these so-called open quantum systems. This is a pedagogical review paper on the rudiments of modern quantum mechanics, which keeps citations to the literature to a minimum and assumes only a knowledge of elementary quantum mechanics. It is meant to prepare the reader for studying the modern research papers at the forefront of this field.

Keywords: Open quantum systems, density matrices, quantum channels

1. Introduction

The twentieth century witnessed the birth of the concept of quantum of energy in 1900, the development of the theory of quantum mechanics in 1927, and finally its application to many branches of science and technology, which started right afterward, has vastly continued up until now. One can fairly say that this was all about quantum mechanics in the bulk, that is, both the theory and its application to other branches of science and technology rested on the presumption that quantum mechanical analysis of atoms, when extended to a macroscopic aggregate of particles by statistical means, produces results which precisely describe the behavior of bulk matter. Thus, quantum mechanics could describe the behavior of electrons in a solid, molecules in a gas, ions in a plasma, and atoms in a Fermi liquid. This tremendous success prevailed throughout the previous century, and it has continued and will continue into the present century as well.

However, over the past five decades, extraordinary developments in atomic, molecular, and optical physics, especially the development of high-intensity short-pulsed lasers, precise measurements of frequencies, and attaining extremely low temperatures, has completely opened a new window for exploration of microscopic phenomena, namely the ability to manipulate individual microscopic objects like atoms, electrons, and photons. This has led in many ways to a completely new perspective on quantum mechanics. First, we will be able to explore the limitations of quantum theory when applied to individual microscopic entities. For example, extremely precise measurements have been performed to set the scale at which the superposition principle is still valid [1] and second, we will be able to envisage completely new applications, like quantum computation [2], quantum communication [3] and quantum simulation [4]. This development has led to a quite new look at the formalism of quantum mechanics, different from what we all had learned in our university curricula. Needless

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[*]Correspondence: vahid.karimipour@gmail.com

to say, this is not by any means a change of the principles of quantum mechanics or a new interpretation of these principles, but only a new perspective that suits the new world of individual microscopic particles. But why this new perspective is needed? To answer this question, it is instructive to remind ourselves of the basic principles of quantum mechanics:

- **A-** The state of a quantum system is defined by a vector $|\psi\rangle$ (actually a ray) in a Hilbert space H,

- **B-** The state $|\psi\rangle$ evolves by a unitary operation $|\psi(t)\rangle = U(t)|\psi(0)\rangle$, where $U(t)$ is a unitary operator. Equivalently the continuous evolution of a state is governed by the Schrodinger equation: $i\hbar\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle$, where H is a Hermitian operator, called the Hamiltonian.

- **C-** To any observable K, there corresponds a Hermitian operator $\hat{K}$ whose eigenvalues are the values we observe in the laboratory when the observable K is measured and each value k_i is obtained by the probability $P_i = |\langle k_i|\psi\rangle|^2$ in which $|k_i\rangle$ is the eigenstate of $\hat{K}$ corresponding to the eigenvalue k_i.

The crucial point is that these principles have been formulated for an isolated system. An isolated system is described by a state vector and undergoes a unitary evolution. In reality, a system is never isolated, and this lack of isolation becomes more pronounced when we focus on a small system consisting of a few atoms, spins, or photons. It is for these reasons that we need a reexamination of these definitions and concepts, which is the aim of this review article. This re-examination shapes the structure of this review. In the subsequent sections, we examine these three principles one by one and see how they are adapted to the study of individual quantum systems, which are no longer isolated but are part of a larger system. Therefore, the paper consists of three main sections, which consider states, dynamics, and measurement in order. The paper is written in a pedagogical and self-contained manner, and no prior knowledge or familiarity with quantum information theory is required except the basic concepts of quantum mechanics.

To keep the paper short, I have left out many important topics from the vast field of quantum computation and quantum information and have focused only on the above three main concepts, namely state, evolution, and measurement. Once the reader grasps these concepts, it is fair to say that he or she is familiar with the basic jargon of the field and can move on to more specialized texts or papers. To proceed further, it is helpful to first gather our notations and conventions.

1.1. Notations and conventions

In this section, we set up notations and conventions that we will use in the rest of this paper. A Hilbert space is denoted by H and the linear space of operators acting on this Hilbert space by $L(H)$. We take all the Hilbert spaces to be finite-dimensional and define a basis for them to be $\{|0\rangle, |1\rangle, \cdots |d-1\rangle\}$ where $d = dim(H)$. When $d = 2$, we call the corresponding object a qubit, which is a shorthand for quantum bit. The basis $\{|0\rangle, |1\rangle\}$ for a qubit is usually called the computational basis, since they are the quantum version of $\{0, 1\}$, which are the states of a classical computational bit. A basis for $L(H)$ is then given by $\{|i\rangle\langle j|, i, j = 0, \cdots d - 1\}$. A Hermitian operator $\hat{A}$ in $L(H)$ is called positive if all its eigenvalues are non-negative, or if $\langle v|\hat{A}|v\rangle \geq 0$ for all $|v\rangle \in H$. The set of positive operators on $L(H)$, which we denote by $L^+(H)$, is no longer a linear subspace of $L(H)$, for the obvious reason that if $\hat{A}$ is positive, then $-\hat{A}$ is not positive. Technically this set is a cone, i.e. if $\hat{A} \in L^+(H)$, then $\lambda\hat{A} \in L^+(H)$ for every positive λ, Figure 1.
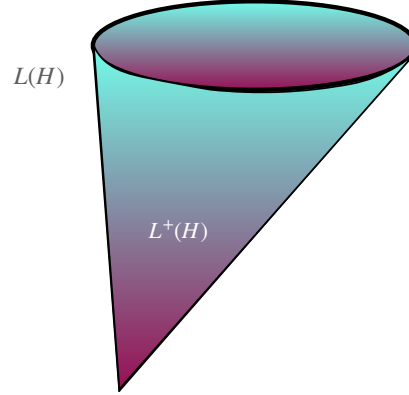
Figure 1. The set of all operators on a Hilbert space H, denoted by $L(H)$, is itself a vector space. However, the set of positive operators on H, denoted by $L^+(H)$, is only a cone.

2. States

In elementary textbooks of quantum mechanics and elementary exposition of postulates of quantum mechanics, states of quantum systems are usually denoted by vectors in a Hilbert space like $|\psi\rangle$ or $|\phi\rangle$. Such a state is the result of careful preparation through a series of filtering operations or measurements. For example, a photon that has passed through a polaroid or a nuclear spin that has passed through a Stern-Gerlach apparatus is in such a state. These states are called pure states. Spin states like $|0\rangle$, $|1\rangle$ or $\alpha|0\rangle + \beta|1\rangle$ are all pure states. However, in most cases, we do not know exactly the state of a quantum system. A beam of photons may have any direction of polarization, a beam of atoms coming out of a furnace can have any direction of spin or angular momentum. In such cases, we only know the probability of the system being in a pure state. If the quantum system is in the state $|\psi_i\rangle$ with probability p_i, then the result of the measurement of any observable M on this quantum system is given by

$$\langle M \rangle = \sum_i p_i \langle \psi_i | \hat{M} | \psi_i \rangle, \tag{2.1}$$

which can be written as

$$\langle M \rangle = \text{Tr}\left(\hat{M} \sum_i p_i |\psi_i\rangle\langle\psi_i| \right) =: \text{Tr}(\hat{M}\rho) \tag{2.2}$$

where the last equality defines the density matrix

$$\rho := \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{2.3}$$

This density matrix has the following properties:

$$\rho^\dagger = \rho, \quad \text{Tr}\rho = 1, \quad \rho \geq 0. \tag{2.4}$$

The last inequality means that for any vector $|v\rangle \in H$, we have

$$\langle v|\rho|v\rangle \geq 0 \qquad \forall v. \tag{2.5}$$

A pure state is thus given by

$$\rho = |\psi\rangle\langle\psi|, \tag{2.6}$$

which has the property $\rho^2 = \rho$. The above intuitive characterization of a density matrix rests on the idea of a quantum system whose pure state is only known statistically. There is another equivalent way to look at a density matrix, namely as the state of a system that is not isolated, as alluded to in the introduction e.g., an atom in an environment. The environment can be as small as another atom or group of atoms in an optical lattice or as large as a photonic bath with billions of photons moving around. Consider, for example, a bi-partite system AB which is in a pure state $|\psi\rangle_{AB} \in H_A \otimes H_B$. Where A denotes the system and B denotes the environment. The basis of H_A is given by $\{|i\rangle, i = 1 \cdots dim(H_A)\}$ and that of H_B is given by $\{|\mu\rangle, \mu = 1 \cdots dim(H_B)\}$. A general state is given by

$$|\psi\rangle_{AB} = \sum_{i,\mu} \psi_{i\mu}|i, \mu\rangle. \tag{2.7}$$

Then, any measurement of an observable on H_A is given by

$$\langle M\rangle_A = \langle\psi|\hat{M} \otimes I|\psi\rangle = \mathrm{Tr}\left((\hat{M} \otimes I)|\psi\rangle_{AB}\langle\psi|\right) = \mathrm{Tr}_A(\hat{M}\rho_A) \tag{2.8}$$

where Tr_A means that we take only on the first subspace and

$$\rho_A = tr_B\left(|\psi\rangle_{AB}\langle\psi|\right) \tag{2.9}$$

or in components

$$(\rho_A)_{ij} = \sum_\mu \psi_{i\mu}\psi_{j\mu}^*. \tag{2.10}$$

It is now easy to show that ρ_A is a matrix with the same properties as in (2.4). Therefore, we come to the definition of a quantum state:

- ■ **A1:** The state of a quantum state is defined by a Hermitian and Positive matrix ρ whose trace is equal to one. If $\rho^2 = \rho$, such a state is called a pure state.

2.1. Qubits and the Bloch sphere

In two dimensions, the space of density matrices has a particularly simple form. Any Hermitian matrix with unit trace can be expanded as

$$\rho = \frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \tag{2.11}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices. The eigenvalues of this matrix are given by $\lambda_\pm = \frac{1}{2}(1 \pm |\mathbf{r}|)$, which shows that $|\mathbf{r}| \leq 1$, in order for ρ to be positive. This implies that there is a one-to-one

correspondence between the density matrices of qubits and the points inside a two-dimensional sphere. Points on the boundary of this sphere correspond to pure states (since one of the eigenvalues are zero and the other is one), and the center of the sphere corresponds to the maximally mixed state $\frac{I}{2}$. The purity of a state ρ is defined as $P(\rho) := \text{Tr}(\rho^2)$. The purity of each point inside the sphere is $P(\rho) = \lambda_+^2 + \lambda_-^2 = \frac{1}{2}(1 + r^2)$, which shows that as we go along a radius toward the boundary of the sphere, the purity increases from the minimum value of zero for the maximally mixed state in the center to the maximum value of one for the pure state at the boundary. Finally, one can show that the pure state $\rho = \frac{1}{2}(I + \mathbf{n} \cdot \boldsymbol{\sigma})$ is nothing but the state $|\mathbf{n}\rangle\langle\mathbf{n}|$, where $|\mathbf{n}\rangle$ is the spin state in the $\mathbf{n}$ direction. i.e. $\mathbf{n} \cdot \boldsymbol{\sigma}|\mathbf{n}\rangle = |\mathbf{n}\rangle$. Figure 2.
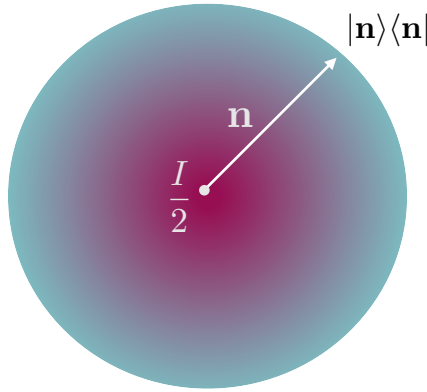
Figure 2.　Bloch sphere. Every point in this sphere corresponds to a qubit state. The boundary is the set of pure states. The center corresponds to the maximally mixed state $\rho = \frac{I}{2}$.

