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Relativistic tunneling through two “transparent” successive barriers

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Abstract: In the case of tunneling of relativistic particles, differently from the nonrelativistic case, a limit of “transparent” barrier can also lead to an apparent “superluminal” behavior when considering the phase time. In this limit, the restricting condition of ”opaque” barrier of the nonrelativistic case is avoided, nevertheless, the very thin width of a single barrier to obtain this “transparent” limit can result in a problem itself, for probing the effect. A combination of two successive transparent barriers can show an apparent “superluminal” behavior along a macroscopic arbitrary distance “L”. Two solutions for energy E above and below the potential square barrier V are found, for both solutions there the apparent superluminal behavior is possible above a threshold of free travelling group velocity (energy) and dependent on the ratio barriers length free path as function of the ratio group velocity - speed of light.

Keywords: Tunneling, relativistic, traversal time

1. Introduction

The traversal time of a particle or a wave packet through a forbidden potential barrier [1–4] has not a unique definition both in nonrelativistic [5] and relativistic case [6]. Different definitions of traversal times have been introduced in the literature [7], each definition could be grouped in four classes, each class has its own peculiarity, unvelling some aspect of the phenomenon but still rising problems on other different facets. No definition of traversal time is universally accepted as unique, anyway some seem more fundamental than others, and one of those, the so called *phase time* is the subject of this article and it will be shown why, in this case, it is specially preferible.

One of the approaches is to use some degree of freedom of the system to define an “internal clock” i.e. some varying physical quantity related to the time spent by a particle inside the barrier. In this class, for example falls the so called *Büttiker-Landauer time* [2] that considers the energy exchanged with a barrier having time-varying height and the *Larmor time* considering the spin flipping inside a magnetic field [8–10]. These times, even if highlighting some interesting physical phenomena, have the problems to be different, depending of ”clocks”, and invasive while changing some degree of freedom of the phenomenon and not clearly related to the effective traversal time or reflection time.

A second class contains the approaches by means of a set of semiclassical trajectories with the calculation of an average tunneling time. Paths can be built through the *Feynman path-integrals* [11], the *Bohm mechanics* [12] or the *Wigner distribution* [13], for example. The inconvenience of such

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approaches is that they have a distribution of complex times even if anyway some real features such as magnitude or real and imaginary parts are related to other definitions of times.

Another approach is the so called *dwell time* that is the ratio between the probability density in the tunneling region and the incoming flux entering the barrier. The problem is that it does not distinguish very well between the transmission and reflection times [14].

Finally, there is the class of times that follow a feature of the wave packet crossing the barrier as a central pick, or a sharp wavefront, etc., relating in time, the ingoing and the outgoing packet. The basic time of this class is the so called *phase time* that uses the definition of group velocity in the stationary-phase approximation.

The stationary phase method was at first introduced by Lord Kelvin [15] and related to group velocity also in the works of Sommerfeld and Brillouin [16] and applied in numerous fields. In our case it can be employed to describe a free wave packet solution of the monodimensional Schrödinger equation

$$\psi(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} G(k,k_0) e^{-\imath(E(k)t+kx-kx_0} \tag{1}$$

with $\hbar = 1$ and $G(k,k_0)$ that physically represents a narrow momentum distribution centered around k_0. This integral can be estimated by finding the value for which the phase has a vanishing derivative, evaluating the integral in the neighborhood of this point since outside of this point the phase is rapidly oscillating giving a null contribute to the integral itself. The movement of the peak coordinate of the wave packet can be obtained by imposing the stationary phase condition $d/dk[Et - k(x-x_0)]|_{k=k_0} = 0$ so that $x_{peak} = x_0 + k_0/mt$ and the peak velocity coincides with the velocity of the particle $v = k_0/m$. This approximation has a limit linked to how much the barrier distorts the wave packet and this distortion is greater for a long "opaque" barrier than for a short "transparent" barrier like in the case of the article. His main positive feature is that it appears to agree rather well to experimental data, more than other times of a different class, including the so called *Hartman effect* [1], an apparent experimental superluminal velocity of a wave packet feature not in contrast with special relativity because not carrying information.

