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A comprehensive review of geometrical thermodynamics: From fluctuations to black holes

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Abstract: This paper presents a comprehensive review of geometrical thermodynamics, which employs geometric concepts to study the thermodynamic properties of physical systems. The review covers key topics such as thermodynamic fluctuation theory, proposed thermodynamic metrics in various coordinate systems, and thermodynamic curvature. Additionally, the paper discusses the geometrical approach to black hole thermodynamics and provides an overview of recent research in this field.

Keywords: Black holes, geometrical thermodynamics, stability

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1. Introduction

Differential geometry is a crucial mathematical discipline that has found numerous applications in physics, particularly in the description of the geometric properties of physical systems. One of its most significant applications is in characterizing the curvature and topology of spacetime to describe the four known interactions of nature. This has yielded significant insights into the fundamental principles of nature and allowed physicists to formulate theories of gravity, such as Einstein's general theory of relativity. Differential geometry also finds use in other areas of physics, including quantum field theory, string theory, and condensed matter physics. By providing a powerful framework for understanding the behavior of massive objects and their interactions with each other in the presence of gravity, differential geometry represents an essential tool for physicists to understand and describe the physical world through the lens of geometry. In this regard, Einstein was the first pioneer of this idea who came up with the relation between the field strength and curvature and proposed the astonishing principle

$$\textit{field strength} \approx \textit{curvature of a Riemannian manifold}$$

to comprehend the physics of the gravitational field [1]. The idea behind this principle can be represented schematically as

$$\textit{metric} \rightarrow \textit{Levi-Civita connection} \rightarrow \textit{Riemann curvature} \approx \textit{gravitational field strength.}$$

In 1953, Yang and Mills introduced a principal fiber bundle associated with a geometric structure to the electromagnetic field. This bundle had the Minkowski spacetime as the base manifold and $U(1)$, representing the internal symmetry of electromagnetism, as the standard fiber [2]. The connection across the fibers was a local cross-section taking values in the algebra of $U(1)$, and the Faraday tensor represented the curvature of this fiber bundle. Their idea was extended to non-Abelian gauge theories by using different connections as local cross-sections. They found that the weak and strong interactions could be represented as the curvature of a principal fiber bundle having a Minkowski base manifold and the standard fiber $SU(2)$ and $SU(3)$, respectively [1]. This construction can be represented schematically as follows

$$\begin{array}{lllll} & \Rsh & \textit{U(1)-connection} & \rightarrow & \textit{U(1)-curvature} \approx \textit{electromagnetic interaction strength} \\ \textit{Minkowski metric} & \rightarrow & \textit{SU(2)-connection} & \rightarrow & \textit{SU(2)-curvature} \approx \textit{weak interaction strength} \\ & \Lsh & \textit{SU(3)-connection} & \rightarrow & \textit{SU(3)-curvature} \approx \textit{strong interaction strength.} \end{array}$$

This line of thought continued until a few decades ago when some physicists, inspired by the central role of curvature in describing field interactions, attempted to extrapolate these ideas to thermodynamic systems and make statements of the following form

$$\textit{thermodynamic interaction} \approx \textit{curvature.}$$

Indeed, this idea led to the challenge of finding an intrinsic geometric formulation for thermodynamics. The starting point is to introduce an equilibrium space, an abstract space whose points can be interpreted as representing the equilibrium states of the system, which can be uplifted to a differential manifold endowed with a metric structure. The existence of such a metric endows spaces of thermodynamic equilibrium states with the notion of a length between the states such that, shorter the distance between a pair of thermodynamic states, the more probable is a fluctuation between them. It is notable that significant information is revealed through the invariants of the geometry, like the curvature scalar. For instance, it has been turned out that the magnitude of the scalar curvature invariant, known as thermodynamic curvature, is proportional to the correlation volume of ordinary thermodynamic system and thus is a measure of interaction strength, and yields fundamental information about inter-particles interaction [3, 4]. More specifically, a negative (positive) sign of the thermodynamic curvature is an indicator of the attractive (repulsive) nature of the interaction between particles, whereas zero value for the thermodynamic curvature means there is no interaction between particles [5–7]. Furthermore, it has been shown that in addition to describing the critical point through its singularity, the scalar curvature also encodes first-order phase transitions in simple fluids, uniquely determines the Widom line in different regimes, and identifies the experimentally determined solid-like patches in the liquid phases [8, 9].

Since the original works by Gibbs [10] and Caratheodory [11], different geometric representations of thermodynamics, depending on the potential chosen, have been explored until now, the most important of which is as follows:

- **Representation based on the Riemannian geometry:** This representation was first applied in thermodynamics and statistical physics by Rao [12], in 1945, using the entropy as thermodynamic potential. This approach which comes from information theory, leads to the Fisher-Rao metric in the thermodynamic limit. In fact, Rao introduced a metric whose components in local coordinates coincide with Fisher's information matrix [12]. Later, his initial work was extended by a number of authors (for a review, see, e.g., [13]). In the continuation of this path, Hessian metrics were postulated to describe the geometric properties of the equilibrium space, commonly known as thermodynamic geometry. In particular, metric structures such as the Hessian of the internal energy and the entropy were proposed by Weinhold [14, 15] and Ruppeiner [3, 16], respectively. It is important to note that Ruppeiner's and Weinhold's metrics are conformally equivalent, with the inverse of the temperature as the conformal factor. Although both metrics have been widely used to study the geometry of the equilibrium space of ordinary systems [17–23], they suffer from the major drawback of not being invariant under Legendre transformations, which means that a given thermodynamical system has different geometrical properties, depending on the thermodynamical potential used.

- **Representation based on the contact geometry:** Contact geometry was first introduced by Hermann [24] into the thermodynamic phase space and later explored by Mrugala [25, 26]. This approach aims to formulate the geometric version of the laws of thermodynamics in a consistent manner. Indeed, contact geometry allows us to consider a Legendre transformation as a coordinate transformation in the phase space [27]. This issue was used by Quevedo [28] to propose the formalism of geometrothermodynamics whose purpose is to incorporate the property of the Legendre invariance in Riemannian structures at the level of the phase space and the equilibrium space. This property guarantees that the thermodynamic characteristics of a system do not depend on the thermodynamic potential used for its description. This geometric setting is widely used to study thermodynamics [29–31], mechanical systems with Rayleigh dissipation [32, 33], statistical mechanics [34] as well as black hole thermodynamics [35].

The geometric theory of thermodynamics offers a robust framework for calculating equations of state without the explicit reliance on underlying microscopic models. Consequently, it can serve as a valuable tool to investigate systems whose microscopic nature is unknown, precluding the use of statistical mechanics to study their properties. Figure 1 provides a visual comparison of the approaches employed by the geometric theory and statistical mechanics in this regard.

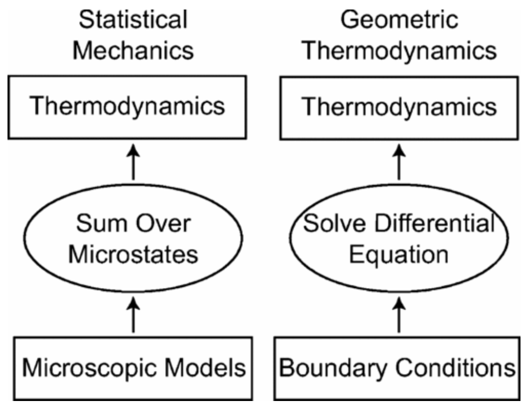

Figure 1: Contrasting philosophies of statistical mechanics and the geometric theory (adapted from [36]).

Since the advent of the geometric representation of thermodynamics, it has proven to be an invaluable tool for uncovering the quantum nature of black holes and gravity. Although a complete understanding of their microscopic degrees of freedom remains elusive, black hole thermodynamics was first explored in the early 1970s by Hawking and Bekenstein. Their pioneering work revealed that black holes are not merely gravity systems but also possess thermal properties. Subsequently, the laws of black hole mechanics were formulated, which correspond to conventional thermodynamics if suitable quantities such as temperature, entropy, energy, etc., are appropriately identified. Based on these discoveries, it was found that black holes radiate as black bodies, possessing characteristic temperatures and entropies [37–41]

$$kT_{\mathrm{H}} = \frac{\hbar\kappa}{2\pi}, \qquad S_{\mathrm{BH}} = \frac{A_{\mathrm{H}}}{4\hbar G}, \tag{1.1}$$

where κ is the surface gravity and A_{H} stands for the area of the horizon, $\hbar$ denotes Planck's constant and G is Newton's constant. However, although it is believed that a black hole does possess thermodynamic quantities and extremely interesting phase structure and a vast amount of studies have been done concerning this issue [42–98], the statistical description of the black hole microstates has not yet been fully understood. Even though a complete quantum gravity theory is still absent, there have been some attempts to understand the microscopic structure of a black hole [99–111]. In this regard, the thermodynamic geometry method has led to many insights into the microstructure of a black hole. In this paper, we aim to provide a review of the studies that have been done on the black hole thermodynamics from the viewpoint of geometry.

The aim of this paper is to provide a comprehensive review of geometrical thermodynamics. We begin in Section 2 by reviewing the fundamentals of thermodynamic fluctuation theory, which serves as the basis for identifying appropriate thermodynamic geometric representations. In Section 3, we explore some essential and fundamental geometries proposed in the context of thermodynamics using the language of differential geometry. This section includes an introduction to Hessian thermodynamic metrics (Section 3.1) and Legendre invariant metrics (Section 3.2). Moreover, we review the application of these metrics to some ordinary thermodynamic systems in Section 3.3. We then discuss and review the concept of thermodynamic curvature, which arises from thermodynamic fluctuation theory, in Section 4. After introducing the geometric perspective of thermodynamics, we shift our focus to studying black holes using geometric thermodynamic approaches. To accomplish this, we first review the laws of black hole mechanics in Section 5.1, followed by the laws of black hole thermodynamics in Section 5.2. Subsequently, we discuss and review the applications of geometric thermodynamics in investigating the nature of black holes in Section 5.3. Finally, we end the paper with a summary in Section 6.

2. Thermodynamic fluctuation theory

The macroscopic physical observables of a system are rooted in microscopic quantities at equilibrium, which typically remain close to their average values. The random deviations from these values that describe the natural behavior of a system are referred to as thermodynamic fluctuations, which are studied within the context of thermodynamic fluctuation theory.

In most statistical mechanical contexts, the Gaussian approximation of thermodynamic fluctuations theory yields the probability of finding a system in a particular thermodynamic state [112]. To illustrate, consider an infinite system of particles and an imaginary open volume with a fixed volume V, into which the particles can move freely in and out, as depicted in Figure 2. This theory provides a suitable framework for determining the probability of finding a specific energy U and a particular number of particles N within the open volume.

However, it has been demonstrated that moving beyond the Gaussian approximation by taking into account covariance, conservation, and consistency results in a fundamentally novel entity: the thermodynamic Riemannian curvature $\mathcal{R}$. This quantity is a thermodynamic invariant that reveals information about interparticle interactions [3]. In statistical mechanics, Boltzmann's expression for

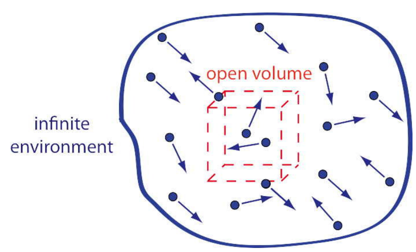

Figure 2: An infinite environment of particles and an open volume, with fixed volume V, into which particles fluctuate in and out (adapted from [113]).

the entropy of a system in the microcanonical ensemble is defined as follows [114]

$$S = k_B \ln \Omega, \tag{2.1}$$

where k_B is the Boltzmann constant and Ω indicates the number of microstates of the system and is the function of extensive quantities like energy, volume, and particle numbers, i.e. $\Omega = \Omega(U, V, N)$. In 1907, Einstein proposed a theory which based on the relation (2.1) is inverted to express the number of states in terms of entropy [114]:

$$\Omega = \exp\left(\frac{S}{k_B}\right). \tag{2.2}$$

This relation was used as the basis of thermodynamic fluctuation theory.

In what follows we begin by studying the thermodynamic fluctuations with a single variable, and explain the problems that lead to efforts to go beyond the Gaussian approximation. We will show that the corrected theory is obtained by introducing a second independent fluctuating variable which results in the appearance of the thermodynamic curvature concept.