2.2. Higher dimensional states

Higher dimensional states can be expanded in a similar way by invoking a basis of $d^2 - 1$ Hermitian traceless matrices, which we normalize as

$$\langle \Gamma_i, \Gamma_j \rangle = (d^2 - d)\delta_{ij}I, \qquad i, j = 1 \cdots d. \tag{2.12}$$

Here, we are taking the dimension of the Hilbert space to be $dim(H) = d$. The states are called qudits for short. A qudit density matrix is expanded as

$$\rho = \frac{1}{d}(I + \mathbf{r} \cdot \Gamma), \tag{2.13}$$

where $\mathbf{r}$ is now a $d^2 - 1$ dimensional vector. In contrast to the qubit case, very little is known about the topology and geometry of the set of qudit states, even when $d = 3$. The reason is the high dimension of $L(H_d)$, which makes the evaluation of eigenvalues and, hence, testing the positivity of the matrices very difficult. We only know that the set $D(H_d)$ is still a convex subset of $L(H_d)$, meaning that if ρ_1 and ρ_2 are two points in this subset, then the line joining them, $\rho(\lambda) = \lambda\rho_1 + (1 - \lambda)\rho_2$ also belongs to $D(H_d)$.

One finds from (2.12) that

$$\text{Tr}(\rho^2) = \frac{1}{d}\big[1 + (d - 1)\mathbf{r} \cdot \mathbf{r}\big]. \tag{2.14}$$

This means that any state, which is pure and, hence, has the property $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho) = 1$, satisfies $\mathbf{r} \cdot \mathbf{r} = 1$ and lies on the $d^2 - 2$ dimensional sphere S_{d^2-2}. However, this is conditioned on the matrix to be a state, that is, it should be positive. There are many points on this sphere, which are not density matrix at all. An example clarifies this point. Consider the matrix $H = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$. The matrix $\rho_1 = \frac{1}{3}(I + H) = |3\rangle\langle 3|$ is a pure state, while the matrix $\rho_2 = \frac{1}{3}(I - H)$ is not a state at all. This is also evident if we count the number of parameters. A pure state $|\psi\rangle = \sum_{i=1}^{d} \psi_i |i\rangle$ has $2d - 2$ real parameters, which is much less than the dimension of the sphere S_{d^2-d}, unless $d = 2$, which is the case of qubits which we saw before. The situation is depicted in Figure 3.
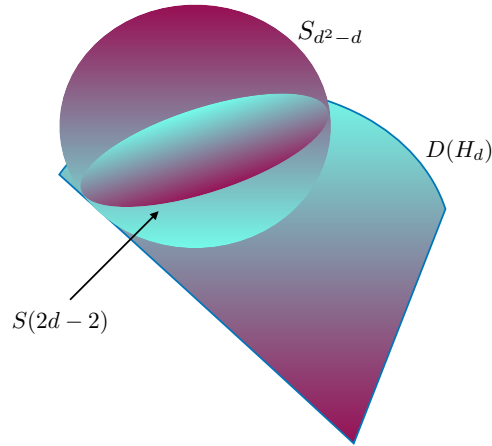
Figure 3. A highly schematic relation between three different sets. The space of qudits $D(H_d)$. The sphere S_{2d-2} is the set of pure states. Part of the sphere S_{d^2-d} lies within the set of states and part of it is outside. In two dimensions, that is for qubits, we have $S_{2d-2} = S_{d^2-d} = \partial D(H_d)$.

2.3. Entangled states

Since 1935 and the seminal papers of Schrodinger [5] and Einstein, Podolsky and Rosen [6], entanglement has been one of the central topics of discussion in quantum theory. Almost all these discussions have been centered around the non-local properties that entangled states bring about for quantum mechanics. A close examination of these non-local properties led Einstein, Podolsky, and Rosen to claim that quantum theory is not yet a complete theory for describing real elements of nature. The culmination of this line of research was the proposal of Bell-inequality [7] in 1965 and its experimental violation [8] in the '70s. It has only been recently since the 1990s that entanglement has been recognized as a resource in quantum technology, and, like any other resource, investigation of its formal and quantitative properties has now come to the forefront of attention. In this section, we will have a very brief discussion of these aspects.

Consider a system of two qubits. A general pure state of this system is written as

$$|\psi\rangle_{AB} = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle. \tag{2.15}$$

Such a state is of product or disentangled form if it can be written as $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$ or $|\psi\rangle = |\phi_1\rangle|\phi_2\rangle$ for

short, where $|\phi_1\rangle = \alpha|0\rangle + \beta|1\rangle$ and $\phi_2\rangle = \gamma|0\rangle + \delta|1\rangle$ are one-qubit states. A simple calculation shows that a state is disentangled only if the condition $ad - bc = 0$ holds. In the space of all two-qubit states parameterized by the complex numbers a, b, c and d, this condition defines a hyperplane, which shows that disentangled states form a subset of measure zero in this space. Thus, mathematically entangled states are the rule rather than the exception. That is, if we randomly, by some computer code, generate a two-qubit state, it will be entangled with almost unit probability. This is also the case from the physical point of view since even if we very carefully prepare two qubits in a product state, ubiquitous interactions in the environment quickly entangle them. We are now faced with the problem of defining a measure of entanglement. A handy measure is $C(\psi) = 2|ad - bc|$, which is zero for product states and one for maximally entangled Bell states like $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. As much as this measure seems plausible, one has still to justify it on physical and operational grounds. In all quantum communication protocols, it is the maximally entangled or Bell states that enable us to do extraordinary tasks, which are otherwise impossible by classical means. Therefore, it is plausible that we ascribe the value of 1 (ebit) to the Bell states and compare all the other states with Bell states. Like any other measure, the basic idea for assessing the amount of entanglement of a state like (2.15) is to see to what extent it can be converted to a Bell state by Local Operations and Classical Communications (LOCC). The constraint of LOCC is very important here, since if the two qubits are allowed to be in one single lab in one location, then one can do unitary operations on both of them and entangle them. Here, we want to see how much entanglement we can distill out of a given state if the two qubits are far apart.

Consider a state like (2.15). If we have a number of N states of this form, we can ask how many Bell pairs we can distill out of these states by LOCC. It has been shown [9] that, in the limit of N, a large (indeed $N \longrightarrow \infty$) number of $NS(\rho_A) = NS(\rho_B)$ Bell states can be distilled, where $S(\rho)$ is the von-Neumann entropy, which is defined as

$$S(\rho) = -Tr(\rho \log \rho).$$

For the state (2.15), a simple calculation shows that the von-Neuman entropy is equal to

$$S(\rho_A) = S(\rho_B) = -\frac{1+p}{2} \log \frac{1+p}{2} - \frac{1-p}{2} \log \frac{1-p}{2} \tag{2.16}$$

where $p = \sqrt{1 - C(\psi)^2}$. For a product state, it turns out that $C = 0$, and, hence, the von-Neumann entropy vanishes, while for a Bell state $C = 1$, which leads to the maximum value of one for the von-Neumann entropy. This shows that the handy measure $C(\psi)$ is indeed a very good measure of entanglement of a pure state. Finally, we should stress that this is all about entanglement of pure states. A discussion of entanglement of mixed states will extend this review beyond its short size limit.

2.4. Similarity of two states

The usual definition of the similarity or overlap between two states is given by $F(\phi, \psi) := |\langle \phi | \psi \rangle|^2$. Here F stands for fidelity or similarity of two states. When we are dealing with mixed states, this definition should be revised. Consider the case where one of the states is pure and the other is mixed, which we denote respectively by $|\phi\rangle$ and ρ. Let ρ have a decomposition $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Then, a natural definition for the fidelity of ρ

and ϕ is to take the average fidelity of all the states ψ_i with ϕ and define

$$F(\phi, \rho) = \sum_i p_i |\langle \phi | \psi_i \rangle|^2. \tag{2.17}$$

In this form, it seems to depend on the particular decomposition that we choose for ρ, which makes it an undesirable definition. However, we can re-write it as

$$F(\phi, \rho) = \sum_i p_i |\langle \phi | \psi_i \rangle|^2 = \sum_i p_i \langle \phi | \psi_i \rangle \langle \psi_i | \phi \rangle = \langle \phi | \rho | \phi \rangle, \tag{2.18}$$

showing explicitly that it is independent of any particular decomposition of ρ. We can rewrite this as $F(\phi, \rho) = Tr(|\phi\rangle\langle\phi|\rho)$ and, hence, may be tempted to extend this definition to the fidelity of two mixed states in the following form

$$F_1(\rho, \sigma) := Tr(\rho\sigma). \tag{2.19}$$

However, this definition is obviously problematic. We expect that the fidelity of any state with itself be the maximum value of unity, while according to this definition, except for pure state, the fidelity of any state with itself is less than 1. In fact for a general qubit state $\rho = \frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma})$, one finds

$$F_1(\rho, \rho) = Tr(\rho^2) = \frac{1}{2}(1 + \mathbf{r} \cdot \mathbf{r}) \tag{2.20}$$

which is obviously less than 1 except for pure states. To overcome this shortcoming, we can improve the definition and try the following

$$F_2(\rho, \sigma) = Tr(\sqrt{\rho}\sqrt{\sigma}). \tag{2.21}$$

Now the shortcoming is removed, and we have $F_2(\rho, \rho) = Tr(\rho) = 1$. However, this definition is also problematic. The point is that while $\sqrt{\rho}$ and $\sqrt{\sigma}$ are Hermitian and positive, their product does not need to be even Hermitian. Therefore, we try our final definition, which is

$$F(\rho, \sigma) = Tr(|\sqrt{\rho}\sqrt{\sigma}|). \tag{2.22}$$

Since $|A| = \sqrt{AA^\dagger}$, this is equivalent to

$$F(\rho, \sigma) = Tr\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right). \tag{2.23}$$

Despite its apparent asymmetry, it can be shown that this is really symmetric, that is, $F(\rho, \sigma) = F(\sigma, \rho)$. This concludes our brief description of states in quantum theory.

3. Dynamics

Here again, we can invoke two different points of view for the study of quantum dynamics. They are similar to those, which we used for the description of quantum states. The first one is based on our ignorance of exactly what kind of unitary evolution our quantum state has gone through, and the second one is based on

the dynamics of a system that is part of a bigger system that is undergoing a unitary evolution. Both points of view lead to the same formalism, which is then generalized to an even more general setting.