The phase time is defined, in the stationary-phase approximation, as the energy derivative of the transmission phase shifts: $\tau_p = \hbar d\alpha/dE$ given the transmission coefficient as $T(k) = \sqrt{T}e^{\imath\alpha}$ [5]. Applying this definition, in the nonrelativistic case, to tunneling through a rectangular potential barrier of height $V_0 > E$ and width "a" the phase time for a wave packet tends, in the limit of "opaque" barrier ($qa \gg 1$ where q is the momentum inside the barrier) to a constant value independent from the width "a" (Hartman effect [1]) so that it could lead to apparent superluminal velocities. Altough the interpretation of this apparent superluminal effect, is not a subject of the present article, it can be demonstrated that this does not violate Einstein's relativity [17]. There are two main problems with the use of phase time with an "opaque" barrier case, the first is that it is a very restricting condition for experimental testing, considering the exponential decay of the amplitude through a tunneling process and the low signal to noise ratio after the barrier, the second is that the shape of the wave packet is deeply deformed after the passage through an opaque barrier so that the causality relation between identical feature of the packet can be questioned [18].

The importance of the *phase time* is also related to the wave packet analogy of tunneling quantum particles and different kind of guided waves, propagating through a forbidden region, of optical, microwave and acoustic nature, and their experimental results [19–22]. In the case of relativistic particles [23] through a barrier of width "a" the phase time has a different expression but it may be still recognized, in this case, a generalized Hartman effect. For example, in the case of two successive barriers [24], the phase time becomes independent, in the limit of "opaque" barrier, both from the width "a" and from the distance L between the barriers. The limit of opaque barrier gives, by definition, strong constraints for experimental probes, because the wave function amplitude decreases exponentially and this is more effective when more than one barrier is considered. For relativistic particles indeed, differently from the case of nonrelativistic particles, it is possible to consider the opposite limit of "transparent" barrier $qa \ll 1$ that leads, as well, to an apparent superluminal result for the phase time [25]. In this article this limit of "transparent" barrier for relativistic particles is applied to a double barrier configuration. The presence of two barriers of width a and distance L, in some conditions, leads to a more evident apparent superluminal behavior where the ratio a/L is a key factor.

2. Phase time in the approximations of "transparent barriers" and relativistic particles

The equations of the momentum outside ($\hbar k$) and inside ($\hbar q$) a potential barrier of height V_0 of a particle of mass m and energy E, are

$$\hbar kc = \sqrt{E^2 - m^2c^4} \tag{2}$$

$$\hbar qc = \sqrt{m^2c^4 - (V_0 - E)^2}. \tag{3}$$

To have a proper tunneling, V_0 must be in the range $E - mc^2 < V_0 < E + mc^2$, because below the lower limit the particle has enough energy to propagate over the potential barrier while, above the upper limit, the barrier can become supercritical and spontaneously emit positrons and electrons in the so called Klein tunneling [26]. In the limit of "transparent" barriers ($qa \ll 1$), the potential satisfies two solutions: for V_0 greater than the total energy E we have solution (a)

$$V_0 \approx E + mc^2 - \frac{(\hbar q)^2}{2m} \quad \text{for} \quad V_0 > E \tag{4}$$

and for V_0 lesser than the total energy E we have solution (b)

$$V_0 \approx E - mc^2 + \frac{(\hbar q)^2}{2m} \quad \text{for} \quad V_0 < E. \tag{5}$$

The expression for the phase time across two potential barriers of width a separated by a vacuum path of length L is, from Lunardi et al. [24]

$$\tau_p = \frac{1}{\hbar c^2}\left\{(kL)\frac{E}{k^2} - \frac{1}{k^2q^2}\frac{h_1}{\Gamma^2 + \Delta^2}\right\}. \tag{6}$$

The expressions for h_1, Γ and Δ are given in Appendix A. The approximation for "transparent" barriers ($qa \ll 1$), at first order, is given by

$$\frac{h_1}{\Gamma^2 + \Delta^2} \approx \left\{\left[(V_0 - E)k^2\left(\frac{1}{\alpha} - \alpha\right)\right] + \left[-mc^2(k^2 + q^2)\left(\frac{1}{\alpha} + \alpha\right)\right]\right\} qa + O[qa]^2, \tag{7}$$

where $\alpha \equiv \frac{k}{q}\frac{(E-V_0+mc^2)}{(E+mc^2)}$. This formula has a nonrelativistic counterpart if we slow down the particle setting $E = E_k + mc^2$, where the kinetic energy $E_k \ll mc^2$. In this case $\alpha = k/q$. Inserting these substitutions in Equation (6) and eliminating the terms containing $1/c^2$, the expression for the nonrelativistic limit of the phase time τ_{nr}, trough two "transparent" barriers of width a at distance L is

$$\tau_{nr} = \frac{m}{\hbar k}\left[L + a\left(3 + \frac{q^2}{k^2}\right)\right]. \tag{8}$$