2.1. One fluctuating variable

Consider a thermodynamic system that exhibits equilibrium properties described by a set of n additive and invariant parameters denoted as $X_1, X_2, ..., X_n$. In statistical mechanics, the understanding of a system's thermodynamic properties involves studying its fundamental equation as a function of the conserved parameters. For instance, in the entropy representation, the fundamental equation can be expressed as

$$S = S(X_1, X_2, ..., X_n). \tag{2.3}$$

For a simple fluid, the fundamental equation takes the form $S = S(U, V)$ where U represents the energy of the fluid, and V denotes the volume of space that the fluid occupies. Since U, V, and S are additive quantities, the fundamental equation can be expressed as

$$S = V\, s(u), \tag{2.4}$$

where s and u are the entropy and energy per volume, respectively.

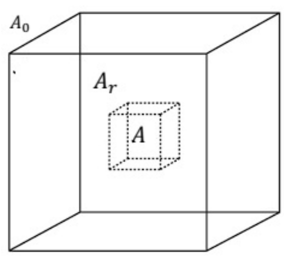

Figure 3: A closed system A_0 consists of an enclosed, open system A and reservoir A_r.

Now, we consider a very large closed thermodynamic system A_0 in thermodynamic equilibrium, with fixed volume V_0 and fixed energy per volume u_0. Suppose that the mentioned thermodynamic system is divided into two subsystems by a fixed and open partition: a finite system A, with fluctuating internal energy per volume u and constant volume V, and the reservoir A_r, with fluctuating internal energy per volume u_r and constant volume V_r as depicted in Figure 3. It is important to mention that due to being open the boundary between A and A_r, these two subsystems exchange energy through fluctuations, which can be described by thermodynamic fluctuation theory. The basic postulate of the microcanonical ensemble is that the occurrence probability of all accessible microstates of A_0 is equal [114]. Hence, the probability of finding the internal energy per volume of A between u and $u + du$ will be proportioned to the microstates' number of A_0 existing in this range:

$$P(u, V)du = C\Omega_0(u, V)du, \tag{2.5}$$

where $\Omega_0(u, V)$ indicates the density of states, and C stands for a normalization factor. Making use of Eq. (2.2), the Einstein's famous relation reduces to

$$P(u,V)du = C\exp\left[\frac{S_0(u,V)}{k_B}\right]du, \tag{2.6}$$

where $S_0(u,V)$ is the entropy of A_0. Statistical mechanics asserts that for a thermodynamic system in the equilibrium state, energy distribution between subsystems occurs in such a way that the system's entropy is maximized. In this context, the maximum value of $S_0(u,V)$ corresponds to the state in which the two parts A and A_r admit the energy $u = u^*$ and $u_r = u_r^*$, respectively. Using the fact that the entropy is additive

$$S_0(u,V) = V s(u) + V_r\, s(u_r), \tag{2.7}$$

and expanding each of entropy densities $s(u)$ and $s(u_r)$ around their maximum energy values, we get

$$\begin{aligned} S_0(u,V) \quad = \quad & \tilde{S}_0 + V s'(u^*)\Delta u + V_r s'(u_r^*)\Delta u_r + V\frac{1}{2!}s''(u^*)(\Delta u)^2 \\ & + V_r\frac{1}{2!}s''(u_r^*)(\Delta u_r)^2 + V\frac{1}{3!}s'''(u^*)(\Delta u)^3 + \cdots , \end{aligned} \tag{2.8}$$

where $\Delta u = u - u^*$ and $\Delta u_r = u_r - u_r^*$. Moreover, the prime denotes differentiation with respect to u, and $\tilde{S}_0$ indicates the entropy of A_0 in equilibrium. Since the first order expansion of the entropy must be zero at the maximum points

$$V s'(u^*)\Delta u + V_r s'(u_r^*)\Delta u_r = 0, \tag{2.9}$$

applying the conservation law of energy, $V\Delta u = -V_r\Delta u_r$, results in the relation $s'(u^*) = s'(u_r^*)$ which forces $u^* = u_r^*$. In addition, additivity of energy requires that $Vu^* + V_r u_r^* = V_0 u_0$, leading to $u^* = u_r^* = u_0$. Therefore,

$$\Delta u = u - u_0, \qquad\qquad \Delta u_r = u_r - u_0. \tag{2.10}$$

According to the energy conservation law, higher-order terms in the expansion of $s(u_r)$ can often be neglected when compared to the corresponding terms in the expansion of $s(u)$. By truncating the series of equation Eq. (2.8) at second order and considering the resulting Gaussian distribution, we can approximate fluctuations in thermodynamic fluctuation theory. This approximation is commonly used in statistical mechanics and has been described by authors such as [114, 115]

$$P_G(u,V)du = C\exp\left[-\frac{V}{2}g(u_0)(\Delta u)^2\right]du, \tag{2.11}$$

where $g(u_0) \equiv \frac{-s''(u_0)}{k_B}$ and due to being proportional to the heat capacity, it is always positive. In order to obtain the normalization factor, one can use the normalization condition $\int P_G(u,V)du = 1$ which results in

$$C = \sqrt{\frac{V\,g(u_0)}{2\pi}}. \tag{2.12}$$

2.1.1. Some shortcomings of thermodynamic fluctuation theory

While Eq. (2.11) is effective, there are limitations when we go beyond the Gaussian approximation. One problem occurs when we change the thermodynamic parameter u to a general parameter $x(u)$. Since the counting of microstates resulting in Eq. (2.6) can be performed equally well with the parameter x, we should get the same expression except that u is replaced by x:

$$P(x,V)dx = C\exp\left[\frac{S_0(x,V)}{k_B}\right]dx. \tag{2.13}$$

However, we obtain a different expression if we make a straightforward coordinate transformation on the left hand side of Eq. (2.6)

$$P(u,V)du = \left[P(u,V)\left(\frac{du}{dx}\right)\right]dx \equiv P(x,V)dx, \tag{2.14}$$

which gives the probability of finding the new parameter in the range from x to $x+dx$. Since entropy is a function of state, then $S_0(u,V) = S_0(x,V)$. Therefore, we end up with

$$P(x,V)dx = C\left(\frac{du}{dx}\right)\exp\left[\frac{S_0(x,V)}{k_B}\right]dx, \tag{2.15}$$

which is inconsistent with Eq. (2.13) because the factor $du = (du/dx)dx$ is not generally a constant and cannot simply be absorbed into the normalization factor. Hence, it can be concluded that the formulation of the thermodynamic fluctuation theory depends on the coordinates used and its equation is not covariant. It is important to mention that this problem does not arise in the Gaussian approximation. One can easily find that the Gaussian approximation will be covariant if the following transformation rule

$$g(x_0) = g(u_0)\left(\frac{du}{dx}\right)^2, \tag{2.16}$$

is used for the function $g(x_0)$. Indeed, since the fluctuations are small in the regime of validity of the Gaussian approximation, the derivative of u with respect to x is constant over this range. Therefore, small fluctuations guarantee the covariance validity of the theory in the Gaussian approximation.

Another class of problems is concerned with the fact that this theory does not provide correct values for the average value of the standard densities in the entropy expansion beyond the Gaussian approximation. To explain more, we define the average value of a thermodynamic function as

$$\langle f\rangle = \int f(u)\,P(u,V)du, \tag{2.17}$$

and hence,

$$\langle \Delta u\rangle = \int \Delta u\,P(u,V)du. \tag{2.18}$$

From a physical point of view, we expect that $\langle\Delta u\rangle = 0$, which results in a conservation rule, for any physically correct probability density $P(u,V)$ and for all V. Although the mentioned conservation rule, i.e. $\langle\Delta u\rangle = 0$, obtains in the Gaussian approximation because the probability density is an even function of $u-u_0$, thermodynamic fluctuation theory violates this rule beyond that approximation. In fact, if the fluctuations are not small enough, we should consider the third-order term in the entropy expansion which leads to $\langle\Delta u\rangle \neq 0$.

2.1.2. Toward a consistent and covariant theory

Now, we try to find an appropriate solution to the problems that have been mentioned. To this end, we take a different approach from Eq. (2.6) because this equation is fundamentally incorrect beyond the Gaussian approximation.

Since the form of the Gaussian approximation Eq. (2.11) is similar to the solution of a diffusion equation, with the role of "time" played by the inverse volume i.e. $t = 1/V$, and with a Dirac delta function peaked at $\Delta u = 0$, one can be guided to find a diffusion-like equation. The most general form of the diffusion equation which maintains normalization can be expressed as [3, 116]

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[K(x)P] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[g^{-1}(x)P\right], \tag{2.19}$$

where $K(x)$ and $g(x)$ are drift term which should be determined. It is important to mention that t is simply a measure of volume and nothing is diffusing in time. Indeed, the *diffusion* reflects the fact that the thermodynamic state of A gets increasingly undetermined by decreasing V. By specifying $K(x)$ and $g(x)$, along with the initial condition and appropriate boundary conditions, one can find the covariant theory [3].

Now we can examine some general consequences of this equation:

• For constant functions $K(x) = K$ and $g(x) = g$ which is justified for small t, this has the exact normalized solution of the form of Gaussian expression

$$P_G(x,t) = \sqrt{\frac{g}{2\pi\, t}}\exp\left[-\frac{1}{2t}\, g(\Delta x - K\, t)^2\right], \tag{2.20}$$

where $\Delta x \equiv x - x_0$, and x_0 is a constant. This solution has the Dirac delta function $\delta(x-x_0)$ starting at $t = 0$.

• If x_a and x_b are the limits of thermodynamic space and hence the smallest and largest values of x, we will arrive at the following relations by successive multiplications by x and integration by parts

$$\frac{d}{dt}\int_{x_a}^{x_b} P\, dx = -(K\, P)\Big|_{x_a}^{x_b} + \frac{1}{2}\frac{\partial}{\partial x}(g^{-1}P)\Big|_{x_a}^{x_b}, \tag{2.21}$$

$$\frac{d}{dt}\langle x\rangle \;=\; \langle K\rangle - (xKP)\Big|_{x_a}^{x_b} + \frac{1}{2}x\frac{\partial}{\partial x}(g^{-1}P)\Big|_{x_a}^{x_b} - \frac{1}{2}(g^{-1}P)\Big|_{x_a}^{x_b}, \tag{2.22}$$

and

$$\frac{d}{dt}\left\langle x^2\right\rangle = \left\langle g^{-1}\right\rangle + 2\,\langle x\, K\rangle + boundry\ \ terms. \tag{2.23}$$

Taking t to be small, and switching back to the u coordinate we find that Eqs. (2.22) and (2.23) reduce to

$$\langle \Delta x\rangle \;=\; \int \Delta x\, P_G(x,t)dx = K\, t, \tag{2.24}$$

and

$$\left\langle (\Delta x)^2\right\rangle = \int (\Delta x)^2\, P_G(x,t)dx = \frac{t(1+g\, K^2\, t)}{g}. \tag{2.25}$$

Comparing Eqs. (2.20) and (2.24) with the corresponding ones from the thermodynamic Gaussian expression Eqs. (2.11), (2.18) shows that $K = 0$ and $t = \frac{1}{V}$. Also, g is fixed by $g(u_0) \equiv \frac{-s''(u_0)}{k_B}$, specifying the width of the fluctuations. It may seem that the function $K(x)$ in Eq. (2.19) is irrelevant due to the fact that $K(u) = 0$ in u coordinates. However, the importance of its non-zero value becomes clear in the topic of coordinate transformation.

- To study about how to transform Eq. (2.19) to another coordinate $\tilde{x} = \tilde{x}(x)$, we first investigate the possibility of transforming $P(x,t)$, $K(x)$, and $g(x)$ such that the form of the partial differential equation is unchanged

$$\frac{\partial \tilde{P}}{\partial t} = -\frac{\partial}{\partial \tilde{x}}\left[\tilde{K}(\tilde{x})\tilde{P}\right] + \frac{1}{2}\frac{\partial^2}{\partial \tilde{x}^2}\left[\tilde{g}^{-1}(\tilde{x})\tilde{P}\right]. \tag{2.26}$$

Since the probability is a scalar quantity independent of the choice of coordinate, it requires $\tilde{P}d\tilde{x} = Pdx$, and thus

$$\tilde{P} = P\left(\frac{dx}{d\tilde{x}}\right). \tag{2.27}$$

Substituting this expression into Eq. (2.26) and after some straightforward calculation we find that the coefficients of the corresponding derivatives of P is equal to Eq. (2.19) if and only if the functions K and g transform as

$$\tilde{g} = g\left(\frac{dx}{d\tilde{x}}\right)^2, \tag{2.28}$$

and

$$\tilde{K} = K\frac{d\tilde{x}}{dx} + \frac{1}{2}g^{-1}\frac{d^2\tilde{x}}{dx^2}. \tag{2.29}$$

Hence, by using the appropriate transformation rules, the partial differential equation Eq. (2.19) will be covariant and in this way, we have solved the problem of fluctuation theory with one variable.