Suppose an experimentalist whom we call Alice sends a bunch of photons into an optical fiber, which is to be received by another experimentalist far away whom we call Bob. The photons usually undergo various types of disturbances inside the cable. There are stress and strain inside the cable, and the polarization of the photons may change in various unknown ways. In its simplest form, the action of the cable on the state of photons can be described as follows:

$$\mathcal{E}(\rho) = \int d\theta P(\theta) U(\theta) \rho U^{\dagger}(\theta) \tag{3.1}$$

where $U(\theta)$ is the rotation operator of the polarization vector in the plane perpendicular to propagation, θ is the angle of rotation and $P(\theta)$ is the probability of rotation by angle θ. Note that even if Alice sends a pure state $|\psi\rangle\langle\psi|$, Bob will receive a mixed state. To be more specific, let $U(\theta)$ be a phase kick $U(\theta) = R_z(\theta) = e^{i\theta\sigma_z}$ and let these phase kicks be applied with a Gaussian probability distribution $P(\theta) = \frac{1}{\lambda\sqrt{4\pi}}e^{-\frac{\theta^2}{4\lambda}}$.

$$\mathcal{E}(\rho) = \frac{1}{\lambda\sqrt{4\pi}} \int R_z(\theta)|\psi\rangle\langle\psi|R_z^{\dagger}(\theta)e^{-\frac{\theta^2}{4\lambda}}\,d\theta. \tag{3.2}$$

Let the input state ρ be a pure state $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = a|0\rangle + b|1\rangle$. Then we will have

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \qquad\qquad |\psi\rangle\langle\psi| = \begin{pmatrix} a\bar{a} & a\bar{b} \\ b\bar{a} & b\bar{b} \end{pmatrix} \tag{3.3}$$

The action of one phase kick on this state is given by

$$R_z(\theta)\rho R_z^{\dagger}(\theta) = \begin{pmatrix} a\bar{a} & a\bar{b}e^{i\theta} \\ b\bar{a}e^{-i\theta} & b\bar{b} \end{pmatrix} \tag{3.4}$$

and the full action of all phase kicks yields the output state as

$$\begin{aligned}
\mathcal{E}(\rho) &= \frac{1}{\sqrt{4\pi}\lambda} \int_{-\infty}^{\infty} \begin{pmatrix} a\bar{a} & a\bar{b}e^{i\theta} \\ b\bar{a}e^{-i\theta} & b\bar{b} \end{pmatrix} e^{-\frac{\theta^2}{4\lambda}}\,d\theta \\
&= \begin{pmatrix} a\bar{a} & a\bar{b}e^{-\lambda} \\ b\bar{a}e^{-\lambda} & b\bar{b} \end{pmatrix}.
\end{aligned} \tag{3.5}$$

Thus, the initial pure state becomes a mixed state, and for large λ, the final state looses its anti-diagonal elements and, hence, its coherence altogether. This is an example of what we call a quantum channel or in more technical term a completely positive trace-preserving (CPT). Complete positivity is a very important restriction on quantum operations, which will be described shortly. In general a CPT map has the form

$$\mathcal{E}(\rho) = \sum_m K_m \rho K_m^{\dagger} \tag{3.6}$$

where K_m's are called Kraus operators of the map, and are constrained by

$$\sum_m K_m^{\dagger} K_m = I. \tag{3.7}$$

Note that the operators K_m need not be unitary themselves, and only the above condition is necessary. This is a generalization of the unitary evolution of the density matrix. In the latter case, we have only one Kraus operator, which, by necessity, due to (3.7) is a unitary matrix. We can now readily verify that a quantum channel or CPT retains all the properties of a density matrix, that is, it retains the Hermiticity, positivity and unit trace of the input state, or, in short, it maps density matrices to density matrices. Condition (3.6) is necessary for the trace-preserving property. One can rewrite (3.1) in the form

$$\mathcal{E}(\rho) = \sum_m \mathrm{Tr}(K_m \rho K_m^\dagger) \frac{K_m \rho K_m^\dagger}{\mathrm{Tr}(K_m \rho K_m^\dagger)} =: \sum_m P_m \rho_m, \tag{3.8}$$

where $P_m := \mathrm{Tr}(K_m \rho K_m^\dagger)$ is a probability, and $\rho_m := \frac{K_m \rho K_m^\dagger}{\mathrm{Tr}(K_m \rho K_m^\dagger)}$ is a state. In this form, the output state is interpreted as a mixture of states with definite probabilities. It is as if the quantum channel has incurred error operators K_m on the state with probabilities P_m. When we think about it in this way, a quantum channel is nothing but the presentation of all the possible errors that can occur and spoil a quantum state when it passes through a medium, either through space (i.e. optical fiber or air...) or time (i.e. when we leave a state on an ion trap or a quantum register for a while.)

Alternatively, we can view the dynamics as the evolution of a quantum system that is part of a bigger system, where the latter undergoes a unitary evolution. Suppose that the system A is part of a larger system AB, which undergoes a unitary transformation U. It is plausible that the system AB is in a pure product state $|\psi\rangle|e\rangle$ where $|\psi\rangle$ is the state of the system A and $|e\rangle$ is an arbitrary state of the environment B. The reason is that when we prepare or measure the state of the system A, both the system and the environment project into their own states. We take the basis of the Hilbert space H_A to be $\{|i\rangle, i = 1 \cdots d\}$ and the basis of the Hilbert space H_B to be $\{|\mu\rangle = 1 \cdots n\}$. Under the unitary dynamics, this product state changes as

$$\rho| \otimes |e\rangle\langle e| \longrightarrow U(\rho \otimes |e\rangle\langle e|)U^\dagger. \tag{3.9}$$

The problem is that while the initial state is in a product form, the final state is no longer product and almost always is in an entangled state. The final state of the system is, therefore, in a mixed state, which can be written as

$$\begin{aligned}
\mathcal{E}(\rho) &= \mathrm{Tr}(U(\rho \otimes |e\rangle\langle e|)U^\dagger) \\
&= \sum_{\mu=1}^{n} \langle\mu|U(\rho \otimes |e\rangle\langle e|)U^\dagger|\mu\rangle \\
&= \sum_{\mu=1}^{n} \langle\mu|U|e\rangle \rho \langle e|U^\dagger|\mu\rangle \\
&= \sum_{\mu=1}^{n} K_\mu \rho K_\mu^\dagger, \tag{3.10}
\end{aligned}$$

where the Kraus operators are $K_\mu = \langle\mu|U|e\rangle$. Note that we have suppressed the subscripts to avoid cluttering of notation. In a more detailed notation, this should be written as $K_\mu = \langle\mu_B|U_{AB}|e_B\rangle$, which makes it manifest that K_μ is an operator acting on the Hilbert space H_A. One can now check that

$$\sum_\mu K_\mu^\dagger K_\mu = \sum_\mu \langle e|U^\dagger|\mu\rangle\langle\mu|U|e\rangle = \langle e|U^\dagger U|e\rangle = I_A,$$

where I_A is the identity operator on the Hilbert space H_A. One can also generalize this to the case where the environment is in a mixed state ρ_e. In that case, the state of the environment can be decomposed as $\rho_e = \sum_\nu |\nu\rangle\langle\nu|$ where $|\nu\rangle$ are unnormalized states. Then by linearity, one finds

$$\mathcal{E}(\rho) = \sum_{\mu,\nu=1}^{n} K_{\mu,\nu}\rho K_{\mu,\nu}^\dagger, \tag{3.11}$$

which again can be written as

$$\mathcal{E}(\rho) = \sum_{a=1}^{n^2} K_a\rho K_a^\dagger, \tag{3.12}$$

where we have used a collective index a instead of the pair (μ,ν). To capitulate, we have

- ■ **B1:**The most general quantum evolution that a quantum system can undergo is a map of the form

$$\rho \longrightarrow \mathcal{E}(\rho) = \sum_m K_m\rho K_m^\dagger \tag{3.13}$$

where $\sum_m K_m^\dagger K_m = I$. In this form, this generalizes the concept of unitary evolution that was listed in the introduction.

3.1. Examples

There are many physically motivated examples of quantum channels, specifically qubit channels, which are of particular interest due to their role in quantum computation and quantum communications. A few of these are as follows:

3.1.1. Bit-Flip and Phase-Flip Channels

This channel is defined as

$$\mathcal{E}(\rho) = (1-p)\rho + pX\rho X \tag{3.14}$$

in which $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the first Pauli matrix[*]. In such a channel, the qubit state remains intact with probability $1-p$ and gets flipped with probability p. States of the form $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ pass through the channel invariably, while other states get mixed. The reader can now show that, under the channel (3.14), the state Bloch vector $\mathbf{r} = (x,y,z)$ changes to $\mathbf{r}' = (x,(1-2p)y,(1-2p)z)$. Thus, the Bloch sphere turns into an elongated ellipsoid along the $x-$ direction, Figure 4. If we change X to Z, the resulting channel is called Phase-flip channel (for the obvious reason that with probability p, it changes the relative phase in $a|0\rangle + b|1\rangle$ to $a|0\rangle - b|1\rangle$). The Bloch sphere is now shrunk in the $z-$ direction.
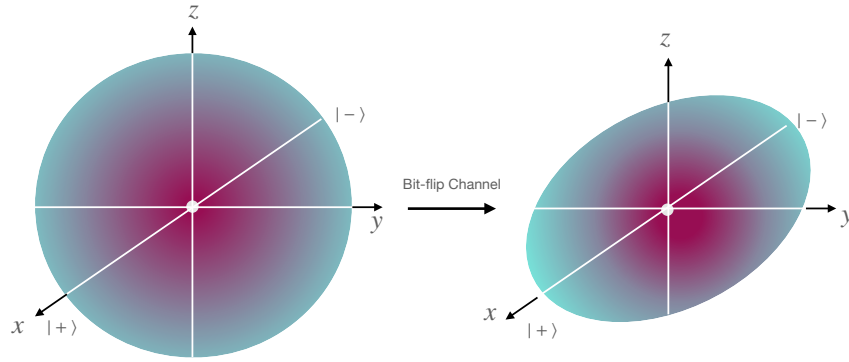
3.1.2. Example two: Depolarizing Channel,

This channel is defined as

$$\mathcal{E}(\rho) = (1-p)\rho + p\frac{I}{2}.$$

---

[*]It is customary in quantum information literature to use X, Y and Z for the Pauli matrices.

Figure 4. The bit-flip channel uniformly shrinks the Bloch sphere in directions perpendicular to the $x-$ axis.