This is consistent with the method to obtain nonrelativistic transit across two barriers [27] if the "transparent barrier" limit is applied. Considering instead the approximation of relativistic particles ($E \gg mc^2$) and "transparent" barriers ($qa \ll 1$) solutions (a) (4) and (b) (5) are given in the following:

2.1. Solution (a) for $E < V_0 < E + mc^2$

For this solution with V_0 greater than E, $\alpha \approx \frac{\hbar^2 kq}{2mE} \ll 1$. Substituting V_0 (4) into (7)

$$\frac{h_1}{\Gamma^2 + \Delta^2} \approx \left[\left(-\frac{\hbar^2 k^2 q^2}{2m} - mc^2 q^2\right)\frac{1}{\alpha}\right] qa \tag{9}$$

that, substituting $\alpha \approx \frac{\hbar^2 kq}{2mE} \ll 1$ becomes

$$\frac{h_1}{\Gamma^2 + \Delta^2} \approx \left(-Ekq - \frac{2m^2c^2E}{\hbar^2}\frac{q}{k}\right) qa; \tag{10}$$

so, the phase time τ_p (6) for this solution becomes

$$\tau_p \approx \left(\frac{L}{c^2} + \frac{a}{c^2} + \frac{2m^2}{\hbar^2 k^2}a\right)\frac{E}{\hbar k}. \tag{11}$$

Since the usual phase velocity of the free particle is $V_\phi = E/(\hbar k)$ the final expression for the phase time for this solution is

$$\tau_p \approx \frac{V_\phi}{c^2}\left[L + a\left(1 + \frac{2m^2c^2}{\hbar^2 k^2}\right)\right]. \tag{12}$$

2.2. Solution (b) for $E - mc^2 < V_0 < E$

For this solution with E greater than V_0, $\alpha \approx \frac{2mc^2}{E}\frac{k}{q} \gg 1$. Substituting V_0 (5) and α into (7)

$$\frac{h_1}{\Gamma^2 + \Delta^2} \approx \left(-\frac{c^2 q\hbar^2 k^3}{E} - \frac{2m^2c^4 qk}{E}\right) qa, \tag{13}$$

finally the expression of the phase time for this solution is

$$\tau_p \approx \frac{V_\phi}{c^2}\left[L + \frac{c^2}{V_\phi^2}a\left(1 + \frac{2m^2c^2}{\hbar^2 k^2}\right)\right] \tag{14}$$

that is very similar to (12) considering that, for relativistic particles, $V_\phi \simeq c$.

3. Conditions for traversal time superluminal behavior

For a free relativistic particle, $c^2/V_\phi = V_g$, where V_g is the group velocity or the so called classical velocity of the particle; then, τ_f could be assumed as the time it would take a free relativistic particle to travel the same path of the example, i.e.

$$\tau_f = \frac{V_\phi}{c^2}\,(L+2a)\,. \tag{15}$$

3.1. Conditions for solution (a)

So a free particle takes a longer time to travel the distance $L+2a$, than the phase time, by an amount Δt

$$\Delta t \equiv \tau_f - \tau_p = \frac{V_\phi}{c^2}a\left[1-\frac{2m^2c^2}{\hbar^2k^2}\right] = \frac{a}{V_g}\left[3-2\frac{c^2}{V_g^2}\right] \tag{16}$$

because $c^2/V_g^2 = (\hbar^2k^2+m^2c^2)/\hbar^2k^2$. The tunneling thus is a kind of accelerator of the motion. It must be recalled that Equation (16) is valid for a relativistic particle with $V_g \simeq c$ and it can be seen from (16) that the time gain of a tunneling relativistic particle with respect to a free particle begins when the velocity is $V_g > \sqrt{\frac{2}{3}}\,c = 0.82\,c$ and reaches the limit of a/V_g as V_g grows toward the limiting value c.