Before ending this subsection, it is worth mentioning that Eq. (2.28) which is matched Eq. (2.16) is exactly the metric transformation in Riemannian geometry. Therefore, this geometry arises naturally from our demand for a covariance fluctuation theory under coordinate transformations. Based on this issue, a thermodynamic Riemannian geometry can be built and system's fluctuations can be investigated according to their geometrical properties.

2.2. Two fluctuating variables

Now, we discuss fluctuations with two independent variables. This situation can occur if the subsystem and its environment exchange particles as well as energy. We will see that the two independent fluctuating variables allow a Riemannian geometry with non-zero thermodynamic curvature.

Consider a pure fluid system of N identical particles in a volume V whose the fundamental equation is [117]

$$S = S(U, N, V). \tag{2.30}$$

Since S, U, N, and V are all additive thermodynamics parameters, we can write

$$S = Vs(u, \rho), \tag{2.31}$$

where $\{s,u,\rho\} \equiv \{\frac{S}{V},\frac{U}{V},\frac{N}{V}\}$ are quantities per volume. Similar to the one variable fluctuating case, we divide a closed, infinite system A_0 into two parts: a finite A and an infinite reservoir A_r, by an imaginary, immovable partition as shown in Figure 1. Both the internal energy U and the number of particles N are allowed to fluctuate [114]. Here, we define the extensive parameters as $\{\mathcal{E}^1,\mathcal{E}^2\} \equiv \{u,\rho\}$, and the corresponding intensive parameters get as

$$\mathcal{I}_\alpha \equiv \frac{\partial\, s}{\partial \mathcal{E}^\alpha}, \tag{2.32}$$

where $\alpha = 1,2$. Basic thermodynamics gives $\{\mathcal{I}_1,\mathcal{I}_2\} = \{\frac{1}{T},-\frac{\mu}{T}\}$, where T is the temperature, and μ is the chemical potential. The properties of A_0 and A_r are denoted by subscripts 0 and r, respectively.

The probability of finding the state of A in the range (u,ρ) to $(u+du,\rho+d\rho)$ is

$$P du d\rho = C \exp\left(\frac{S_0}{k_B}\right) du d\rho, \tag{2.33}$$

where C is a normalization factor, and S_0 can be written as

$$S_0 = V s(u,\rho) + V_r s(u_r,\rho_r). \tag{2.34}$$

There will be a maximum value for S_0 in the state where $\mathcal{E}^\alpha = \mathcal{E}_r^\alpha = \mathcal{E}_0^\alpha$. By applying the same argument as used in Sec. II.A, and expanding the entropies in Eq. (2.34) around this maximum in powers of the differences $\Delta\mathcal{E}^\alpha = \mathcal{E}^\alpha - \mathcal{E}_0^\alpha$ and $\Delta\mathcal{E}_r^\alpha = \mathcal{E}_r^\alpha - \mathcal{E}_0^\alpha$, we get

$$\begin{aligned} \Delta S_0 \quad = \quad & V\mathcal{I}_\mu \Delta\mathcal{E}^\mu + V_r \mathcal{I}_{r\mu}\Delta\mathcal{E}_r^\mu + V\frac{1}{2!}\frac{\partial \mathcal{I}_\mu}{\partial \mathcal{E}^\nu}\Delta\mathcal{E}^\mu\Delta\mathcal{E}^\nu \\ & + V_r\frac{1}{2!}\frac{\partial \mathcal{I}_{r\mu}}{\partial \mathcal{E}_r^\nu}\Delta\mathcal{E}_r^\mu\Delta\mathcal{E}_r^\nu + \cdots , \end{aligned} \tag{2.35}$$

where ΔS_0 is the difference between S_0 and its maximum value, and summation over repeated indices is assumed. A necessary condition for maximum entropy, i.e. $\mathcal{I}_\alpha = \mathcal{I}_{r\alpha}$, is given by the conservation of both energy and particles which requires that $V\Delta\mathcal{E}^\alpha = -V_r\Delta\mathcal{E}_r^\alpha$. Also, by neglecting the second quadratic term of Eq. (2.35) compared with the first one, this equation reduces to

$$\frac{\Delta S_0}{k_B} = -\frac{V}{2} g_{\mu\nu}\Delta\mathcal{E}^\mu\Delta\mathcal{E}^\nu, \tag{2.36}$$

where the symmetric intensive matrix

$$g_{\mu\nu} \equiv -\frac{1}{k_B}\frac{\partial^2 s}{\partial\mathcal{E}^\mu\partial\mathcal{E}^\nu}, \tag{2.37}$$

must be positive-definite because the entropy has a maximum value in equilibrium. By substituting the above equation into Eq. (2.33) and calculating the normalization factor, we arrive at the Gaussian approximation as follows [114]

$$P_G\, d\mathcal{E}^1 d\mathcal{E}^2 = \left(\frac{V}{2\pi}\right)\exp\Big(-\frac{V}{2}g_{\mu\nu}\Delta\mathcal{E}^\mu\Delta\mathcal{E}^\nu\Big)\sqrt{g}\, d\mathcal{E}^1 d\mathcal{E}^2, \tag{2.38}$$

where g is the determinant of the matrix $g_{\alpha\beta}$. Regarding the transformation law for a first rank contravariant tensor

$$\Delta\mathcal{E}^{\mu} = \frac{\partial \mathcal{E}^{\mu}}{\partial x^{\alpha}} \Delta x^{\alpha}, \tag{2.39}$$

it can easily be shown that the exponential part of Eq. (2.38) behaves like a scalar quantity under the coordinate transformation. This invariant treatment should be maintained because the probability of fluctuation between two states and the entropy difference should not depend on the coordinates used to describe the system. Since the part of $g_{\mu\nu}\Delta\mathcal{E}^{\mu}\Delta\mathcal{E}^{\nu}$ has the look of the distance between thermodynamic states in the form of a Riemannian metric, it leads us to define a new and useful component for our discussion named *thermodynamic length*

$$\Delta\ell^2 \equiv g_{\mu\nu}\Delta\mathcal{E}^{\mu}\Delta\mathcal{E}^{\nu}, \tag{2.40}$$

which is a positive-definite quantity. Indeed, the above relation represents a well-defined Riemannian metric in the space of thermodynamic state, as described in the Appendix A. From a physical standpoint, the interpretation of the distance between two thermodynamic states in this space is clear: *the distance represents the measure of probability in thermodynamics*. The greater the probability of a fluctuation occurring between two states, the closer together they are considered to be. It is worth noting that the dimension of the squared thermodynamic length is inversely proportional to volume, which differs from the Riemannian geometry in general relativity (GR) where the dimension of the squared length is typically measured in meters squared.

3. Thermodynamic metrics in different coordinate systems

After familiarizing ourselves with the concept of geometrical thermodynamics, we will now examine some important and fundamental geometries proposed in the context of thermodynamics using the language of differential geometry. Differential geometry has been extensively applied in physics, chemistry, and engineering, and has found broad applications in thermodynamics, offering an alternative way to describe thermodynamic systems. To begin, we introduce the equilibrium space, which is an abstract space whose points can be interpreted as representing the equilibrium states of the system. Various attempts have been made to provide a suitable thermodynamic metric, some of which we will review below.

3.1. Hessian thermodynamic metrics

In the previous section, we discussed the origin of the concept of geometrical thermodynamics in the context of thermal fluctuations. Drawing on the fluctuation theory of equilibrium states of physical systems, we define the appropriate thermodynamic metric as the Hessian metric, which is based on the Hessian of the thermodynamic potentials. Here, we will examine the issue that different metrics have been proposed depending on the parameters of the thermodynamic phase space and introduce some of the provided metrics:

3.1.1. Ruppeiner metric

The Ruppeiner metric naturally appears in thermodynamic fluctuation theory and is based on the line element given in Eq. (2.40) [3, 16]

$$\Delta\ell^2 = g^{R}_{\mu\nu}\Delta\mathcal{E}^{\mu}\Delta\mathcal{E}^{\nu}. \tag{3.1}$$

An important point that should be noted is that we are not constrained to express this metric in a specific coordinate and different metrics can be defined with different choices of thermodynamic coordinates. Indeed, using the fact that the value of $\Delta\ell^2$ must be independent of the coordinates used to calculate one can explore the form of metric in other coordinate systems. Keeping this point in mind, we will find the form of the Ruppeiner metric in some thermodynamic coordinate systems:

► $(u, \mathcal{E}^i)$ **fluctuation coordinate**

As explained earlier, working in the systems whose coordinates are extensive parameters, i.e. $(u, \mathcal{E}^1, \mathcal{E}^2, ..., \mathcal{E}^r)$*, leads to Eq.

$$g^R_{\mu\nu} = -\frac{1}{k_B}\frac{\partial^2 s}{\partial\mathcal{E}^\mu\partial\mathcal{E}^\nu}, \tag{3.2}$$

which indeed is the Hessian matrix of the thermodynamic entropy (2.37) [3, 16]. Moreover, the thermodynamic stability necessitates the definite positivity of $g^R_{\mu\nu}$.

► $(T, \mathcal{E}^i)$ **fluctuation coordinate**

Another coordinate system is defined based on the intensive as well as extensive parameters. In this case, making use of Eqs. (2.37) and (2.40) along with Eq. (2.32) and the transformation law (2.39) we get

$$\Delta\ell^2 = -\frac{1}{k_B}\Delta\mathcal{I}^\mu\Delta\mathcal{E}_\mu. \tag{3.3}$$

Moreover, to express the metric completely in terms of the intensive parameters, we use

$$\Delta\mathcal{E}^\mu = \frac{\partial\mathcal{E}^\mu}{\partial\mathcal{I}^\alpha}\Delta\mathcal{I}^\alpha, \tag{3.4}$$

and substituting it into Eq. (3.3) yields

$$\Delta\ell^2 = \frac{1}{k_B}\frac{\partial^2\phi}{\partial\mathcal{I}^\mu\partial\mathcal{I}^\nu}\Delta\mathcal{I}^\mu\Delta\mathcal{I}^\nu, \tag{3.5}$$

where ϕ is the Legendre transformation of the entropy relative to all extensive parameters of the system (except volume) and reads as

$$\phi(\mathcal{I}^0, \mathcal{I}^1, ..., \mathcal{I}^r) = s - \mathcal{I}^\mu\mathcal{E}_\mu. \tag{3.6}$$

Working in the entropy representation, the intensive parameters, given by Eq. (2.32), can be obtained as follows

$$\{\mathcal{I}^0, \mathcal{I}^i\} = \{\frac{1}{T}, -\frac{\mu^i}{T}\}, \tag{3.7}$$

*The zeroth component is internal energy and other components denote other extensive parameters of the system in which μ^i is the chemical potential of the i^{th} fluid component. Using the above relation and doing the following substitution into Eq. (3.3)

$$\Delta\mathcal{E}^0 = \Delta u = T\,\Delta s + \sum_{i=1}^{r}\mu^i\Delta\mathcal{E}^i, \tag{3.8}$$

$$\Delta\mathcal{I}^0 = -\frac{1}{T^2}\Delta T, \tag{3.9}$$

$$\Delta\mathcal{I}^i = \frac{\mu^i}{T^2}\Delta T - \frac{1}{T}\Delta\mu^i, \qquad 1 \le i \le r \tag{3.10}$$

we obtain

$$\Delta\ell^2 = \frac{1}{k_B T}\Delta T\Delta s + \frac{1}{k_B T}\sum_{i=1}^{r}\Delta\mu^i\Delta\mathcal{E}^i. \tag{3.11}$$

Let us find the form of the metric in $(T, \mathcal{E}^1, \mathcal{E}^2, ..., \mathcal{E}^r)$ coordinate system. To this end, we use the following relation

$$\Delta s = \frac{\partial s}{\partial T}\Delta T + \sum_{i=1}^{r}\frac{\partial s}{\partial\mathcal{E}^i}\Delta\mathcal{E}^i, \tag{3.12}$$

$$\Delta\mu^i = \frac{\partial\mu^i}{\partial T}\Delta T + \sum_{i=1}^{r}\frac{\partial\mu^i}{\partial\mathcal{E}^i}\Delta\mathcal{E}^i, \tag{3.13}$$

as well as the Maxwell relation

$$\frac{\partial s}{\partial\rho^i} = -\frac{\partial\mu^i}{\partial T}, \tag{3.14}$$

and substituting Eq. (3.14) into Eq. (3.11), we get

$$\Delta\ell^2 = \frac{1}{k_B T}\frac{\partial s}{\partial T}(\Delta T)^2 + \frac{1}{k_B T}\sum_{i,j=1}^{r}\frac{\partial\mu^i}{\partial\mathcal{E}^j}\Delta\mathcal{E}^i\Delta\mathcal{E}^j. \tag{3.15}$$

► $(T, \mathfrak{I}^i)$ **fluctuation coordinate**

As another example, we try to express the metric in $\mathfrak{I} = (T, \mu^1, \mu^2, ..., \mu^r)$ coordinate which are the intensive parameters in the energy representation, i.e. $\mathfrak{I}^0 = \frac{\partial u}{\partial s} = T$ and $\mathfrak{I}^i = \frac{\partial u}{\partial\mathcal{E}^i} = \mu^i$. Making use of the following relations

$$\Delta s = \frac{\partial s}{\partial T}\Delta T + \sum_{i=1}^{r}\frac{\partial s}{\partial\mu^i}\Delta\mu^i, \tag{3.16}$$

$$\Delta\mathcal{E}^i = \frac{\partial\mathcal{E}^i}{\partial T}\Delta T + \sum_{j=1}^{r}\frac{\partial\mathcal{E}^i}{\partial\mu^j}\Delta\mu^j, \tag{3.17}$$

Table 1: Thermodynamic potentials and the Riemannian line elements in four coordinate systems.