In such a channel the qubit state remains intact with probability $1-p$ and gets completely mixed with probability p. This usually happens when a photon passes through an optical fiber which due to stress, strain and distortion of the fiber, the polarization state of the photon changes in unknown ways.
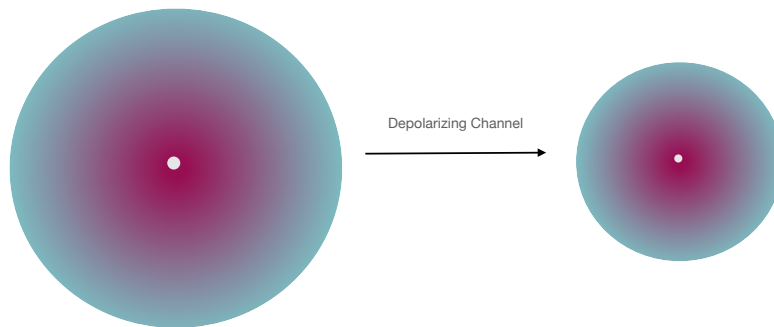
When extended to all positive operators, this channel should be written as

$$\mathcal{E}(\rho) = (1 - p)\rho + p\text{Tr}(\rho)\frac{I}{2}.$$

When extended to $d-$ dimensions, the depolarizing channel is defined as

$$\mathcal{E}(\rho) = (1 - p)\rho + p\text{Tr}(\rho)\frac{I}{d}.$$

The Bloch sphere uniformly shrinks under the depolarizing channel, Figure 5.

Figure 5. The depolarizing channel uniformly shrinks the Bloch sphere in all directions.

The above channels have the special property that they map maximally mixed states into maximally mixed states, i.e. $\mathcal{E}(\frac{I}{2}) = \frac{I}{2}$. Such channels are called unital in addition to (3.7) having the property that $\sum_m K_m K_m^\dagger = I$. The next example is a non-unital channel.

3.1.3. The amplitude-damping (AD) channel

This example deals with probabilistic photon loss to the environment. It is assumed that a single mode of light in a cavity has been so attenuated that it contains either no photon or one photon and looses its photon with some probability to the environment. The quantum channel, which describes this probabilistic photon loss is defined as

$$\mathcal{E}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger, \tag{3.15}$$

where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \cos\theta \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & \sin\theta \\ 0 & 0 \end{pmatrix}. \tag{3.16}$$

Under this channel, a state $|0\rangle\langle 0|$ remains invariant, since there is no photon to be lost to the environment, while a state $|1\rangle\langle 1|$ transforms to $\sin^\theta |0\rangle\langle 0| + \cos^2\theta |1\rangle\langle 1|$. Thus, a single photon may be lost to the environment with probability $\sin^2\theta$. Note that $A_0^\dagger A_0 + A_1^\dagger A_1 = I$ as it should be, but $A_0 A_0^\dagger + A_1 A_1^\dagger \neq I$, which means that this is an example of a non-unital channel. Obviously, a uniform mixture in the form $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ will no longer be invariant under this channel; hence, completely mixed states are not m ped to themselves under the AD non-unital channel.

3.1.4. State transfer through spin chains

A quantum computer consists of arrays of qubits in a row or in a plane. It may be necessary to transfer a qubit from one part of this quantum computer to another place. By the qubit, we mean not its physical identity but its quantum state. A naive way for doing this is to swap the qubit state one by one by acting the swap operator P on adjacent qubits, which acts on basis states as $P|\alpha, \beta\rangle = |\beta, \alpha\rangle$. Note that this two-qubit operator swaps arbitrary states, as a simple computation shows

$$P|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle, \tag{3.17}$$

where $|\psi\rangle$ and $|\phi\rangle$ are superpositions of basis states $|0\rangle$ and $|1\rangle$. This, however, requires individual addressing of qubits, which is not feasible without distorting the final state due to the accumulation of noise. In other words, a succession of these actions may destroy the coherence of the quantum state altogether. To remedy this, a novel approach was first suggested in [10] and developed in many other works [11–13]. This approach is based on the observation that an array of stationary qubits, no matter how they are realized, can always be modeled by a Hamiltonian, which governs the nearest-neighbor interaction of spins. Such a Hamiltonian is nothing but the Heisenberg Hamiltonian expressed as

$$H = -J \sum_{i=1}^{N-1} \mathbf{S}_i \cdot \mathbf{S}_{i+1} - B \sum_{i=1}^{N} S_{i,z}. \tag{3.18}$$

This is, of course, the simplest Hamiltonian, and depending on the actual physical constructions of qubits and the degree of our control on them, one can think of other variations of this Hamiltonian, i.e. where the couplings are not isotropic or even where the couplings differ between different spins. In any case, the basic idea is to use the dynamics of the spin chain itself to transfer the state from one end to another. When $J > 0$, which corresponds to the so-called Ferromagnetic interaction, the ground state is a product state of all spins up, depicted by

$$|g.s\rangle = |0, 0, \cdots 0\rangle =: |\mathbf{0}\rangle \tag{3.19}$$

Now, if the first spin is loaded with a given state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$, the total state of the chain, namely $|\Psi(0)\rangle = |\phi, 0, 0, \cdots 0\rangle$ is no longer an eigenstate of the Heisenberg Hamiltonian and will evolve to $|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$ which spreads over all spins or qubits. The density matrix of the last spin is given by $\rho_N(t) = Tr_{\hat{N}}(|\Psi(t)\rangle\langle\Psi(t)|)$, where $Tr_{\hat{N}}$ means a trace over all spins except the last one. Therefor this is a channel which turns the input state into a mixed state, i.e.

$$\mathcal{E} : |\phi\rangle\langle\phi| \longrightarrow \rho_N(t) := Tr_{\hat{N}}(|\Psi(t)\rangle\langle\Psi(t)|). \tag{3.20}$$

In general, the output state has a fidelity with the input state which varies with time, but at a specific time t_0, this overlap is a maximum and if the last spin is detached from the other qubits at this specific time, we will have a state which is very much similar to the input state. This is depicted symbolically in Figure 6.
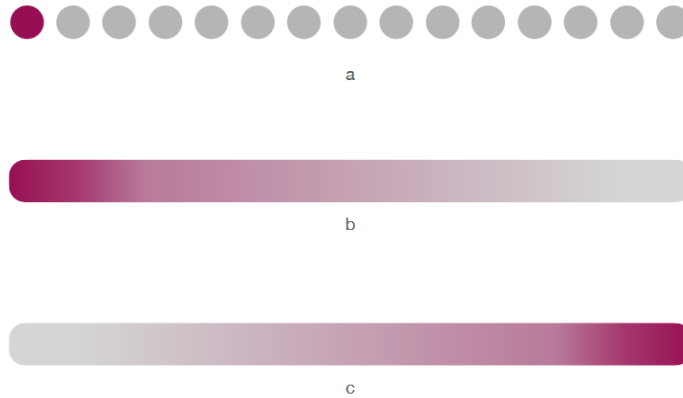
Figure 6. Schematic description of quantum state transfer. a- The chain is in its ground state, which is a product state of spin ups $|0\rangle^{\otimes N}$, and the left hand qubit is in an arbitrary state. b- After a while, this arbitrary state diffuses into the whole chain by entanglement. c-At a particular time t_0, the state of the qubit appears almost at the right end of the chain. This state is not fully separable from the rest of the chain, but it has a high fidelity with the original state, which was to be transferred.

In this way, we have achieved a high-fidelity state transfer through the dynamics of spin chain. Over the course of years, this idea has been developed in many diverse direction, to achieve perfect fidelity by using chains with engineered coupling constants [14], and by adding some external but global control in one [15] and two and three dimensional lattices [16]. There has also been attempts to use this method for generation of arbitrary multi-qubit states over the spin chain, like the so-called W states [17]. Finally, there are works, which use this technique to design static quantum circuits in exactly the same way that classical circuits are constructed by a combination of gates and wires in space [18, 19].

This last example concludes our list of examples, but by no means exhausts the vast number of physically relevant examples of quantum channels and the infinite number of mathematically consistent completely positive trace-preserving maps.

3.2. Continuous time evolution

Up to now, we have discussed the evolution of a quantum system in finite time. This is the generalization of the unitary evolution $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ to open quantum systems. One may ask how the Schrodinger equation $i\hbar\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle$ is generalized in this case. This is a complex question the precise answer of which depending on the assumptions and approximations used leads to slightly different answers. Here, for simplicity,

we ignore these complications and refinements and provide a simple and yet fairly common answer to this question. We consider an infinitesimal time ϵ and write the evolution equation as

$$\rho(t+\epsilon) = \sum_{m=1}^{n} K_m(t,\epsilon)\rho(t)K_m^\dagger(t,\epsilon). \tag{3.21}$$

We now want to guess the dependence of the Kraus operators on the parameter ϵ. We note that the Kraus operators come from the following equation

$$K_m(t,\epsilon) = \langle m|U(t,t+\epsilon)|0\rangle. \tag{3.22}$$

In particular we have

$$K_0(t,\epsilon) = \langle 0|U(t,t+\epsilon)|0\rangle = \langle 0|I - i\epsilon H(t)|0\rangle = I - i\epsilon L_A(t). \tag{3.23}$$

Note that the operator L_A is not Hermitian and can be written as

$$L_A = H_0 + iL_0 \tag{3.24}$$

in which H_0 and L_0 are Hermitian. Now, we use the trace-preserving property

$$K_0^\dagger K_0 + \sum_m K_m^\dagger K_m = I$$

and by substituting K_0 in this relation, we obtain

$$2\epsilon L_0 + \sum_{m \neq 0} K_m^\dagger K_m = 0. \tag{3.25}$$

This relation shows that the operators $K_{m \neq 0}$ are proportional to $\sqrt{\epsilon}$. Hence, we can write

$$K_m = \sqrt{\epsilon}L_m. \tag{3.26}$$

Substituting all the Kraus operators we obtain

$$\frac{d}{dt}\rho(t) = -i[H_0, \rho] + \sum_m L_m \rho L_m^\dagger - \frac{1}{2}L_m^\dagger L_m \rho - \frac{1}{2}\rho L_m^\dagger L_m. \tag{3.27}$$

This equation is called the Lindblad equation or more completely as the GKLS equation for its independent discoverers, Gorini, Kowakowski, Lindblad, and Sudarshan [20, 21]. The first term of this equation shows the unitary evolution of the quantum system, but the later terms show that the state of quantum system jumps stochastically between different levels. As a simple example consider a two-level atom with energy levels $|g\rangle$ and $|e\rangle$. The Hamiltonian is given by $H = \omega\sigma_z = \omega(|g\rangle\langle g| - |e\rangle\langle e|)$. However, there are also jump operators, which act on the system like

$$\sigma_+ = \gamma|e\rangle\langle g|, \qquad \sigma_- = \gamma|g\rangle\langle e|. \tag{3.28}$$

which, according to the Lindblad equation, cause a jump between the energy levels. Therefore, the density matrix evolves according to

$$\frac{d}{dt}\rho = -i\omega[\sigma_z, \rho] + \gamma\sigma_+\rho\sigma_- + \gamma\sigma_-\rho\sigma_-\gamma\rho. \tag{3.29}$$

One can solve this equation by taking $\rho(t) = \frac{1}{2} \begin{pmatrix} 1 + z(t) & x(t) - iy(t) \\ x(t) + iy(t) & 1 - z(t) \end{pmatrix}$ and solve the simple equations

for the entries. The result will be

$$
\begin{aligned}
x(t) &= e^{-\gamma t}(x(0)\cos 2\omega t + y(0)\sin 2\omega t) \\
y(t) &= e^{-\gamma t}(-x(0)\sin 2\omega t + y(0)\cos 2\omega t) \\
z(t) &= e^{-\gamma t}z(0).
\end{aligned}
\tag{3.30}
$$

This shows that the Bloch vector rotates around the thrid axis with frequency ω and at the same time shrinks at a rate γ tending at very long time to the maximally mixed state $\rho(t \longrightarrow \infty) = \frac{1}{2}I$.