The time gain could be such that the motion could be defined superluminal in the sense considered by the Hartman effect: defining the traversal velocity V_T as the traveled path $L+2a$ divided by the phase time τ_p, then

$$V_T = \frac{L+2a}{\tau_p} = \frac{L+2a}{\tau_f-\Delta t} \simeq \frac{L+2a}{\tau_f} + \frac{L+2a}{\tau_f^2}\Delta t \tag{17}$$

that, in terms of free propagating group velocity V_g and of barriers length a, becomes

$$V_T = V_g + V_g\frac{a}{L+2a}\left[3-2\frac{c^2}{V_g^2}\right]. \tag{18}$$

Let us now consider the conditions on V_g and a such that the traversal velocity V_T tends toward the speed of light in vacuum c. Setting $V_T \to c$, $V_g = \beta c$ and $a = \delta L$, the (18) becomes

$$\delta = \frac{\beta^2-\beta}{2+2\beta-5\beta^2}. \tag{19}$$

In Figure 1, δ vs β is plotted. The curve (a) shows the values for which $V_T \to c$. The region on the right of the curve is the region of superluminality. There is no solution for $\beta = (1+\sqrt{11})/5 = 0.8633$ so there is no superluminal effect for $V_g \le 0.8633c$, whatever be the barrier length a. Conversely, for $V_g \ge 0.8633c$ there are values of $\delta \equiv a/L$ for which $V_T \ge c$.

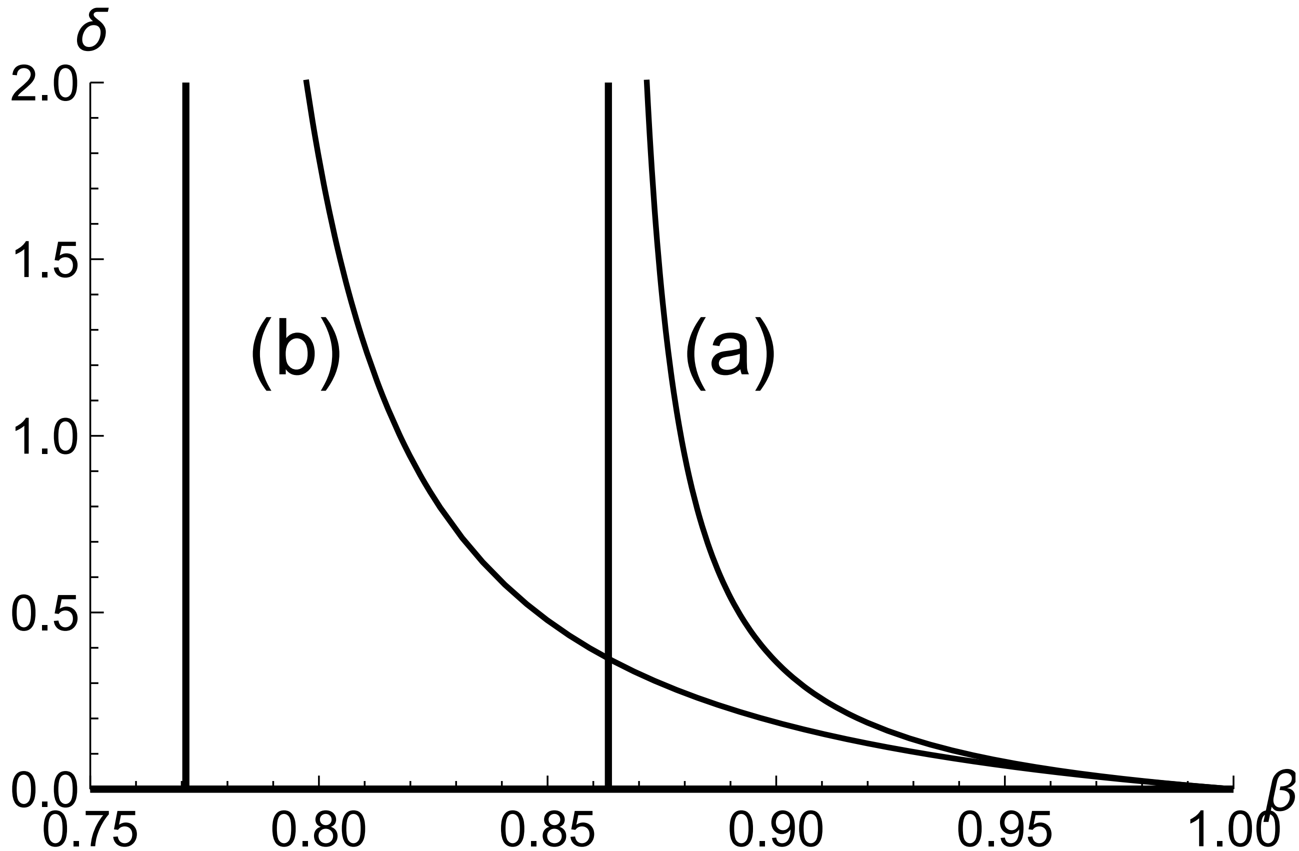

Figure 1. The curves show values of $\delta \equiv a/L$ and $\beta \equiv V_g/c$ for which the traversal velocity $V_T \to c$. The region on the right of the curves is superluminal while on the left is subluminal. The curve (a) is for the energy E lower than the potential barrier V_0 (case a), while the curve (b) is for the case where the energy E is greater than the potential barrier V_0 (case b). The two vertical asymptotes for curves (b) and (a) are at positions $\beta = 0.7709$ and $\beta = 0.8633$, respectively.