Coordinates	***Potential***	***Line element*** $(\Delta\ell)^2$
$\mathcal{E}=(u,\mathcal{E}^1,\mathcal{E}^2,...,\mathcal{E}^r)$	s	$-\frac{1}{k_B}\frac{\partial^2 s}{\partial\mathcal{E}^\alpha\partial\mathcal{E}^\beta}$
$\mathcal{I}=\{\frac{1}{T},-\frac{\mu^1}{T},-\frac{\mu^2}{T},...,-\frac{\mu^r}{T}\}$	$\phi(\mathcal{I})=s-\mathcal{I}^\mu\mathcal{E}_\mu$	$\frac{1}{k_B}\frac{\partial^2\phi}{\partial\mathcal{I}^\mu\partial\mathcal{I}^\nu}\Delta\mathcal{I}^\mu\Delta\mathcal{I}^\nu$
$\mathfrak{I}=(T,\mu^1,\mu^2,...,\mu^r)$	$\omega(\mathfrak{I})=u-Ts-\sum_{i=1}^r\mu^i\mathcal{E}^i$	$-\frac{1}{k_BT}\frac{\partial^2\omega}{\partial\mathfrak{I}^\mu\partial\mathfrak{I}^\nu}\Delta\mathfrak{I}^\mu\Delta\mathfrak{I}^\nu$
$(T,\mathcal{E}^1,\mathcal{E}^2,...,\mathcal{E}^r)$	$f=u-Ts$	$\frac{1}{k_BT}\frac{\partial s}{\partial T}(\Delta T)^2+\frac{1}{k_BT}\sum_{i,j=1}^r\frac{\partial\mu^i}{\partial\mathcal{E}^j}\Delta\mathcal{E}^i\Delta\mathcal{E}^j$

and substituting them into Eq. (3.11) leads to

$$\Delta\ell^2=-\frac{1}{k_BT}\frac{\partial^2\omega}{\partial\mathfrak{I}^\mu\partial\mathfrak{I}^\nu}\Delta\mathfrak{I}^\mu\Delta\mathfrak{I}^\nu, \tag{3.18}$$

where

$$\omega(\mathfrak{I}^0,\mathfrak{I}^1,...,\mathfrak{I}^r) = u-Ts-\sum_{i=1}^r\mu^i\mathcal{E}^i \tag{3.19}$$

$$= -\phi(\mathcal{I}^0,\mathcal{I}^1,...,\mathcal{I}^r)T. \tag{3.20}$$

A summary of the mentioned metrics is tabulated in the Table 1. It should be noted that in the all selected frameworks, the volume of the system is considered as a quantity whose value is fixed and does not fluctuate. Moreover, as several examples, the thermodynamic metric elements $g_{\alpha\beta}$ are evaluated in the state $x^\alpha=x_0^\alpha$ for a number of coordinate systems which are listed in Table 2.

3.1.2. Weinhold metric

An alternate geometric method, using the idea of conformal mapping from the Riemannian space to thermodynamic space, was proposed by Weinhold in 1975 [15] in which a metric is introduced in the space of equilibrium states of thermodynamic systems as the Hessian of the internal energy u

$$g^W_{\mu\nu}=\frac{1}{k_B}\frac{\partial^2 u}{\partial\mathscr{E}^\mu\partial\mathscr{E}^\nu}, \tag{3.21}$$

where $\mathscr{E}=(s,\mathcal{E}^1,\mathcal{E}^2,...,\mathcal{E}^r)$. Notably, this metric can also be obtained from the Ruppeiner metric via a transformation of fluctuation coordinates. In addition, the Weinhold geometry is conformally related to the Ruppeiner geometry with the temperature being the conformal factor [3, 16, 119, 120],

$$ds^2_{(R)}=\frac{1}{T}\,ds^2_{(W)}\,. \tag{3.22}$$

Table 2: Thermodynamic potentials and the thermodynamic metric elements in four coordinate systems (adapted from [6]).

Coordinates	***Potential***	$\{g_{11}, g_{12}, g_{22}\}$
$\{u, \rho\}$	$s(u, \rho)$	$-\frac{1}{k_B}\left\{\frac{\partial^2 s}{\partial u^2}, \frac{\partial^2 s}{\partial u\,\partial\rho}, \frac{\partial^2 s}{\partial\rho^2}\right\}$
$\{s, \rho\}$	$u(s, \rho)$	$\frac{1}{k_B T}\left\{\frac{\partial^2 u}{\partial s^2}, \frac{\partial^2 u}{\partial s\,\partial\rho}, \frac{\partial^2 u}{\partial\rho^2}\right\}$
$\{T, \rho\}$	$f(T, \rho) = u - Ts$	$\frac{1}{k_B T}\left\{-\frac{\partial^2 f}{\partial T^2}, 0, \frac{\partial^2 f}{\partial\rho^2}\right\}$
$\{T, \mu\}$	$\Omega(T, \mu) = u - Ts - \mu\rho$	$-\frac{1}{k_B T}\left\{\frac{\partial^2 \Omega}{\partial T^2}, \frac{\partial^2 \Omega}{\partial T\,\partial\mu}, \frac{\partial^2 \Omega}{\partial\mu^2}\right\}$

Weinhold showed that the laws of thermodynamics assure that $g^W_{\mu\nu}$ has the positivity required of a metric or first fundamental form on the surface of thermodynamics states. This property enabled him to rederive thermodynamic relations using simple geometric arguments [121]. Such a metric gives us a way to define distances and angles and, therefore, it enables us to study the geometry of the surface [19]. This metric turns out to be positive as a consequence of the second law of thermodynamics.

Salamon et al., considered the Weinhold metric to study the physical significance of thermodynamic length [121]. They found that the local meaning of $g^W_{\mu\nu}$ is the distance between the energy surface and the linear space tangent to this surface at some point where $g^W_{\mu\nu}$ is evaluated. In Ref. [20], the authors generalized the physical interpretation of thermodynamic length to a two-dimensional thermodynamic system with constant heat capacity and showed that in an isochoric thermodynamic system with two degrees of freedom, thermodynamic length is related to the heat flux of a quasi-static process at a constant mole number. A relation between thermodynamic length and work for an isentropic Ideal and quasi-Ideal Gas along isotherms was found in Ref. [18]. In fact, the thermodynamic length was considered as a measure of the amount of work done by the system along isotherms. After that in Ref. [19] a generalization of this relation was provided for unfixed temperature and also found that the thermodynamic length of an isentropic Ideal or quasi-Ideal Gas measures the difference of the square roots of the energies of two given states. Naturally, if there is no work received or done by such a system then the length of the path is zero. The Weinhold approach has been intensively used to study, from a geometrical point of view, the properties of the space generated by Weinhold's metric [122, 123], the chemical and physical properties of various two-dimensional thermodynamic systems [17–21], and the associated Riemannian structure [16, 30, 124]. Despite the efforts conducted in the context of the Weinhold metric, the geometry based on this metric seems physically meaningless in the context of purely equilibrium thermodynamics.

3.1.3. Shortcomings of Hessian metrics

Weinhold and Ruppeiner metrics have been widely used to describe the phase structure of condensed matter systems [14–16, 22, 125–131]. However, in some cases, these metrics have shown inconsistency with each other [132–136]. One reason for this discrepancy is due to their dependence on the choice of thermodynamic potential. Specifically, Hessian metrics lack Legendre invariance, which results in the loss of geometric structure when a different thermodynamic potential is employed to describe equilibrium states. To address this problem, Quevedo *et al.* proposed a Legendre invariant metric formalism [28] that preserves geometric structure regardless of the chosen thermodynamic potential. This formalism will be further discussed in the following section.

3.2. Legendre invariant metric

Invariance under Legendre transformations is a fundamental property of thermodynamics, and it follows that any thermo-geometric metric should maintain this invariance. Legendre-invariant metrics are important because they convey the essential message that the thermodynamic properties of physical systems do not depend on the choice of thermodynamic potential from a geometric standpoint. To devise a thermo-geometric metric in a Legendre-invariant manner, one must incorporate concepts from differential geometry, such as contact manifolds, metrics, and Riemannian geometry, as well as Legendre transformations. Each of these concepts is explained in detail in the appendix. By taking a rigorous and systematic approach to the geometrization of thermodynamic systems, researchers can develop Legendre-invariant metrics that provide critical insights into the behavior of thermodynamical systems.

3.2.1. Quevedo metric

As it was already mentioned, neither Weinhold nor Ruppeiner metric is formulated in a Legendre-invariant way which makes them inappropriate for describing the geometry of thermodynamic systems. By using purely mathematical considerations and geometric objects, Quevedo put forward the idea of geometric thermodynamic and showed that both Weinhold and Ruppeiner formalisms can be unified into a single approach called geometrothermodynamics (GTD) [28]. The main motivation for introducing the formalism of GTD was to formulate a geometric approach which takes into account the fact that in ordinary thermodynamics the description of a system does not depend on the choice of the thermodynamic potential, i. e., it is invariant with respect to Legendre transformations.

One of the primary geometric components of GTD is the *phase manifold* which is essential in order to have Legendre transformations as diffeomorphisms. The triad $(\mathcal{T}, \theta, G)$ is called phase manifold, or generally a *Riemannian contact manifold*, if the following conditions are satisfied (see appendices B and C for more detail):

- $\mathcal{T}$ is a (2n+1)-dimensional manifold which coordinatized by the set $Z^A = \{\Phi, E^a, I^a\}$ with $A = 0, ..., 2n$, and $a = 1, ..., n$. Here Φ is the thermodynamic potential, and E^a and I^a denote the extensive and intensive variables, respectively.
- θ is a non-vanishing differential contact one-form defined on the cotangent manifold $T^*(\mathcal{T})$, satisfying the condition $\theta \wedge (d\theta)^n \neq 0$, that according to Darboux theorem always exists in any odd dimensional manifold and can be expressed as follows

$$\theta = d\Phi - I_a dE^a. \quad \text{(Darboux theorem)} \tag{3.23}$$

This expression for the contact one-form θ is clearly invariant under the Legendre transformations given in (C.9 - C.11), i.e.

$$\theta \rightarrow \bar{\theta} = d\bar{\Phi} - \bar{I}_a d\bar{E}^b. \tag{3.24}$$

- G is a Legendre invariant Riemannian metric on $\mathcal{T}$, i.e. $G = G_{AB} dZ^A dZ^B$. Indeed, demanding Legendre invariance of the metric G results in a set of algebraic equations for the components G_{AB}, whose solutions can be split into three different metrics [137], namely

$$G^I = (d\Phi - I_a dE^a)^2 + (\xi_{ab} E^a I^b)(\delta_{cd} dE^c dI^d), \tag{3.25}$$

$$G^{II} = (d\Phi - I_a dE^a)^2 + (\xi_{ab} E^a \, I^b)(\eta_{cd} dE^c dI^d), \tag{3.26}$$

$$G^{III} = (d\Phi - I_a \, dE^a)^2 + \sum_{a=1}^{n} E_a I_a \, dE^a \, dI^a, \tag{3.27}$$

which the first and the second relations are invariant under total Legendre transformations while the third one is invariant with respect to partial Legendre transformations. Here ξ_{ab} is a diagonal constant $(n \times n)$-matrix, $\delta_{ab} = \mathrm{diag}(1, \cdots, 1)$ and $\eta_{ab} = \mathrm{diag}(-1, \cdots, 1)$. It is notable that G^I and G^{II} are used to describe systems with first order and second order phase transitions, respectively.