3.3. Mathematical considerations

The theory of completely positive maps is a very rich subject in mathematics and has been under investigation since the 70's [23], well before the advent of the theory of quantum computation and quantum information. Despite this long history, the theory is still under intensive investigation. In this subsection, we will briefly discuss some of the basic concepts.

3.3.1. Affine transformation

One can ask what the action on the Bloch sphere will be for a general CPT map. The answer is an affine transformation. To see this, we note that a CPT map should be convex-linear and; therefore, it should satisfy

$$
\mathcal{E}(\lambda\rho_1 + (1 - \lambda)\rho_2) = \lambda\mathcal{E}(\rho_1) + (1 - \lambda)\mathcal{E}(\rho_2).
\tag{3.31}
$$

In terms of the Bloch vectors, this means that such a map induces a convex-linear map $\mathbf{r} :\longrightarrow f(\mathbf{r})$ on the Bloch sphere, that is

$$
f(\lambda\mathbf{r}_1 + (1 - \lambda)\mathbf{r}_2) = \lambda f(\mathbf{r}_1) + (1 - \lambda)f(\mathbf{r}_2),
\tag{3.32}
$$

which limits the form of this map to an affine transformation

$$
f(\mathbf{r}) = M\mathbf{r} + \mathbf{t}
\tag{3.33}
$$

where $\mathbf{M}$ and $\mathbf{t}$ are a real matrix and a real vector, respectively. The singular value decomposition of $\mathbf{M}$ in the form $M = S\Lambda T$, where S and T are rotation matrices and $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$ is a real diagonal matrix, indicates that, under the map f and hence $\mathcal{E}$, the Bloch sphere is first rotated, then stretched differently in various directions and then rotated again and finally translated by the vector $\mathbf{t}$, such that the final shape obtained is still inside the original Bloch sphere, Figure 7.
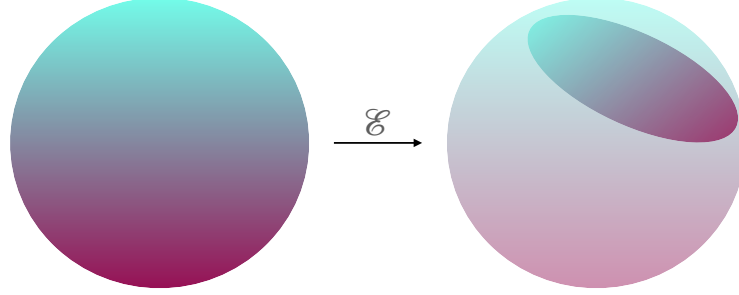
More precisely, let $\mathcal{E}$ be a channel, which transforms the states as follows:

$$
\mathcal{E}\left(\frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma})\right) = \frac{1}{2}\left(I + (\mathbf{Mr} + \mathbf{t}) \cdot \boldsymbol{\sigma}\right).
$$

Consider now the map

$$
\mathcal{E}_c(\rho) := U\mathcal{E}(V\rho V^{-1})U^{-1},
$$

Figure 7. The action of a qubit channel on the Bloch sphere. The Bloch sphere is rotated and stretched differently in various directions and finally is displaced from the center. For unital channels, the final displacement is absent.

where U and V are two unitary operators. We know that under unitary conjugations, the Pauli matrices transform as follows:

$$V\sigma_i V^{-1} = T_{ij}\sigma_j, \quad U\sigma_i U^{-1} = S_{ij}\sigma_j, \tag{3.34}$$

where S and T are rotation matrices which are representations of U and V respectively. This means that

$$V(\mathbf{r}\cdot\boldsymbol{\sigma})V^{-1} = T^{-1}\mathbf{r}\cdot\boldsymbol{\sigma}, \quad U(\mathbf{r}\cdot\boldsymbol{\sigma})U^{-1} = S^{-1}\mathbf{r}\cdot\boldsymbol{\sigma}. \tag{3.35}$$

Therefore, the action of $\mathcal{E}_c$ is found to be

$$
\begin{aligned}
\mathcal{E}_c(I + \mathbf{r}\cdot\boldsymbol{\sigma}) &= U\mathcal{E}(I + T^{-1}\mathbf{r}\cdot\boldsymbol{\sigma})U^{-1} \\
&= U\Big(I + (MT^{-1}\mathbf{r} + \mathbf{t})\cdot\boldsymbol{\sigma}\Big)U^{-1} = \big(I + (S^{-1}MT^{-1}\mathbf{r} + S\mathbf{t})\cdot\boldsymbol{\sigma}\big).
\end{aligned} \tag{3.36}
$$

If we now choose the unitary matrices U and V such that $S^{-1}MT^{-1} = \Lambda$, where Λ is a diagonal matrix, we have shown that for any qubit channel $\mathcal{E}$, with affine matrix M, there is a canonical representation $\mathcal{E}(\rho) = U^{-1}\mathcal{E}_c(V^{-1}\rho V)U$ such that the affine transformation of $\mathcal{E}_c$ is given by a diagonal matrix $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$. Note that, for unital channels, the affine transformation becomes a linear transformation since $\mathbf{t} = 0$. We should note, however, that not every affine transformation corresponds to a legitimate quantum channel. The necessary but not sufficient conditions are that all $|\lambda_i|$ should be less than or equal to one. For unital channels, the necessary and sufficient condition is that the vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ lies inside the tetrahedron shown in Figure 8. For non-unital channels, more conditions are imposed [22].

3.3.2. Complete positivity

A quantum state $\rho \in D(H_A)$ is a positive unit-trace operator. Any quantum operation that we do in the laboratory, should change this state to another state with the same property. Denote this operation by $\mathcal{E} : D(H_A) \longrightarrow D(H_B)$, where we allow for the two Hilbert spaces to be different. The freedom to change dimensions stems from a physical requirement, namely operations where we adjunct another quantum system

$$\boldsymbol{v}_0 = (1, \quad 1, \quad 1)$$
$$\boldsymbol{v}_1 = (1, -1, -1)$$
$$\boldsymbol{v}_2 = (-1, 1, -1)$$
$$\boldsymbol{v}_3 = (-1, -1, 1)$$
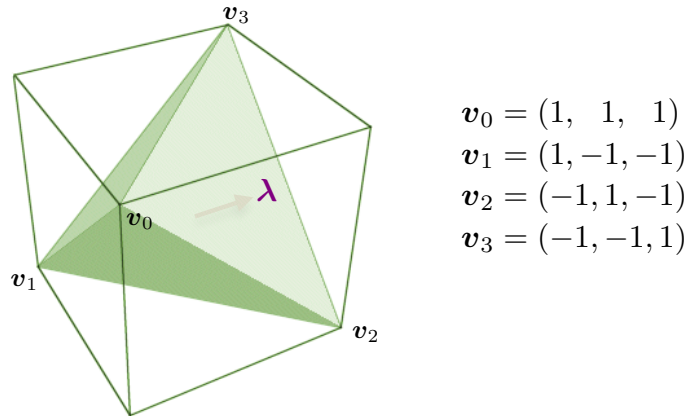
Figure 8. A necessary and sufficient condition for a unital qubit channel to be a completely positive map is that the eigenvalues $\{\lambda_i\}$ lie within the tetrahedron shown above.

to our old system (i.e. adding an atom or ion) or removing or ignoring part of the system (i.e. ignoring an atom in a two-atom system). Whatever this operation is, we know that the new state $\mathcal{E}(\rho)$ should also be a quantum state. Therefore, the map $\mathcal{E}$ is a map that preserves Hermiticity, positivity, and the trace of its argument. We call such a map a positive trace-preserving map. Based on the linearity of quantum mechanics, we make an important assumption that this map is linear. However, not every positive trace-preserving operation is a legitimate quantum operation that can be implemented in the laboratory. Such an operation must satisfy an extra condition, which is called complete positivity, which we now define.

Consider Figure 9 where Alice is doing a quantum operation (evolution, measurement, etc) on her qubit A. But this qubit may be entangled with some other quantum system C on which she has no control or even she is not aware. Nevertheless, her quantum operation must change the state of the combined quantum system AC into a legitimate state of the same combined system.
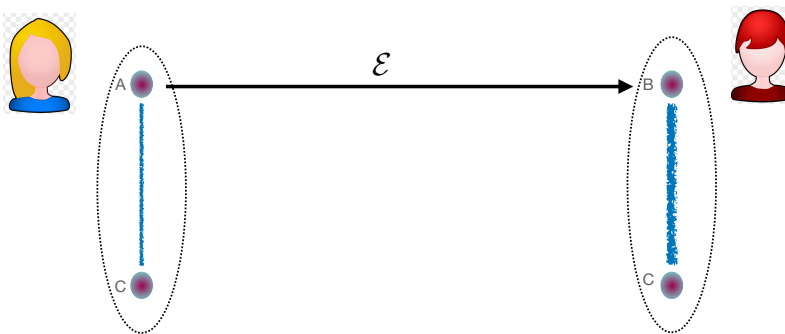
Figure 9. A completely positive map: Alice is doing a quantum operation $\mathcal{E}$, on her quntum system A which may be entangled with another system C. The whole operation $\mathcal{E} \otimes I$ should change quantum states to quantum states, even if Alice has no control over the system C or is not even aware of its presence.

This means that when we extend the map $\mathcal{E}$ in the following form

$$\mathcal{E}_A \otimes I_C : L^+(H_A \otimes H_C) \longrightarrow L^+(H_B \otimes H_C) \tag{3.37}$$

it still should be a positive map. The crucial point is that while the map $\mathcal{E}_A \otimes I_C$ is in product form, not all states in $L^+(H_A \otimes H_C)$ are in product form, and, in fact, all of them except a set of measure zero are entangled. Therefore, there is no guarantee that the map $\mathcal{E}_A \otimes I_C$, when acting on these larger set of states, still preserves its positivity. An example clarifies this and shows that there are positive maps, which are not completely positive. The simplest example is the transpose operator $T : \rho \longrightarrow \rho^T$, which obviously is positive and trace-preserving. The extension of transpose, which is called partial transpose, changes the components as

$$T \otimes I : \rho_{ij,kl} \longrightarrow \rho_{kj,il}. \tag{3.38}$$

It is now easy to see that this extension is not positive. A simple counter example is given by the positive matrix

$$\rho = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.39}$$

with eigenvalues $(0, 0, 0, 1)$ which under partial transpose is turned into

$$\rho = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{3.40}$$

with eigenvalues $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2})$. Another operation, which is positive but not completely positive and, hence, not implementable in the laboratory, is the operation which turns every state into its orthogonal: $\rho \longrightarrow \rho^\perp$. The action of such a map on pure state is to turn any state $\rho = \frac{1}{2}(I + \mathbf{n} \cdot \boldsymbol{\sigma})$ to $\rho^\perp = \frac{1}{2}(I - \mathbf{n} \cdot \boldsymbol{\sigma})$. On the Bloch sphere this means that every point on the surface of the Bloch sphere is mapped to its diagonally opposite point, that is:

$$(n_x, n_y, n_z) \longrightarrow (-n_x, -n_y, -n_z). \tag{3.41}$$

To prove that such an operation is not a CPT, it is enough to compose it with a $\pi-$ rotation around the $y-$ axis, which we know is a legitimate quantum operation. This amounts to the following transformation on the unit vectors on the Bloch sphere

$$(n_x, n_y, n_z) \longrightarrow (-n_x, -n_y, -n_z) \longrightarrow (n_x, -n_y, n_z). \tag{3.42}$$

But, in view of the properties of Pauli matrices, i.e. $(\sigma_x^T = \sigma_x, \sigma_y^T = -\sigma_y, \sigma_z^T = \sigma_z)$ and the expression $\rho = \frac{1}{2}(I + \mathbf{n} \cdot \boldsymbol{\sigma})$, this combined operation is nothing but the transpose operation $\rho \longrightarrow \rho^T$, which we know is not a CPT and, hence, not a legitimate quantum operation. Therefore, the original operation $\rho \longrightarrow \rho^\perp$ could not have been a CPT.