3.2. Conditions for solution (b)

In the case in which $E > V_0 > E - mc^2$, the gain in time is

$$\triangle t \equiv t_f - t_p = \frac{aV_g}{c^2} \tag{20}$$

so the traversal velocity becomes

$$V_T = V_g + \frac{a}{L + 2a}\frac{V_g^3}{c^2}, \tag{21}$$

and, differently from the previous case, the traversal velocity V_T is always greater than the group velocity V_g. Proceeding then like in the case (a), defining the ratios $\delta \equiv a/L$ and $\beta \equiv V_g/c$ and setting the limit $V_T \to c$, the corresponding of Equation (19) is, in this case,

$$\delta = \frac{1 - \beta}{\beta^3 + 2\beta - 2}. \tag{22}$$

There is not a possible apparent superluminal behavior of the traversal velocity for $\delta \leq 0$ thus for $\beta \leq [(9+\sqrt{105})^{2/3} - 2 \cdot 3^{1/3}]/[3^{2/3}(9+\sqrt{105})^{1/3}]$ i.e. for $V_g \leq 0.7709c$ while for values above this limit, the apparent superluminal behavior is represented by the space on the right of the curve (b) in Figure. 1.

The meaning of the curves is this: if, for example, we consider the energetic situation of curve (b) and set up $\delta = 1$ and, it is sent against the barriers, a particle with speed such that $\beta = 0.8$, the curve is not matched so the resulting traversal velocity V_T is below c. At about $\beta = 0.8177$ (solution of Equation (22) with $\delta = 1$) the curve (b) is matched and $V_T = c$. So, finally, in the condition of curve (b), for values $\delta = 1$ and $\beta > 0.8177$ the traversal velocity becomes superluminal $V_T > c$. Thus the curves must be seen as a guide to set parameters to obtain superluminal behavior.

4. Conclusion

Considering the *phase time* of relativistic particles passing through a two forbidden barriers of width a and distance L in regime of "transparent" barrier approximation, $qa \ll 1$ an apparent superluminal behavior is found defining the traversal velocity as path divided by *phase time*. In the two cases of energy slightly above and under the barrier potential height, thresholds and conditions for superluminal behavior are found with the former case more favorable than the latter, with equivalent conditions, depending, the gain in time, on the energy of the particle and proportional to the width a of the barriers.

Appendix A

In this appendix the symbols Γ, Δ and h_1 inserted in the expression (6) for the phase time, are explicitly defined.

$$\Gamma \equiv 8\alpha^2 \cosh(2qa) - 4(1+\alpha^2)^2 \sin^2(kL) \sinh^2(qa). \tag{23}$$

$$\Delta \equiv 4\alpha(1-\alpha^2)\sinh(2qa) + 2(1+\alpha^2)^2 \sin(2kL)\sinh^2(qa). \tag{24}$$

$$\begin{aligned} h_1 \equiv\ & \Delta\{2(1+\alpha^2)[(1+\alpha^2)Eq^2(2kL)\sin(2kL) + \\ & -4\alpha^2 mc^2(k^2+q^2)\cos(2kL)]\sinh^2(qa) - 4\alpha^2 mc^2(k^2+q^2) \\ & [(1+\alpha^2) + (3-\alpha^2)\cosh(2qa)] + k^2(2qa)E - V_0) \\ & [(1+\alpha^2)^2\cos(2kl) - (1-6\alpha^2+\alpha^4)]\sinh(2qa))\} + \\ & +\Gamma\{-4\alpha(1-\alpha^2)k^2(2qa)(E-V_0)\cosh(2qa) + 2(1+\alpha^2) \\ & [(1+\alpha^2)Eq^2(2kL)\cos(2kL) + 4\alpha^2 mc^2(k^2+q^2)\sin(2kL)] \\ & \sinh^2(qa) + [4\alpha(1-3\alpha^2)mc^2(k^2+q^2) - (1+\alpha^2)^2 \\ & k^2(2qa)(E-V_0)\sin(2kL)]\sinh(2qa)\}. \end{aligned} \tag{25}$$

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