In the context of thermodynamics, any Riemannian contact manifold $(\mathcal{T}, \theta, G)$ whose components are Legendre invariant is called a *thermodynamic phase space* (phase manifold) that can be considered as the starting point for a description of thermodynamic systems using geometric concepts.

On the other hand, the space of *thermodynamic equilibrium states* (equilibrium manifold) is an n-dimensional Riemannian submanifold $\mathcal{E} \subset \mathcal{T}$, with a nondegenerate metric g and the extensive variables E^a as the coordinates, defined by means of a smooth embedding mapping $\varphi: \ \mathcal{E} \ \longmapsto \mathcal{T}$, i.e. $\varphi : (E^a) \longmapsto (Z^A(E^a)) = (\Phi(E^a), E^a, I^a(E^a))$, if the condition

$$\varphi^*(\theta) = \varphi^\star(d\Phi \; - \; I_a \, dE^a) = 0, \tag{3.28}$$

is satisfied, where $\varphi^\star$ indicates the pullback of φ. Moreover, a thermodynamic metric g is induced in the equilibrium manifold $\mathcal{E}$, which we demand to be compatible with the metric G on $\mathcal{T}$, by means of $g = \varphi^*(G)$. Therefore, regarding (3.25)-(3.27), the following relation can be obtained for the thermodynamic metric g

$$g_{ab}^{I/II} = \beta_\Phi \Phi \xi_a{}^c \frac{\partial^2 \Phi}{\partial E^b \partial E^c}, \tag{3.29}$$

where $\xi_a{}^c = \delta_a{}^c = \mathrm{diag}(1, \cdots, 1)$ for g^I and $\xi_a{}^c = \eta_a{}^c = \mathrm{diag}(-1, 1, \cdots, 1)$ for g^{II}. Besides, the constant β_Φ indicates the degree of homogeneity of the thermodynamic potential Φ [138]. The third metric of the equilibrium space can be written as

$$g^{III} = \sum_{a=1}^{n} \left(\delta_{ad} E^d \frac{\partial \Phi}{\partial E^a} \right) \delta^{ab} \frac{\partial^2 \Phi}{\partial E^b \partial E^c} dE^a dE^c \; . \tag{3.30}$$

Note that $\Phi(E^a)$ represents a fundamental thermodynamic equation which must be known explicitly. Φ can be either the entropy or the internal energy of the system [117].

Considering condition (3.28) leads us to the following relations

$$d\Phi = \delta_{ab} I^a dE^b, \tag{3.31}$$

$$\frac{\partial \Phi}{\partial E^a} = \delta_{ab} I^b, \tag{3.32}$$

which correspond to the *first law of thermodynamics* and the standard conditions of thermodynamic equilibrium, respectively [117]. Eq. (3.32) also means that the intensive thermodynamic variables are dual to the extensive ones. In this construction, the *second law of thermodynamics* implies that the fundamental equation satisfies the condition [117, 139]

$$\pm \frac{\partial^2 \Phi}{\partial E^a \partial E^b} \geq 0, \qquad \text{(convexity condition)} \tag{3.33}$$

where the sign depends on the thermodynamic potential. For example, if Φ is identified as the entropy, the sign is positive whereas it should be negative in the case of Φ being the internal energy of the system [117].

It is worth mentioning that as the thermodynamic potential is a homogeneous function of its arguments, it satisfies the homogeneity condition

$$\Phi(\lambda E^a) = \lambda^\beta \Phi(\lambda E^a), \tag{3.34}$$

for constant parameters λ and β. Making use of the condition (3.32) along with differentiating the homogeneity condition with respect to λ, one can easily get the following expression for the Euler's identity

$$\beta \Phi(E^a) = \delta_{ab} I^b E^a, \tag{3.35}$$

which the result has been evaluated at $\lambda = 1$. In addition, calculating the exterior derivative of Euler's identity and using the first law of thermodynamics (3.31) gives rise to the generalized Gibbs-Duhem relation

$$(1 - \beta) I^a dE^b \delta_{ab} + E^a dI^b \delta_{ab} = 0. \tag{3.36}$$

It is important to note that the classical expressions for Euler's identity and Gibbs-Duhem relation will be obtained by instituting $\beta = 1$ into the above equations.

3.3. Calculation of thermodynamic metrics for ordinary thermodynamic systems

In order to apply the geometric representation, which was explained throughout the previous parts, to some ordinary thermodynamic systems, we consider ideal gas and real gas with 5-dim phase space with variables (U, S, T, p, V), and study them in more detail.

3.3.1. Ideal gas

First, we consider the ideal gas, as the simplest example of a thermodynamic system with two degrees of freedom. In this case, the equation of state is given by

$$PV = Nk_B T, \tag{3.37}$$

where k_B indicates the Boltzmann constant and N is the number of molecules. Its internal energy is a function of the temperature and reads as

$$U_{\text{ideal}}(T) = n\,C_V T, \tag{3.38}$$

in which $n \equiv \frac{N}{N_A}$ is defined as the number of moles of the gas with N_A denoting Avogadro's number. Besides, C_V indicates the heat capacity at constant volume. Making use of the first law of thermodynamics, $dQ = TdS = dU + PdV$, one can immediately get

$$dS = nC_V \frac{dU}{U} + nR\frac{dV}{V}, \tag{3.39}$$

or equivalently

$$S_{\text{ideal}}\,(U,\,V) = n\,C_V \ln{(U)} + nR\ln{(V)} + \mathcal{S}_0, \tag{3.40}$$

where $\mathcal{S}_0$ is an integration constant and $R = N_A k_B$ denotes the gas constant. This relation is known as the fundamental relation in the entropy representation. Working in the energy representation we get

$$U_{\text{ideal}}(S, V) = V^{-\frac{R}{C_V}}\, e^{\frac{S-\mathcal{S}_0}{nC_V}}. \tag{3.41}$$

Having the above information, we try to examine thermodynamic metrics in different representations:

- ***Ruppeiner metric:*** Recalling that the Ruppeiner metric is defined as $g^R_{\mu\nu} \overset{\text{def}}{=} -\frac{1}{k_B}\frac{\partial^2 S}{\partial E^\mu \partial E^\nu}$, with $E = \left(E^1,\, E^2\right) = (U,\, V)$, the metric components can be obtained as

$$\begin{aligned} g^R_{11}\,(U,\,V) &= \frac{n\,C_V}{k_B U^2},\\ g^R_{12}\,(U,\,V) &= g_{21}\,(U,\,V) = 0,\\ g^R_{22}\,(U,\,V) &= \frac{nR}{k_B V^2}. \end{aligned} \tag{3.42}$$

Hence

$$g^R_{\text{ideal}} = \frac{nC_V}{k_B U^2} dU^2 + \frac{n\,R}{k_B V^2} dV^2. \tag{3.43}$$

- ***Weinhold metric:*** Working in the energy representation and using the relation $g^W_{\mu\nu} \overset{\text{def}}{=} \frac{1}{k_B}\frac{\partial^2 U}{\partial\theta^\mu \partial\theta^\nu}$ with $\left(\theta^1,\,\theta^2\right) = (S,\,V)$, the components of Weinhold's metric will be given as follows

$$\begin{aligned} g^W_{11}\,(S,\,V) &= \frac{1}{k_B n^2 C_V^2}\, e^{\frac{S}{nC_V}} V^{-\frac{R}{C_V}},\\ g^W_{12}\,(S,\,V) &= g_{21}\,(U,\,V) = -\frac{RV}{nC_V^2} e^{\frac{S}{nC_V}} V^{-\frac{C_P}{C_V}},\\ g^W_{22}\,(S,\,V) &= \frac{RC_P}{C_V^2} e^{\frac{S}{nC_V}} V^{-(1+\frac{C_P}{C_V})}, \end{aligned} \tag{3.44}$$

and as a result

$$g^{W}_{\text{ideal}} = e^{\frac{S}{nC_V}} \left[\frac{V^{-\frac{R}{C_V}}}{k_B n^2 C_V^2}\, dS^2 + \frac{RC_P}{C_V^2} V^{-(1+\frac{C_P}{C_V})}\, dV^2 - \frac{RV}{nC_V^2} V^{-\frac{C_P}{C_V}}\, dSdV \right]. \tag{3.45}$$

• ***Quevedo Metric:*** In the case of GTD, by using the most general metric (3.30) and working in the entropy representation, the components of Quevedo's metric will take the following form

$$g^{Q}_{11}\,(U,\,V) = \frac{(nC_V)^2}{U^2}, \tag{3.46}$$

$$\tag{3.47}$$

$$g^{Q}_{12}\,(U,\,V) = g^{ideal}_{21}\,(U,\,V) = 0, \tag{3.48}$$

$$\tag{3.49}$$

$$g^{Q}_{22}\,(U,\,V) = \frac{(nR)^2}{V^2}. \tag{3.50}$$

$$\tag{3.51}$$

Therefore,

$$g^{\text{Q}}_{\text{ideal}} = \frac{(nC_V)^2}{U^2} dU^2 + \frac{(nR)^2}{V^2} dV^2. \tag{3.52}$$

3.3.2. Van der Waals gas

Now, we consider a more realistic model of a gas known as the Van der Waals Gas whose equation of state is given by

$$\left(P + \frac{n^2}{V^2} a\right)(V - nb) = nRT, \tag{3.53}$$

where a and b are positive constants, indicating the characteristic of the particular gas under consideration†. Moreover, its internal energy reads as

$$U_{\text{VdW}}(T) = n\,C_V T - n^2 \frac{a}{V} + U_0. \tag{3.54}$$

Using the first law of thermodynamics

$$dS = \frac{n\,C_V}{U + n^2 \frac{a}{V}} dU + \frac{n\,R}{V - n\,b} dV, \tag{3.55}$$

the fundamental equation for the Van der Waals gas in the entropy representation is given by

$$S_{\text{VdW}}\,(U,\,V) = nC_V \ln\left(U + n^2 \frac{a}{V}\right) + n\,R \ln\left(V - n\,b\right) + \mathcal{C}_r, \tag{3.56}$$

†Indeed, a is a characteristic of molecule interaction in the gas and b is characteristic of the part of the volume occupied by the molecules [117].

with $\mathcal{C}_r$ an integration constant. Setting $\mathcal{C}_r = 0$ and solving the above relation for U we get the internal energy of Van der Waals gas as a function of S and V as follows

$$U_{VdW}(S,V) = (V - n\,b)^{-\frac{R}{C_V}}\, e^{\frac{S}{n\,C_V}} - n^2 \frac{a}{V}. \tag{3.57}$$

Considering the mentioned information for the Van der Waals gas as well as Eqs. (3.2), (3.21) and (3.30), some thermodynamic metrics will be obtained as follows:

- ***Ruppeiner metric:***

$$\begin{aligned} g_{\rm UU}^{\rm VdW(R)}(U,\,V) &= n\,C_V \frac{V^2}{k_B\,(an^2 + U\,V)^2}, \\ g_{\rm UV}^{\rm VdW(R)}(U,\,V) &= g_{\rm VU}^{\rm VdW}(U,\,V) = -\frac{an^3 C_V}{k_B\,(an^2+UV)^2}, \\ g_{\rm VV}^{\rm VdW(R)}(U,\,V) &= \frac{nR}{k_B V^2}\frac{V^2}{(V-bn)^2} - \frac{an^3 C_V}{k_B V^2}\frac{(an^2+2UV)}{(an^2+UV)^2}. \end{aligned} \tag{3.58}$$