In view of its very definition and the arbitrariness of the extension, it may seem that the characterization of CPT maps may be extremely difficult if not impossible. However, an important theorem due to Choi [23] paves the way for this characterization. For simplicity, we take $H_A = H_B$ here.

Theorem: The extension of a map $\mathcal{E} : L^+(H) \longrightarrow L^+(H)$ to any Hilbert space in the form $\mathcal{E} \otimes I : L^+(H \otimes H') \longrightarrow L^+(H \otimes H')$ is positive if and only if it its extension in the form $(\mathcal{E} \otimes I) : L^+(H \otimes H) \longrightarrow L^+(H \otimes H)$ is positive.

A CPT map can be characterized in a very efficient way as follows by defining what is called the Choi matrix of the map [23]:

Definition: Let $\mathcal{E} : L^+(H) \longrightarrow L^+(H)$ be a CPT map. Its Choi matrix is defined as

$$C_{\mathcal{E}} := d(\mathcal{E} \otimes I)(|\phi^+\rangle\langle\phi^+|) = \sum_{i,j=1}^{d} \mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j| \in L(H) \otimes L(H) \tag{3.43}$$

where $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i,i\rangle$.

One can then see that the Choi matrix has the following properties:

- by definition of complete positivity of the map $\mathcal{E}$, $C_{\mathcal{E}}$ is a positive operator.

- $\mathrm{Tr}_2 C_{\mathcal{E}} = \mathcal{E}\left(\sum_i |i\rangle\langle i|\right) = \mathcal{E}(I)$

- $\mathrm{Tr}_1 C_{\mathcal{E}} = \sum_{i,j} Tr\left(\mathcal{E}|i\rangle\langle j|\right) \otimes |i\rangle\langle j| = \sum_{i,j} Tr\left(|i\rangle\langle j|\right) \otimes |i\rangle\langle j| = \sum_i |i\rangle\langle i| = I$

- $\mathrm{Tr}_{1,2} C_{\mathcal{E}} = d,$

where the indices 1 and 2 indicate on which of the factors we are taking the trace, $d = dim(H)$ and in the second item we have used the trace-preserving property of the map. Once we have the Choi matrix, we can find the action of the map on any state as follows: First we note that

$$_2\langle i|C_{\mathcal{E}}|j\rangle_2 = \mathcal{E}(|i\rangle\langle j|), \tag{3.44}$$

and therefore for any state $|\psi\rangle = \sum_i \psi_i |i\rangle \in H$

$$\mathcal{E}\left(|\psi\rangle\langle\psi|\right) = \sum_{i,j} \psi_i \psi_j^* \,_2\langle i|C_{\mathcal{E}}|j\rangle_2 = \,_2\langle\psi^*|C_{\mathcal{E}}|\psi^*\rangle_2. \tag{3.45}$$

We now note that since $\frac{1}{d}C_{\mathcal{E}}$ is a positive operator and, hence, $\frac{1}{d}C_{\mathcal{E}}$ is a density matrix, it has a convex decomposition in terms of pure states. Let us write this as

$$C_{\mathcal{E}} = \sum_m |K_m\rangle\langle K_m| \tag{3.46}$$

where $|K_m\rangle$'s are in general non-orthonormal states. This means that

$$\mathcal{E}(\rho) = \sum_m \,_2\langle\psi^*|K_m\rangle\langle K_m|\psi^*\rangle_2 \tag{3.47}$$

but $\langle\psi^*|K_m\rangle$ is linear with respect to $|\psi\rangle$ and hence can be written as a linear operator

$$\langle\psi^*|K_m\rangle = K_m|\psi\rangle. \tag{3.48}$$

Combining this with the previous equation, we find

$$\mathcal{E}(|\psi\rangle\langle\psi|) = \sum_m K_m|\psi\rangle\langle\psi|K_m^\dagger. \tag{3.49}$$

This can be extended by linearity to

$$\mathcal{E}(\rho) = \sum_m K_m\rho K_m^\dagger. \tag{3.50}$$

which is nothing but the Kraus decomposition of the CPT map $\mathcal{E}$. In this way it has been proved that any CPT map can be written as (3.50).

3.3.3. Vectorization and the dynamical matrix

There is a canonical one-to-one correspondence between matrices and vectors, which allows us to look at quantum channels in a different way which is instructive in some ways. Let $\rho = \sum_{i,j} \rho_{ij}|i\rangle\langle j| \in L^+(H)$ be a density matrix (or simply a positive matrix). One can map this to a vector $|\rho\rangle \in H \otimes H$ as follows:

$$\rho = \sum_{i,j} \rho_{ij}|i\rangle\langle j| \longrightarrow |\rho\rangle = \sum_{i,j} \rho_{i,j}|i,j\rangle. \tag{3.51}$$

This process is called vectorization and turns out to be useful as we will see. First we note that this correspondence is linear in the sense that for any two matices A and B and any complex number $\alpha \in C$,

$$|\alpha A + B\rangle = \alpha|A\rangle + |B\rangle. \tag{3.52}$$

Moreover for any three matrices $A, B, C \in L(H)$

$$|ABC^T\rangle = (A \otimes C)|B\rangle. \tag{3.53}$$

This is verified by straightforward calculation as follows:

$$
\begin{aligned}
|ABC^T\rangle &= \sum_{i,j}(ABC^T)_{i,j}|i,j\rangle = \sum_{i,j,k,l} A_{ik}B_{kl}C_{jl}|i,j\rangle \\
&= \sum_{i,j,k,l}(A_{ik}C_{jl})B_{kl}|i,j\rangle = \sum_{i,j,k,l}(A \otimes C)_{ij;kl}B_{kl}|i,j\rangle = (A \otimes C)|B\rangle.
\end{aligned} \tag{3.54}
$$

This means that one can look at a CPT map as a linear operation on the vectorized form of matrices. That is:

$$|\mathcal{E}(\rho)\rangle = |\sum_m K_m\rho K_m^\dagger\rangle = \sum_m (K_m \otimes K_m^*)|\rho\rangle =: \Phi_\mathcal{E}|\rho\rangle. \tag{3.55}$$

The linear operator

$$\Phi_\mathcal{E} = \sum_m K_m \otimes K_m^* \tag{3.56}$$

is called the dynamical matrix of the CPT map $\mathcal{E}$. The dynamcial matrix has the obvious property that

$$\Phi_{\mathcal{E}\circ\mathcal{F}} = \Phi_\mathcal{E}\Phi_\mathcal{F}. \tag{3.57}$$

Therefore the composition of two quantum channels transforms to the simple multiplication of their dynamical matrices. We can now derive a simple relation between the Choi matrix and the Dynamical matrix. In view of (3.46) we note that

$$C_{\mathcal{E}\,ij,rs} = \langle ij| \sum_m |K_m\rangle\langle K_m|rs\rangle = \sum_m (K_m)_{ij}(K_m)_{rs}{}^* \tag{3.58}$$

while in view of (), we note that

$$(\Phi_{\mathcal{E}})_{ij,rs} = \langle ij| \sum_m K_m \otimes K_m^* |rs\rangle = \sum_m (K_m)_{ir}(K_m)^*_{js} \tag{3.59}$$

This allows us to find a simple relation between the two matrices. If we define the **Reshuffling** operation R as

$$(A^R)_{ij,rs} = A_{ir,js}, \tag{3.60}$$

then the relation between the dynamical and the Choi matrix is nothing but a reshuffling operation, namely

$$\Phi_{\mathcal{E}}{}^R = C_{\mathcal{E}}. \tag{3.61}$$

It is in order to study a simple example. Consider the bit-flip channel, which we re-write here for convenience

$$\mathcal{E}(\rho) = (1-p)\rho + pX\rho X^\dagger. \tag{3.62}$$

For this channel we find

$$\Phi_{\mathcal{E}} = (1-p)I + pX \otimes X = \begin{pmatrix} 1-p & & & p \\ & 1-p & p & \\ & p & 1-p & \\ p & & & 1-p \end{pmatrix} \tag{3.63}$$

and the Choi matrix is

$$C_{\mathcal{E}} = \Phi_{\mathcal{E}}{}^R = \begin{pmatrix} 1-p & & & 1-p \\ & p & p & \\ & p & p & \\ 1-p & & & 1-p \end{pmatrix} \tag{3.64}$$

It is seen that $\mathrm{Tr}_2 C_{\mathcal{E}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_1$ which every channel should satisfy and $\mathrm{Tr}_1 C_{\mathcal{E}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ which every unital channel (like the bit-flip channel) should satisfy.

3.4. Inversion of quantum channels

It is a direct consequence of unitarity that the quantum evolution of a closed system can be reversed in time. If an initial state $|\psi(0)\rangle$ evolves to a final state $|\psi(t)\rangle$ by the unitary operator $U(t)$, the final state $|\psi(t)\rangle$ can also evolve to the initial state $|\psi(0)\rangle$ by the unitary operator $U^\dagger(t)$. If the unitary operator is implemented by the Hamiltonian H, the inverse unitary operator is implemented by the Hamiltonian $-H$. Therefore, by suitably changing the coupling constant of the Hamiltonian, one can exactly invert a unitary evolution. In practice this may not be so easy to do this, i.e. it may not be so easy to change a ferromagnetic coupling to

an anti-ferromagnetic coupling, but in principle this is possible and this is what matters and makes the unitary evolution reversible. The path of a quantum state in the Hilbert space traced by the state $|\psi(\tau)\rangle$ from $\tau = 0$ to $\tau = t$ can be exactly traversed in the reverse direction. In this sense, unitary evolution is deterministic, and it is only in the process of measurement that the probabilistic nature of quantum formalism shows itself. However, we should note that measurement is not entirely a quantum mechanical process, as it involves the interaction of the microscopic quantum system and the macroscopic apparatus which is governed by classical mechanics. And this is indeed the weakest part of the formalism of quantum mechanics, which has led to decades of controversy and a multitude of interpretations.