- ***Weinhold metric:***

$$\begin{aligned} g_{\rm SS}^{\rm VdW(W)}(S,\,V) &= \frac{1}{k_B n^2 C_V^2} e^{\frac{S}{nC_V}} (V-nb)^{-\frac{R}{C_V}}, \\ g_{\rm SV}^{\rm VdW(W)}(S,\,V) &= g_{\rm VS}^{\rm VdW(W)}(U,\,V) = -\frac{R}{k_B n C_V^2} e^{\frac{S}{nC_V}} (V-nb)^{-(1+\frac{R}{C_V})}, \\ g_{\rm VV}^{\rm VdW(W)}(S,\,V) &= \frac{R}{k_B C_V^2} e^{\frac{S}{nC_V}} (C_V+R)(V-nb)^{-(2+\frac{R}{C_V})} - n^2 \frac{2a}{k_B V^3}, \end{aligned} \tag{3.59}$$

- ***Quevedo Metric:***

$$g_{\rm UU}^{\rm VdW(Q)}(U,\,V) = \left(nC_V \frac{U}{U+n^2\frac{a}{V}}\right)^2 \frac{nC_V V^2}{(an^2+UV)^2}, \tag{3.60}$$

$$g_{\rm UV}^{\rm VdW(Q)}(U,\,V) = g_{\rm VU}^{\rm VdW(Q)}(U,\,V) = -\frac{1}{2}\frac{an^3 C_V}{(an^2+UV)^2}$$

$$\left[\left(nC_V \frac{U}{U+n^2\frac{a}{V}}\right)^2 + \left(nR\frac{V}{V-nb} - nC_V \frac{n^2\frac{a}{V}}{U+n^2\frac{a}{V}}\right)^2\right], \tag{3.61}$$

$$g_{\rm VV}^{\rm VdW(Q)}(U,\,V) = -\left(nR\frac{V}{V-nb} - nC_V \frac{n^2\frac{a}{V}}{U+n^2\frac{a}{V}}\right)^2$$

$$\left[-\frac{nR}{V^2}\frac{V^2}{(V-bn)^2} + \frac{an^3 C_V}{V^2}\frac{(an^2+2UV)}{(an^2+UV)^2}\right]. \tag{3.62}$$

In the limit $a \to 0,\ b \to 0$, the obtained results for the Van der Waals gas will reduce to the ideal gas.

4. Thermodynamic curvature

According to what was discussed, based on the thermal fluctuation between equilibrium states, a new method was proposed to study the thermodynamics of physical systems using Riemannian geometry. Given a metric in Riemannian geometry, one of the immediate objects of interest is its Ricci curvature scalar $\mathcal{R}$ [140], which is known as the thermodynamic Riemannian curvature in the context of thermodynamic systems. $\mathcal{R}$ is the only geometrical scalar invariant in thermodynamics that has a volume dimension and as long as we are dealing with a two-dimensional hypersurface, all the information lies in this quantity and hence plays a fundamental role.

To go further, we examine the scalar curvature for some ordinary thermodynamic systems and then discuss some interesting connections between curvature and interactions in more detail.

• ***Ideal gas:*** As a first example, let us consider the ideal gas. Making use of the results of thermodynamic metric for ideal gas as well as using (A.12) one can easily find that the thermodynamic curvature is zero for a monoatomic ideal gas system. Since the ideal gas is a noninteracting gas, this result suggests that $\mathcal{R}$ can be some type of measure of interactions between particles.

• ***Van der Waals gas:***

As a second example, we investigate the scalar curvature of the the Van der Waals gas in different geometrical representation:

▶ **Weinhold representation:**

Using Eq. (3.59) and (A.12), the scalar curvature in the Weinhold representation will be obtained as following form

$$\mathcal{R}^W = \frac{aRV^3}{C_V(pV^3 - aV + 2ab)^2}. \tag{4.1}$$

As this relation shows the scalar curvature goes to zero when $a \to 0$ or $V \to \infty$. Since the quantity $\frac{a}{V^2}$ characterizes the attractive interaction within a system, scalar curvature seems to be a measure of the attraction among particles while its dependence on the parameter b is more quantitative then qualitative. Besides, the scalar curvature diverges at the critical points determined by the algebraic equation $pV^3 - aV + 2ab = 0$, which is exactly the equation that determines the location of first order phase transitions of the van der Waals gas [117]. Consequently, a first order phase transition can be interpreted geometrically as a curvature singularity. This is in accordance with our intuitive interpretation of thermodynamic curvature.

▶ **Ruppeiner representation:**

In order to work in this geometry, we consider the fluctuation coordinates to be the temperature T and the volume V. Therefore, the line element of the Ruppeiner geometry will be as follows [141]

$$dl^2 = \frac{3}{2T^2}dT^2 + \frac{TV^3 - 2a(V-b)^2}{TV^3(V-b)^2}dV^2. \tag{4.2}$$

Employing the (A.12), we get the scalar curvature as

$$\mathcal{R}^R = \frac{4a(V-b)^2\left(a(V-b)^2 - TV^3\right)}{3\left(2a(V-b)^2 - TV^3\right)^2}. \tag{4.3}$$

It is easy to check that the denominator of the scalar curvature $\mathcal{R}$ vanishes at the critical point. Clearly, the condition $V = b$ results in $\mathcal{R} = 0$. Indeed, at this point all the volume is taken up by the fluid molecules so that no extra room remains. Therefore, the total fluid becomes a rigid body and no interaction can exist between the fluid molecules, which is consistent with the result that $\mathcal{R} = 0$ corresponds to vanishing interaction [142].

► **Quevedo representation:**

Using the components of Quevedo metric for the Van der Waals gas as well as (A.12), the corresponding curvature scalar can be written as [143]

$$\mathcal{R}^Q = \frac{\mathcal{N}_{VdW}(U,V)}{(3V(V-b)(U+PV))^3\left(\frac{3}{2}(V-b)(PV^3 - aV + 2ab)\right)^2}, \tag{4.4}$$

where the function $\mathcal{N}_{vdW}(U,V)$ is a polynomial which is different from zero at points where the denominator vanishes. Similar to the two previous representations, there exist curvature singularities at those points where the condition $PV^3 - aV + 2ab = 0$ is satisfied, which as mentioned earlier, these are the points where first order phase transitions occur in the van der Waals gas.

In addition to the mentioned results regarding the scalar curvature in the cases of ideal gas and Van der Waals gas, some calculations done on models with critical points which represent that $|\mathcal{R}|$ diverges near these points in the same way as the correlation volume ξ^d, where d is the spatial dimension of the system and ξ is the correlation length [3, 16, 22]. These properties, i.e. the connection between curvature and interactions, suggest that $\mathcal{R}$ *can be considered as a measure of interparticle interaction.* Indeed, the calculations indicate that the thermodynamic curvature yields a measure of the smallest volume in which the thermal fluctuation theory is effective. We expect this volume to be ξ^d near criticality. Therefore, these results lead us to interpret $|\mathcal{R}|$ as being proportional to the correlation volume

$$|\mathcal{R}| \sim \xi^d, \tag{4.5}$$

which has been confirmed near critical points, where ξ^d is large enough to contain many particles, for several statistical mechanics models [22, 126–128, 144–148]

An important point that should be mentioned is that $\mathcal{R}$ is a signed quantity and various calculations reveal that the sign of $\mathcal{R}$ corresponds to whether interactions are effectively attractive or repulsive (The sign of $\mathcal{R}$ is subject to convention and we use Weinberg's sign convention [149]). For fluid and solid systems, an overall pattern is that $\mathcal{R}$ is negative for systems where attractive interparticle interactions dominate, and positive where repulsive interactions dominate (see Figure 4 for a better understanding). For systems with no statistical mechanical interactions (e.g., an ideal gas), the scalar curvature vanishes, and consequently, the geometry of the associated two-dimensional space is flat. Therefore, the sign of $\mathcal{R}$ alone provides direct information about the properties of the interaction among particles. For this reason, the thermodynamic length is considered a measure of

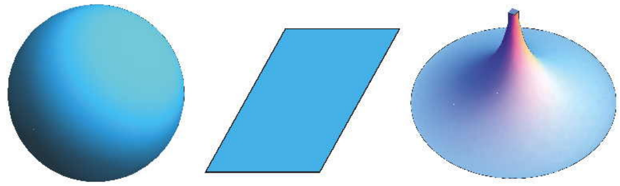

Figure 4: Surfaces related to constant Ricci curvature scalar R: the sphere, the plane, and the pseudosphere. $R<0$ represents attractive interparticle interactions dominate. $R>0$ is related to repulsive interactions dominant. Regardless of the sign convention for R, attractive and repulsive interactions correspond to the geometry of the sphere and pseudosphere, respectively [113].

statistical mechanical interactions within a thermodynamic system. These results have been reviewed in [3, 22, 23].

Here, we try to find the relation between the scalar curvature and phase transitions. To this end, we take a system whose fluctuating coordinates are T and some extensive quantities $\mathcal{E}$ and examine the scalar curvature concerning the associated metric. Therefore, we work with metric (3.15) and we consider the metric is two-dimensional and diagonal, for simplicity,

$$d\ell^2 = \frac{1}{k_B T}\frac{\partial s}{\partial T} dT^2 + \frac{1}{k_B T}\frac{\partial \mu}{\partial \mathcal{E}} d\mathcal{E}^2. \tag{4.6}$$

Since the heat capacity at constant $\mathcal{E}$ is

$$C_{\mathcal{E}} = T\left(\frac{\partial S}{\partial T}\right)_{\mathcal{E}}, \tag{4.7}$$

the line element can be expressed as

$$d\ell^2 = \frac{C_{\mathcal{E}}}{k_B T^2} dT^2 + \frac{(\partial_{\mathcal{E}}\mu)_T}{k_B T} d\mathcal{E}^2, \tag{4.8}$$

and the scalar curvature is easily obtained as follows

$$\begin{aligned} \mathcal{R} \;&=\; \frac{1}{2C_{\mathcal{E}}^2(\partial_{\mathcal{E}}\mu)^2}\Big\{T(\partial_{\mathcal{E}}\mu)\Big[(\partial_{\mathcal{E}}C_{\mathcal{E}})^2 + (\partial_T C_{\mathcal{E}})(\partial_{\mathcal{E}}\mu - T\partial_{T,\mathcal{E}}\mu)\Big] \\ &+\; C_{\mathcal{E}}\Big[(\partial_{\mathcal{E}}\mu)^2 + T\left((\partial_{\mathcal{E}}C_{\mathcal{E}})(\partial_{\mathcal{E},\mathcal{E}}\mu) - T(\partial_{T,\mathcal{E}}\mu)^2\right) + 2T(\partial_{\mathcal{E}}\mu)(-(\partial_{\mathcal{E},\mathcal{E}}C_{\mathcal{E}}) + T(\partial_{T,T,\mathcal{E}}\mu))\Big]\Big\}. \end{aligned} \tag{4.9}$$

This relation represents that $\mathcal{R}$ can potentially diverge at $(\partial_{\mathcal{E}}\mu)_T = 0$, which is the critical point related to a Van der Waals phase transition, or $C_{\mathcal{E}} = 0$. Hence, the scalar curvature of the Ruppeiner geometry experiences divergent behaviour at the critical point of the phase transition. This property provides a possible connection between the scalar curvature and the correlation length, which goes to infinity at the critical point according to phase transition theory.

Before concluding this section, it is important to mention the analysis of several studies conducted on this subject.

• In a captivating research conducted by Mirza et al. [150], they investigated the thermodynamic curvature of the Ising model on a kagome lattice under the influence of an external magnetic field. The findings of the study demonstrated that the curvature displays a singularity at the critical point, consistent with the Fisher expression. Furthermore, it was observed that the scalar curvature in this model deviates from the conventional scaling R.

• In Ref. [151], the authors investigated the thermodynamic geometry of an ideal $q-$deformed boson and fermion gas. They observed that the statistical interaction between $q-$deformed bosons is attractive, while for $q-$deformed fermions it is repulsive. Additionally, they found that the $q-$deformed fermion gas may be more stable than the $q-$deformed boson gas regardless of the deformation parameter's value. The authors' analysis suggests that a large value of the deformation parameter ($q < 1$) makes the $q-$deformed boson gas more stable compared a small value, indicating that an ideal boson gas is more stable than its deformed counterpart. Furthermore, they examined the singular point of the thermodynamic curvature and discovered new insights into the condensation of $q-$deformed bosons. They concluded that the singular point coincides with the critical point, where phase transitions like Bose-Einstein condensation occur.

• The thermodynamic geometry of an ideal gas, where particles follow non-Abelian statistics, was investigated in Ref. [152]. It was discovered that the behavior of the gas is dependent on the statistical parameter α. When α exceeds $\frac{1}{4}$, condensation does not occur. However, for $0 \leq \alpha \leq \frac{1}{4}$, the non-Abelian statistics resemble bosonic behavior and do not exhibit exclusion properties. At a specific fractional parameter value and given fugacity values, the thermodynamic curvature exhibits a singularity, indicating the occurrence of condensation. This observation suggests the presence of Bose-Einstein condensation in non-Abelian statistics.

• The study on the thermodynamic geometry of fractional exclusion statistics has been expanded to higher dimensions and other types of fractional statistics, such as ideal Haldane, Gentile, and Polychronakos fractional exclusion statistics gas [153]. The findings indicate that gases become less stable in higher dimensions. In the case of an ideal gas following Gentile statistics, the thermodynamic curvature is predominantly positive in the classical limit, reflecting the attractive statistical interaction among Gentileons. However, beyond the classical limit, particles obeying Haldane fractional statistics and Gentile statistics exhibit a Fermi surface, indicating that the repulsive statistical interaction becomes dominant at low temperatures. While the definitions of Haldane and Polychronakos fractional exclusion statistics coincide in the classical limit, their differences become apparent as we move further away from this limit. In the low-temperature regime, the exclusion property associated with particles obeying Hald statistics takes precedence, implying that Bose-Einstein condensation does not occur for particles governed by Haldane fractional statistics.

• A thorough examination of the nonperturbative thermodynamic curvature of a two-dimensional ideal anyon gas was conducted in Ref. [154]. The study revealed that, under low temperature conditions and with a constant particle count, the thermodynamic curvature converges to that of a fermion gas. This implies that, at $T = 0$, particles with general exclusion statistics display a Fermi surface.

• In Ref. [155], the thermodynamic curvature of the anyon gas was investigated using the Ruppeiner approach in both classical and quantum limits. The study explored the concept of "anyons," which refers to particles with fractional statistics in two-dimensional systems, and its relevance to the theory of fractional quantum Hall effect. The results revealed that the behavior of the ideal anyonic gas in the classical limit depends on the value of the statistical parameter α. When $\alpha < \frac{1}{2}$ ($\alpha > \frac{1}{2}$), the scalar thermodynamic curvature is positive (negative), indicating an attractive (repulsive) statistical interaction. Consequently, the ideal anyonic gas is referred to as "Bose-like" ("Fermi-like") for $\alpha < \frac{1}{2}$ ($\alpha > \frac{1}{2}$). In the case of $\alpha = \frac{1}{2}$, the equation of state resembles that of an ideal classical gas, resulting in a thermodynamic curvature of zero. In the quantum limit, the zero point of the thermodynamic curvature shifts from $\alpha = \frac{1}{2}$ to lower values, suggesting that quantum corrections alter the value of α. This implies that the anyon gas behaves as a free noninteracting gas when quantum corrections are applied.

In Ref. [155], the thermodynamic curvature of the anyon gas[‡] was studied with the use of the Ruppeiner approach for both classical and quantum limits. The results showed that the ideal anyonic gas in the classical limit can have two different behavior depending on the value of statistical parameter α. For $\alpha < \frac{1}{2}$ ($\alpha > \frac{1}{2}$), scalar thermodynamic curvature is positive (negative), meaning that the statistical interaction is attractive (repulsive). Thus, the ideal anyonic gas is called "Bose-like" ("Fermi-like") for $\alpha < \frac{1}{2}$ ($\alpha > \frac{1}{2}$). For $\alpha = \frac{1}{2}$, the equation of state is like that of an ideal classical gas where its thermodynamic curvature is zero. In the quantum limit, the zero point of the thermodynamics curvature is shifted from $\alpha = \frac{1}{2}$ to the lower numbers, indicating that quantum corrections change the value of α. This means that the anyon gas is a free noninteracting gas by applying quantum corrections.

5. Application to black holes

This section aims to review the application of thermodynamic geometries to black hole issues.

Understanding the meaning of the event horizon of black holes helps us gain a better understanding of black hole physics and its connection to thermodynamics. Since gravity always attracts, self-gravitating systems tend to grow rather than shrink. As we know, nothing - not even light - can escape the event horizon of a black hole, making the horizon akin to a one-way asymmetric surface: objects can enter but not exit, leading to an increase in the surface area. This aligns with the second law of thermodynamics, which states that the total entropy of a system will increase in a physical process and never decrease. In fact, there is an asymmetric tendency for a one-way increase in entropy, since the area of the event horizon is proportional to entropy. Thus, the size of a black hole cannot decrease in any process.

The properties of a black hole are independent of the details of the collapsing matter, and this universality arises from the fact that black holes can be the thermodynamic limit of the underlying quantum gravitational degrees of freedom. This suggests that the classical and semiclassical features of black holes can provide important information about the nature of quantum gravity. The lack of experimental or observational results presents a major challenge in constructing a theory of quantum gravity. Therefore, understanding the fundamental laws of black hole mechanics may serve as a necessary (if not sufficient) constraint on the theory of quantum gravity.

The classical (non-quantum) characteristics of black holes go back to the solution of Einstein's field equations by Schwarzschild [156]. This solution is obtained by assuming a static, uncharged, spherically symmetric point mass M, located at a central singularity. Einstein's field equations can

[‡]The concept of "anyons" or particles with fractional statistics in two-dimensional systems has applications in the theory of fractional quantum Hall effect [158].

also be solved by adding charge Q (the Reissner Nordström solution), angular momentum J (the Kerr solution), or with all three quantities (M, J, Q) (the Kerr-Newman solution). If the collapsing matter is sufficiently dense, we inevitably approach one of these solutions. A question arises: Is it possible to add enough charge to a black hole to make it explode outward under its own electrostatic repulsion? Or can enough angular momentum be added to a black hole so that it tears apart under its own spin, as in Figure 5? According to cosmic censorship, both scenarios - or any combination of them - are forbidden. Black holes are considered extremal if they are as close as possible to these mechanical limits.

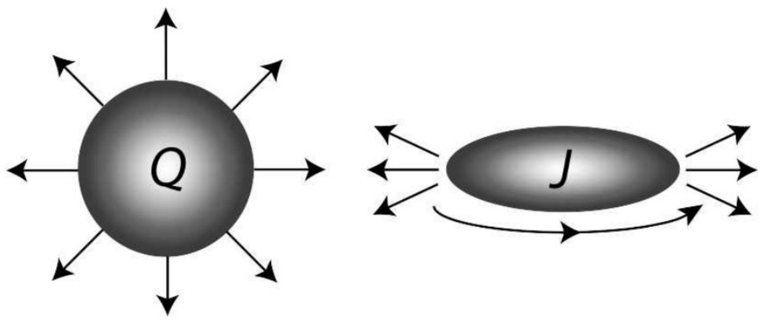

Figure 5: The left graph shows a charged extremal black hole with a high charge Q which is on the verge of exploding under its own electrostatic repulsion. The graph on the right displays a rotating black hole with a very large angular momentum J on the verge of pulling apart [113].

A principle often cited in black hole physics is the ”no-hair theorem” [156], which states that stationary solutions in GR can be fully described by their mass, spin, and electric charge. Other properties, such as geometry and magnetic moment, are uniquely determined by these three parameters, and all other information - referred to as ”hair” - about the matter that formed or falls into a black hole disappears behind its event horizon, becoming permanently inaccessible to external observers. After collapsing matter forms a black hole, there is a short period of settling during which information related to the black hole's creation disappears. Remarkably, the final equilibrium state of a black hole depends solely on mass, electric charge, and angular momentum. This reduction in complexity is essential for black hole thermodynamics.

The no-hair conjecture rules out any possibility of a distribution of black hole equilibrium microstates. The distribution of microstates is necessary for statistical mechanics and thermodynamic fluctuations [157].

5.1. Black Hole thermodynamics

When a system is in thermal equilibrium, it is possible to predict its behavior by determining only a handful of known properties such as internal energy, entropy, temperature, volume, angular momentum, total mass, etc. Such systems which is in interaction with the outside environment follow four laws, which are known as thermodynamic laws [159]:

- **Zeroth Law:** The temperature T is constant for a system in thermal equilibrium.

- **First Law:** When an infinite amount of work W is performed on a system at temperature T, the variation of the entropy S and energy E of the system is given by

$$\delta E = T\delta S + \delta W. \tag{5.1}$$

- **Second Law:** The entropy of an isolated system can never decrease over time.

- **Third Law:** It is not possible to cool a system to an absolute zero temperature [160].

Studies in black hole mechanics have shown that a black hole behaves like a thermodynamic system, with the area of the event horizon as the entropy and a geometric quantity called surface gravity as the black hole's temperature. Black hole thermodynamics appears when quantum fields propagate on a background black hole spacetime. Thermodynamics of black holes suggest that there exists an underlying microstructure of spacetime.

According to Hawking's area theorem the total surface area of the event horizon of the black hole never decreases in any "classical" processes [161, 162]. But a more physical derivation of the theorem occurs by considering a quasistationary process in which the second law can be shown to hold as a consequence of the first law and the null energy condition. In such a process, positive energy matter can be created by a black hole independent of the details of the gravity action. From what was expressed, energy can flow not only into black holes but also out of them. In other words, a black hole can act as an intermediary in energy exchange processes. For the unchanged horizon area, energy extraction is maximally efficient, but processes during which the horizon area increases are irreversible. The analogy with thermodynamic behavior was noted when Bekenstein employed the area theorem and proposed that the area of the black hole is indeed related to the thermodynamic entropy of the event horizon [38]

$$S \equiv \frac{1}{4\hbar G} \times \text{Area of horizon}. \tag{5.2}$$

This analogy was vigorously pursued as soon as it was recognized at the beginning of the 1970s, although there were several obvious flaws at first [163]:

a) the black hole temperature vanishes.

b) entropy is dimensionless, whereas horizon area is a length squared.

c) the area of every black hole is separately nondecreasing, while only the total entropy is nondecreasing in thermodynamics.

By 1975, it was understood that these flaws in black hole physics could be addressed by incorporating quantum theory. As we know, the thermodynamics of black holes is based on applying quantum field theory in curved spacetime [39, 164, 165]. The nature of black hole radiation is strongly related to the quantum gravity effects and pair creations. This suggests the black hole is a quantum mechanical system like any other quantum system. In some respects, a black hole plays in gravitation the same role as an atom plays in quantum mechanics [166]. This analogy suggests that contrary to classical general relativity in which the mass spectrum of black holes is a continuum, in quantum theory the black hole mass spectrum must be discrete. It means that the black hole horizon area has a discrete eigenvalue spectrum and is in fact quantized [167–169]. The quantization of black holes was first proposed by Bekenstein in the early 1970s. According to his opinion, the nonextremal BH horizon area behaves as a classical adiabatic invariant and therefore it should exemplify a discrete eigenvalue spectrum with quantum transitions [167, 168]. This strongly suggests the quantum mechanical properties of black holes in different physical aspects. When a black hole captures or releases a massive point particle, its mass increases or decreases, respectively, which directly affects its horizon area. For an uncharged particle absorption process, the BH horizon area increase is as [167, 168]

$$\Delta A = 8\pi\mu b, \tag{5.3}$$

where ΔA is the BH horizon area change. μ and b are, respectively, the particle rest mass and particle finite proper radius. According to the uncertainty principle, the particle cannot be localized to better than a Compton wavelength. Therefore, the radial position for the particle's center of mass is subject to an uncertainty of $b \geq \xi\hbar/\mu$ [166]. Hence, the smallest possible increase in horizon area is [167, 168, 170, 171]

$$(\Delta A)_{min} = 8\pi\xi\hbar = \alpha l_p^2, \tag{5.4}$$

where ξ is a number of order unity and $l_p = \sqrt{G\hbar}/c^3$ is the Planck length in gravitational units $G = c = 1$. For the BH's charged particle absorption process, the BH horizon area increase is defined as [167, 168]

$$\Delta A = 4l_p^2. \tag{5.5}$$

From what was expressed, the quantization of horizon area in equal steps brings to mind a horizon formed by patches of equal area αl_p^2 which get added one at a time. In quantum theory, degrees of freedom independently reveal distinct states. Since the patches are all equivalent, each has the same number of quantum states, say k. Therefore, the total number of quantum states on the horizon is [166]

$$N = k^{A/\alpha l_p^2}. \tag{5.6}$$

The statistical (Boltzmann) entropy associated with the horizon is lnN or [166]

$$S_{BH} = \frac{Alnk}{\alpha l_p^2}. \tag{5.7}$$

Comparing the above equation with Hawking's coefficient in the black hole entropy calibrates the constant α [169]:

$$\alpha = 4lnk. \tag{5.8}$$

In the present work, we would like to investigate the classical aspects of the theory. Since these form the basis of quantum black hole thermodynamics, their study is crucial. However, it is also interesting to understand what can be inferred without invoking quantum theory, as this may provide insight into the deeper origins of gravity.