When it comes to the evolution of open quantum systems, we can ask if the dynamics is reversible or not? Let a quantum system undergoes an evolution or a CPT map $\rho \longrightarrow \mathcal{E}(\rho)$. We ask if there is another CPT map $\mathcal{E}^{-1}$, which turns back the final state into the initial state? If there is a CPT map such that $\mathcal{E}^{-1} \circ \mathcal{E} = Id$? The answer is that this is generally impossible.

The reason is intuitively clear and simple. Since the quantum system is no longer closed, during the evolution part of the information leaks into the environment and disperses and it never comes back to the quantum system. Technically, this manifests itself in a mathematical property of quantum channels, which states that any CPT map decreases the distance between input states, that is for any two input states ρ_1 and ρ_2 and any quantum channel $\mathcal{E}$, we have

$$d(\mathcal{E}(\rho_1), \mathcal{E}(\rho_2)) \leq d(\rho_1, \rho_2). \tag{3.65}$$

Here the distance between two states is defined as

$$d(\rho_1, \rho_2) = \frac{1}{2} Tr\big(|\rho_1 - \rho_2|\big) \tag{3.66}$$

and for any matrix A, $Tr|A|$ is defined as $Tr|A| = \sqrt{Tr(A^\dagger A)}$.

A moment of reflection shows why this property should hold. Two pictures when transmitted through a communication line loose some of their characteristic features and become a little bit similar to each other. The longer the transmission line and consequently the more the errors, they lose more features and they become more similar to each other. This one-way property immediately prohibits any inversion of a CPT map, since the existence of an inverse means that we have the means to increase the distance of the final states and return them to the initial states.

The reader may wonder why, to show this impossibility, we have not invoked the property of entropy and a statement like "Since the information of a state leaks into the environment any channel increases the entropy of the input state and hence an inverse of a quantum channel cannot exist ". The answer is that this statement is not true! There are quantum channels that increase the input entropy and there are also channels that decrease the input entropy or keep it constant. The leakage of the information has nothing to do with the entropy difference between the input and output. As examples, consider the following:

a) A unitary map $\rho \longrightarrow U\rho U^\dagger$, keeps the entropy constant since $S(\rho) = S(U\rho U^\dagger)$, due to the cyclic property of the trace.

b) A depolarizing channel $\rho \longrightarrow (1-p)\rho + p\frac{I}{2}$, increases the entropy of any state since it shrinks the length of the Bloch vector and brings it closer to the center.

c) A channel like $\rho \longrightarrow Tr(\rho)|\phi\rangle\langle\phi|$ where $|\phi\rangle$ is a fixed state, decreases the entropy of any input state to zero since the output state is pure. To ensure that this later channel is a CPT map and is not an ill-defined map, we can show that it has a Kraus decomposition as follows:

$$Tr(\rho)|\phi\rangle\langle\phi| = \sum_i \langle i|\rho|i\rangle|\phi\rangle\langle\phi| = \sum_i |\phi\rangle\langle i|\rho|i\rangle\langle\phi| = \sum_i K_i \rho K_i^\dagger \qquad (3.67)$$

with $K_i = |\phi\rangle\langle i|$.

Now that an exact inversion is impossible, we can ask to what extent we can achieve an approximate inversion. To give meaning to this approximate inversion, we can invoke the concept of average input-output fidelity. Let a pure state $|\psi\rangle$ enters a quantum channel $\mathcal{E}$ which turns it into $\rho = \mathcal{E}(|\psi\rangle\langle\psi|)$. For an identity channel, the input and output are the same, that is $\rho = |\psi\rangle\langle\psi|$. A channel generally distorts the input state. The proximity of a channel to the identity channel can be measured by an average of input-output fidelity defined as

$$\overline{F}_\mathcal{E} := \int d\psi \langle\psi|\mathcal{E}(|\psi\rangle\langle\psi|)|\psi\rangle. \qquad (3.68)$$

This quantity can be calculated in closed form for any channel. Here we show how it can be calculated for qubit channels [24] and refer the reader to [25] for higher dimensional channels. If we think of Bloch sphere we note that a pure state can be written as

$$|\psi\rangle\langle\psi| = \frac{1}{2}(I + \mathbf{n} \cdot \boldsymbol{\sigma}) \qquad (3.69)$$

where $\mathbf{n}$ is a unit vector on the surface of Bloch sphere. The channel $\mathcal{E}$ turns this state into a mixed state ρ which corresponds to a vector inside the Bloch sphere $\mathbf{r} = M\mathbf{n} + \mathbf{t}$. For a given input state, the fidelity between the input and the output state is given by

$$F(|\psi\rangle, \rho) = \langle\psi|\rho|\psi\rangle = Tr(|\psi\rangle\langle\psi|\rho) = \frac{1}{4}Tr\Big[(I + \mathbf{n} \cdot \boldsymbol{\sigma})((I + \mathbf{r} \cdot \boldsymbol{\sigma}))\Big]. \qquad (3.70)$$

Using the properties of Pauli matrices, this reduces to

$$F(|\psi\rangle, \rho) = \frac{1}{2}(1 + \mathbf{n} \cdot \mathbf{r}) = \frac{1}{2}(1 + \mathbf{n} \cdot M\mathbf{n} + \mathbf{n} \cdot \mathbf{t}) \qquad (3.71)$$

Taking the average and noting that $\int d\psi \, F = \int d\mathbf{n} \, F$ is an integral over surface of Bloch sphere with normalization $\int d\mathbf{n} = 1$ and properties

$$\int d\mathbf{n} \, n_i = 0, \qquad \int d\mathbf{n} \, n_i n_j = \frac{1}{3}\delta_{ij}. \qquad (3.72)$$

we find

$$\overline{F}_\mathcal{E} = \frac{1}{2}\left(1 + \frac{1}{3}Tr(M)\right). \qquad (3.73)$$

This analysis depends very much on the existence of Bloch sphere for qubits. For higher dimensional states, there is no Bloch sphere and the geometry of quantum states is not known. However, a much lengthier analysis in d dimensions [25] leads to the following equation

$$\overline{F}_{\mathcal{E}} = \frac{1}{d}\left(1 + \frac{1}{d+1}Tr(M)\right) \tag{3.74}$$

For a given channel, this average fidelity is generally less than one. The meaning of a quasi-inverse is to combine this channel with another one which brings this average fidelity as much as possible close to one. By this we mean that a quasi-inverse $\mathcal{E}^{qi}$ when combined with $\mathcal{E}$ first of all has a higher average fidelity than the original channel and second, it does this in an optimal way. More concretely the quasi-inverse has the following single inequality:

$$\overline{F}_{\mathcal{E}^{qi}\circ\mathcal{E}} \geq \overline{F}_{\mathcal{E}'\circ\mathcal{E}} \qquad \forall\, \mathcal{E}' \tag{3.75}$$

where $\mathcal{E}'$ is any channel especially including the identity channel. The next step is to see how the affine transformation of a map behaves when it is combined with another map. It is straightforward to see that when two maps $\mathcal{E}$ and $\mathcal{E}'$ corresponding respectively to affine maps (M, t) and (M', t') are combined, the resulting map $\mathcal{E}' \circ \mathcal{E}$ corresponds to the affine map $(M'M, M't + t')$. However, the point is that the average fidelity depends only on the matrix M. Therefore our objective should be to find a matrix that maximizes the following quantity

$$Tr(M^{qi}M), \tag{3.76}$$

which at first sight seems a straightforward problem in matrix algebra. However, the difficulty lies in the restriction that the matrix M^{qi} should correspond to an affine matrix of a legitimate quantum channel and not every matrix has this property. This problem has been solved for qubit channels in [24] and for higher dimensional channels in [25]. We refer the interested reader to these references for details. Here we suffice to mention the final results. For qubit channel, the quasi-inverse $\mathcal{E}^{qi}$ is always a unitary map: $\rho \longrightarrow U\rho U^{\dagger}$ which can be uniquely calculated in terms of the original channel. For higher-dimensional channels, the quasi-inverse need not always be a unitary map and indeed explicit examples of channels have been shown to exist for which the quasi-inverse is not unitary, an example being the so-called Landau-Streater channel which is defined as

$$\mathcal{E}_{LS}(\rho) = \frac{1}{j(j+1)}\left[J_x\rho J_x + J_y\rho J_y + J_z\rho J_z\right] \tag{3.77}$$

where $J_{x,y,z}$ are the representations of the generators of the SU(2) algebra in the spin j representation. This channel is its own quasi-inverse.

4. Measurement

A measurement is nothing but an experimental setup (mirrors, lenses, beam splitters, electric or magnetic fields, etc) which separates the particles in one way or another. Depending on our physical theory we assign a property to particles that have been separated and give meaning to each class of particles (having spin up or down, etc). Therefore a measurement can be most appropriately described as a collection of projectors

$$\{\Pi_i, i = 1\cdots r\}, \qquad \Pi_i\Pi_j = \delta_{ij}\Pi_i, \qquad \sum_i \Pi_i = I,$$

which turn a state $|\psi\rangle\langle\psi|$ into an incoherent statistical mixture of states $\rho_i = \frac{\Pi_i|\psi\rangle\langle\psi|\Pi_i}{\langle\psi|\Pi_i|\psi\rangle}$ with probability $P_i = \langle\psi|\Pi_i|\psi\rangle$. A non-selective measurement is thus described by the operation

$$|\psi\rangle\langle\psi| \longrightarrow \sum_{i=1}^{r} P_i \rho_i = \sum_{i=1}^{r} \Pi_i |\psi\rangle\langle\psi| \Pi_i, \qquad (4.1)$$

which by linearity is extended to

$$\rho \longrightarrow \mathcal{E}(\rho) = \sum_{i=1}^{r} P_i \rho_i = \sum_{i=1}^{r} \Pi_i \rho \Pi_i. \qquad (4.2)$$

This already shows that a non-selective measurement is also a CPT map. In practice an experimentalist may not be able to do an orthogonal or projective measurement, for example, the resolution of the apparatus (i.e. in a Stern Gerlach experiment) may not be high enough. In such cases we are dealing with non-orthogonal measurements and the projectors Π_i are replaced with operators M_i and the measurement process implements the following operation on the density matrix:

$$\rho \longrightarrow \mathcal{E}(\rho) = \sum_{i=1}^{r} M_i \rho_i M_i^\dagger, \qquad \sum_i M_i^\dagger M_i = I. \qquad (4.3)$$

To see how this happens, consider for example the case where the detectors are so distant that each of the particle beams after separation undergoes independent unitary operations (by stray magnetic fields in the apparatus) which cause the beams to partially arrive at the wrong detectors. In such a case, the projectors Π_i are replaced by $M_i = U_i \Pi_i$, where U_i denotes the unitary evolution on the $i-$th beam. Then, the whole measurement is described by

$$\rho \longrightarrow \sum_i U_i \Pi_i \rho \Pi_i U_i^\dagger = \sum_i M_i \rho M_i^\dagger, \qquad (4.4)$$

where $M_i = U_i \Pi_i$ are no longer projectors ($M_i^\dagger M_j \neq \delta_{ij} M_i$), but they still preserve the probability ($\sum_i M_i^\dagger M_i = I$).

Another even more common setup, which dictates non-orthogonal measurement, is when we are not able to precisely address an individual atom (i.e. in an array of atoms) and can only do a projective measurement on a larger system that includes that atom. This projective measurement on the larger system appears to be a non-orthogonal measurement on the small system. This is schematically shown in figure (10). To be more specific, we consider a simple example. Let a spin 1/2 particle be in the state $|\psi\rangle = a|+\rangle + b|-\rangle$ in which the states $|+\rangle$ and $|-\rangle$ are the eigenstates of the $S_z = \frac{1}{2}\sigma_z$ operator. It may not be so easy for us to measure this particle in the basis of S_z, however, due to experimental feasibility, we may be able to measure the total spin of this particle and a particle adjacent to it. Let the second particle be in the state $|+\rangle$. Then, the state of the two particles is given by

$$|\Psi\rangle = a|+,+\rangle + b|-,+\rangle. \qquad (4.5)$$

To determine the outcome of the measurement of the total spin operator S^2, we expand this state in terms of the eigenvectors of total spin

$$|\Psi\rangle = \alpha|1,1\rangle + b\frac{1}{\sqrt{2}}(|1,0\rangle - |0,0\rangle), \qquad (4.6)$$

where we are using the notation $|s, m\rangle$ with $S^2|s, m\rangle = s(s+1)|s, m\rangle$ and $S_z|s, m\rangle = m|s, m\rangle$. Since we are doing a projective measurement on total spin, the results will be

$$P(1) = a^2 + \frac{b^2}{2} \qquad \longrightarrow \qquad \frac{1}{\sqrt{a^2 + \frac{b^2}{2}}}(a|1, 1\rangle + \frac{b}{\sqrt{2}}|1, 0\rangle)$$

$$P(0) = \frac{b^2}{2} \qquad \longrightarrow \qquad |0, 0\rangle. \tag{4.7}$$

By a simple rewriting in terms of the original notation, we will have

$$P(1) = a^2 + \frac{b^2}{2} \qquad \frac{1}{\sqrt{a^2 + \frac{b^2}{2}}}\left(a|++\rangle + \frac{b}{2}(|+,-\rangle + |-,+\rangle)\right)$$

$$P(0) = \frac{b^2}{2} \qquad \frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle) \tag{4.8}$$

In order to see what the state of the original particle is after each outcome, we have to take the trace of the above states over the second particle. The result will be

$$P(1) = a^2 + \frac{b^2}{2} \qquad \rho_1 = \frac{1}{a^2 + \frac{b^2}{2}} \begin{pmatrix} a^2 + \frac{b^2}{4} & \frac{ab}{2} \\ \frac{ab}{2} & \frac{b^2}{4} \end{pmatrix}$$

$$P(0) = \frac{b^2}{2} \qquad \rho_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.9}$$

and we clearly see that the states ρ_0 and ρ_1 are no longer are orthogonal to each other.
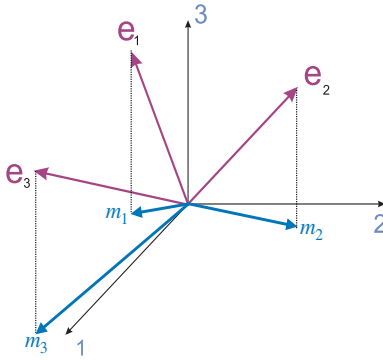
Figure 10. An orthogonal measurement on a large system seems as a non-orthogonal measurement on its subsystem. The number of measurement operators can thus be greater than the dimension of the Hilbert space of the subsystem. Here, the measurement on the large system is orthogonal along the states $\{e_1, e_2, e_3\}$. This measurement induces a non-orthogonal measurement along the vectors $\{m_1, m_2, m_3\}$ whose number is larger than the 2, which is the dimension of the subspace.

Finally an experimentalist may only be interested in the statistics of measurements (number of clicks of each detector) and not the states afterward. This is what happens in almost any measurement. Then the measurement is defined simply by a set of positive operators $\{E_i, i = 1 \cdots M\}$, subject to the condition

$\sum_i E_i = I$. The statistics of the measurement is obtained from $P(i) = Tr(E_i\rho)$ and the condition $\sum_i E_i = I$ guarantees preservation of probabilities. Such a measurement which is indeed the most general and the most common type of measurement is called a positive operator valued measurement or POVM for short. Note that the number of POVM elements E_i may be larger than than the dimension of the Hilbert space of the system we are measuring. Hence, we reformulate the axiom of measurement as follows:

- ■ C1-For a system in state ρ, a measurement is defined by a POVM $\{E_i\}$, ($E_i \geq 0, \ \sum_i E_i = 1$) where the probability of the outcome $P(i)$ is given by $P(i) = Tr(E_i\rho)$.

5. Conclusion

Inspired by the emerging field of quantum science and technology, we have revisited the basic principles of quantum mechanics in a way that is used to study single quantum entities in the quantum world, i.e. atoms, ions, and photons. These objects are the main ingredients that make quantum technology possible: They store information, carry information between different places, and process information in the form which is called quantum computation. This review article has aimed to expose young graduate or undergraduate students to the modern principles of quantum mechanics in the way which is necessary for following the up-to-date literature on the subject. These principles are summarized here for easy reference. For motivations, examples, and elucidations, the reader is referred to the main text.

- ■ **A1:** The state of a quantum state is defined by a Hermitian and Positive matrix ρ whose trace is equal to one. If $\rho^2 = \rho$, such a state is called a pure state.

- ■ **B1:**The most general quantum evolution that a quantum system can undergo is a map of the form

$$\rho \longrightarrow \mathcal{E}(\rho) = \sum_m K_m \rho K_m^\dagger \tag{5.1}$$

where $\sum_m K_m^\dagger K_m = I$. In this form this generalizes the concept of unitary evolution that was listed in the introduction.

- ■ C1-For a system in state ρ, a measurement is defined by a POVM $\{E_i\}$, ($E_i \geq 0, \ \sum_i E_i = 1$) where the probability of the outcome $P(i)$ is given by $P(i) = Tr(E_i\rho)$.

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References

[1] Y.Y. Fein, P. Geyer, P. Zwick, F. Kialka, S. Pedalino, M. Mayor, S. Gerlich and M. Arndt, Quantum superposition of molecules beyond 25 kDa, Nature Physics **15**, 1242–1245 (2019). doi: 10.1038/s41567-019-0663-9

[2] P.W. Shor Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer, SIAM J. Comput. **26**, 1484-1509 (1997). doi: 10.1137/S0097539795293172

[3] C.H. Bennett and G. Brassard, Quantum cryptography: Public key distribution and coin tossing, Theoretical Computer Science **560**, 7–11 (2014). doi: 10.1016/j.tcs.2014.05.025

[4] I.M. Georgescu, S. Ashhab and F. Nori, Quantum simulation, Rev. Mod. Phys. **86**, 153 (2014). doi:10.1103/RevModPhys.86.153

[5] E. Schrödinger, Discussion of probability relations between separated systems, Mathematical Proceedings of the Cambridge Philosophical Society. **31**, 555–563 (1935). doi: 10.1017/S0305004100013554

[6] A. Einstein, B. Podolsky and N. Rosen, Can Quantum-Mechanical Description of Physical Reality be Considered Complete?, Physical Review **47**, 777–780 (1935). doi: doi.org/10.1103/PhysRev.47.777

[7] J. S. Bell, On the Einstein Podolsky Rosen Paradox, Physics Physique Fizika **1**, 195–200 (1964). doi: 10.1103/PhysicsPhysiqueFizika.1.195

[8] A. Aspect, Bell's inequality test: more ideal than ever, Nature **398**, 189-190 (1999). doi: 10.1038/18296

[9] C. Bennett, H. Bernstein, S. Popescu and B. Schumacher, Concentrating Partial Entanglement by Local Operations, Phys. Rev. A **53**, 2046-2052 (1996). doi: 10.1103/PhysRevA.53.2046

[10] S. Bose, Quantum communication through an unmodulated spin chain, Phys. Rev. Lett. **91**, 207901 (2003). doi:10.1103/PhysRevLett.91.207901

[11] A. Bayat and V. Karimipour, Thermal effects on quantum communication through spin chains, Phys. Rev. A **71**, 042330 (2005). doi: 10.1103/PhysRevA.71.042330

[12] A. Bayat and V. Karimipour, Transfer of d-level quantum states through spin chains by random swapping, Phys. Rev. A **75**, 022321 (2007). doi: 10.1103/PhysRevA.75.022321

[13] D. Burgarth and S. Bose, Conclusive and arbitrarily perfect quantum-state transfer using parallel spin-chain channels, Phys. Rev. A **71**, 052315 (2005). doi: 10.1103/PhysRevA.71.052315

[14] M. Christandl, N. Datta, A. Ekert, and A.J. Landahl, Perfect State Transfer in Quantum Spin Networks, Phys. Rev. Lett. **92**, 187902 (2004). doi: 10.1103/PhysRevLett.92.187902

[15] P. J. Pemberton-Ross and A. Kay, Perfect Quantum Routing in Regular Spin Networks, Phys. Rev. Lett. **106**, 020503 (2011). doi: 10.1103/PhysRevLett.106.020503

[16] V. Karimipour, M. Sarmadi Rad, and M. Asoudeh, Perfect quantum state transfer in two- and three-dimensional structures, Phys. Rev. A **85**, 010302 (2012). doi: 10.1103/PhysRevA.85.010302

[17] M. Moradi and V. Karimipour, Generation of quantum states by the dynamics of spin chains: Analytical solution, Phys. Rev. A **99**, 052115 (2019). doi: 10.1103/PhysRevA.99.052115

[18] K.F. Thompson, C. Gokler, S. Lloyd and P.W. Shor, Time independent universal computing with spin chains: Quantum Plinko machine, IOP Publishing **18**, 073044 (2016). doi: 10.1088/1367-2630/18/7/073044

[19] S. Seifnashri, F. Kianvash, J. Nobakht and V. Karimipour, Time-independent quantum circuits with local interactions, Phys. Rev. A **93**, 062342 (2016). doi: 10.1103/PhysRevA.93.062342

[20] G. Lindblad, On the generators of quantum dynamical semigroups, Comm. Math. Phys. **48**, 119 (1976). doi: 10.1007/BF01608499

[21] V. Gorini, A. Kossakowski, and E. Sudarshan, Completely positive dynamical semigroups of N-level systems, J. Math. Phys. **17**, 821 (1976). do: 10.1063/1.522979

[22] M.B. Ruskai, S.J. Szarek and E. Werner, An analysis of completely-positive trace-preserving maps on M2, Linear Algebra and its Applications **347**, 159-187 (2002). doi: 10.1016/S0024-3795(01)00547-X

[23] M.D. Choi, Completely Positive Linear Maps on Complex Matrices, Linear Algebra and its Applications **10**, 285–290 (1975). doi: 10.1016/0024-3795(75)90075-0

[24] V. Karimipour, F. Benatti, and R. Floreanini, Quasi-inversion of qubit channels, Phys. Rev. A **101**, 032109 (2020). doi: 10.1103/PhysRevA.101.032109

[25] F. Shahbeigi, K. Sadri, M. Moradi, K. Życzkowski and V. Karimipour, Quasi-inversion of quantum and classical channels in finite dimensions, J. Phys. A: Math. Theor. **54**, 345301 (2021). doi: 10.1088/1751-8121/ac13db