5.2. The four laws of black hole mechanics

The surprising similarities between the geometrical properties of black holes and classical thermodynamic variables suggest an in-depth connection between the laws of black hole mechanics and the laws of thermodynamics. Employing such a connection, one is able to identify the geometrical features of black holes with their thermodynamic counterparts. In fact, the four laws of black hole mechanics are similar to the four laws of thermodynamics if one considers a correspondence between the temperature T and surface gravity κ of the black holes and also between entropy S and the black hole area A. Before formulating the four laws of black hole mechanics, Bekenstein suggested that the entropy of a black hole is proportional to the horizon area A ($S = \eta A$). Although he could not find the value of η exactly, he provided heuristic arguments to conjecture its value to be $\frac{\ln 2}{8\pi}$ [172]. By discovering Hawking radiation and the fact that the radiation has a thermal spectrum, Hawking found that his remarkable discovery did make Bekenstein's idea consistent, of a finite entropy proportional to black hole area, though not conjectured value $\eta = \frac{\ln 2}{8}$, but $\eta = \frac{1}{4}$ [173]. In the following, we review the four laws of black hole mechanics:

5.2.1. Surface gravity and the zeroth law of black hole mechanics

To explain the zeroth law of black hole mechanics, we need to know the notions of stationary, static, and axisymmetric black holes and also the notion of a Killing horizon. A black hole in asymptotically flat spacetime (M, g_{ab}) is stationary if there is a one-parameter group of isometries on (M, g_{ab}) generated by a Killing vector field χ^a which is unit timelike at infinity [174]. A stationary black hole is said to be static if the Killing field χ^a is hypersurface orthogonal. The black hole is called axisymmetric if one can find a one-parameter group of isometries that correspond to rotations at infinity. A black hole is stationary-axisymmetric if it has "$t - \varphi$ orthogonality property"[§]. This property holds for all stationary-axisymmetric BH solutions to the vacuum Einstein or Einstein-Maxwell equations [175].

A null hypersurface $\mathcal{K}$ is called Killing horizon if Killing vector field χ^a is normal to it i.e. $\chi^a \chi_a = 0$. For stationary black holes, event horizon $\mathcal{H}$ is a Killing horizon. For a static black hole, the static Killing field t^a is normal to the horizon, while for a stationary-axisymmetric black hole with the "$t - \varphi$ orthogonality property", Killing vector field χ^a is defined as $\chi^a = t^a + \Omega \varphi^a$ which becomes null on the $\mathcal{H}$ [176]. Here the constant Ω is the angular velocity of the horizon and φ^a denotes the Killing vector associated with the axial symmetry. Since $\nabla^a(\chi^b \chi_b)$ is normal to $\mathcal{K}$, the vector fields must be proportional at every point on $\mathcal{K}$. Therefore, one can introduce a function, κ known as the surface gravity defined as [163]

$$\kappa^2 = -\frac{1}{2}(\nabla^a \chi^b)(\nabla_a \chi_b). \tag{5.9}$$

Here, one can provide an equivalent definition of the surface gravity such that κ is the magnitude of the acceleration, with respect to Killing time, of a stationary zero angular momentum particle just outside the horizon [163]. In fact, it is the same force per unit mass that should be employed on a particle at infinity to continue its movement on a path. Although κ is defined locally on the horizon, it turns out that it is always constant over the entire event horizon of a stationary black hole. Since the surface gravity is proportional to the temperature of black holes, this resembles the zeroth law of thermodynamics which states that temperature T is constant throughout a system in thermal equilibrium [164]. It is worth mentioning that there are two independent versions of the zeroth law of black hole mechanics. The first one was proposed by Carter, stating that the surface gravity κ, has to be constant over its event horizon $\mathcal{H}$ for all static black holes or stationary-axisymmetric with the $t - \varphi$ orthogonality property. This result is purely geometrical without using field equations. The second one is related to Bardeen, Carter, and Hawking's idea which expresses that if Einstein's

[§]A black hole has the "$t - \varphi$ orthogonality property" if the 2-planes defined by χ^a and the rotational Killing field φ^a are orthogonal to a family of 2-dimensional surfaces.

equation holds with the matter stress-energy tensor satisfying the dominant energy condition, then κ is uniform on any Killing horizon. Evidently, in the second version, the existence of the $t-\varphi$ orthogonality property is omitted, instead, the field equations of GR are used. A property of the surface gravity is that it illustrates a proportionality factor between the affinely parametrized null geodesics that generate the event horizon and the Killing vector field parametrization. Then each component of the affinely parametrized null geodesic generators k^a is defined as $\lambda k^a = \frac{1}{\kappa}\chi^a$, where λ is the affine parameter. It should be noted that near the event horizon k^a will locally be parallel to the Killing vector field χ^a, but that the two vector fields will have different parametrizations [159].

5.2.2. First law of black hole mechanics

It would be interesting to study how the event horizon area is related to the other properties of a stationary black hole such as mass, angular momentum, and surface gravity. Also, it is very important to investigate how a small change in one of these properties will cause changes in the others. To do so, we first introduce the Raychaudhuri equation, defined as [159]

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}k^a k^b, \tag{5.10}$$

in which θ is expansion/contraction of volume which is given by the divergence of k^a defined as $\theta = \nabla_a k^a$ and σ_{ab} is a shear tensor that describes distortions in shape with no change in volume and is represented by a symmetric tensor which is trace free as

$$\sigma_{ab} = \theta_{ab} - \frac{1}{3}\theta h_{ab}, \tag{5.11}$$

in which $h_{ab} = g_{ab} - k_a k_b$. The vorticity tensor ω_{ab}, describing the rotation of an area element of the null hypersurface is given by

$$\omega_{ab} = k_{[a;b]} - \dot{k}_{[a}k_{b]}, \tag{5.12}$$

where $k_{[a;b]} = \frac{1}{2}(k_{a;b} - k_{b;a})$. Using the expression for the Ricci tensor contained in the Einstein equation, the last term of Eq. (5.10) can be written as

$$R_{ab}k^a k^b = 8\pi\left[T_{ab}k^a k^b + \frac{1}{2}\right]. \tag{5.13}$$

Since the quantity $T_{ab}k^a k^b$ is the local energy density as measured by an observer with 4-velocity k^a, it is expected that this quantity is non-negative everywhere for time-like and null k^a. Although internal stresses in a matter can have a negative contribution to T, for a physically reasonable matter, this contribution is always much smaller than the mass and momentum terms which contribute positively to $T_{ab}k^a k^b$. So, one can assume that $T_{ab}k^a k^b \geq -\frac{T}{2}$. This reveals the fact that $R_{ab}k^a k^b \geq 0$. The terms $\sigma_{ab}\sigma^{ab}$ and $\omega_{ab}\omega^{ab}$ are also greater than or equal to zero. If a small amount of matter drops into a black hole, the local value of T_{ab} near the black hole surface changes as δT_{ab}. The resulting change in the black hole area is governed by Eq. (5.10). Since changes θ^2, $\sigma_{ab}\sigma^{ab}$, and $\omega_{ab}\omega^{ab}$ can be neglected to first order in δT_{ab}, the Raychaudhuri equation is simplified as

$$\frac{d\theta}{d\lambda} = -8\pi\delta T_{ab}k^a k^b. \tag{5.14}$$

As was mentioned in the discussion of surface gravity, $k^a = \frac{1}{\kappa\lambda}\chi^a = \frac{1}{\kappa\lambda}(t^a + \Omega\varphi^a)$. Here, evaluating the change δT_{ab} on the black hole is a matter of integrating both sides of the Raychaudhuri equation over the black hole event horizon surface and overall future values of λ [159]

$$\begin{aligned} \kappa \int d^2S \int_0^\infty \lambda \frac{d\theta}{d\lambda} &= -8\pi \int_0^\infty \int d^2S\delta T_{ab}k^a k^b \qquad (5.15) \\ \kappa \int d^2S(\theta\lambda)\Big|_0^\infty - \kappa \int d^2S \int_0^\infty \theta d\lambda &= -8\pi \left(\int_0^\infty d\lambda \int d^2S\delta T_{ab}t^a k^b + \Omega \int_0^\infty dV \int d^2S\delta T_{ab}\varphi^a k^b \right). \end{aligned}$$

Now we need to evaluate these integrals exactly. The boundary term on the left-hand side can be taken to be zero by taking the surface out to a region of spacetime where expansion is insignificant. Also, the second term is the integral of the expansion of each infinitesimal area element of the event horizon over the surface of the event horizon. This is the same as the infinitesimally small change in event horizon surface area δA which is caused by δT_{ab}. On the other hand, the first integral on the right is an integral of the T_{00} component, which is called the mass M in special relativity (it comes from the fact that t^a and k^a both point forward in time). In the second term of the right side, the term $T_{ab}\varphi^a k^a$ is a projection onto the time-φ component of T_{ab}, which is the negative of the angular momentum J. Thus, for a far distant observer where space is approximately flat, Eq. (5.15) can be written as

$$\kappa\delta A = 8\pi(\delta M - \Omega\delta J). \qquad (5.16)$$

Eq. (5.16) is called the first law of black hole mechanics which is analogous to the first law of thermodynamics defined as

$$T\delta S = \delta E + \delta W. \qquad (5.17)$$

It is worth mentioning that the zeroth and first laws of black hole mechanics are concerned with equilibrium or quasi-equilibrium processes. That is, they concern stationary black holes, or adiabatic changes from one stationary black hole to another.

5.2.3. Second law of black hole mechanics

The second law of black hole mechanics is Hawking's area theorem [161], which based on the horizon area A of a black hole can never decrease assuming Cosmic Censorship and a positive energy condition. The Cosmic Censor Conjecture states that a complete gravitational collapse of a body results in a singularity, covered by an event horizon [177]. In other words, every singularity must possess an event horizon that hides the singularity from view. Assuming the Weak energy condition and the Cosmic Censor Conjecture, it has been found [178] that the area of a future event horizon never decreases in asymptotically flat spacetime. The second law of black hole mechanics is similar to the second law of thermodynamics, stating that the entropy S of a closed system cannot decrease.

Now, we explore the relation of Hawking's area theorem to the second law of thermodynamics. According to the area theorem, if the spacetime on and outside the future event horizon is a regular predictable space, and for arbitrary null k^a, the null energy condition, $T_{ab}k^a k^b$ is satisfied by the stress tensor, then the area of spatial cross-sections of the event horizon is nondecreasing. As was mentioned in the discussion of the first law, the Raychaudhuri equation is defined by Eq. (5.10). Also, it was discussed that $R_{ab}k^a k^b$, $\sigma_{ab}\sigma^{ab}$ and $\omega_{ab}\omega^{ab}$ are greater than or equal to zero. These constraints help us to arrive at an important restriction on the expansion of geodesics, namely

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \quad \Longrightarrow \quad \int_{\theta_0}^{\theta} \frac{d\theta}{\theta^2} \leq -\int_0^{\lambda} \frac{1}{2} d\lambda \quad \Longrightarrow \quad \frac{1}{\theta(\lambda)} \geq \frac{1}{\theta_0} + \frac{1}{2}\lambda. \tag{5.18}$$

In the above relation, if one takes θ_0 to be negative, then by increasing λ, moving forward along the null geodesic, a point will be reached where the right side of the inequality would become zero. Thus, to hold the inequality, the left side must be zero at the same point. This reveals an infinite expansion of nearby geodesics which is highly unphysical and is in contradiction with some of the basic properties of black holes. Hence, the only choice is to consider positive θ_0, in which case θ will be non-negative for all λ. Since increasing λ means moving forward in time to a time-like observer, this restriction implies that the event horizon area observed by an observer far from the black hole should never decrease with time. This statement is known as the second law of black hole mechanics.

- ***Generalized second law:***

Classically, there is a serious difficulty with the ordinary second law of thermodynamics when a black hole is present. If a black hole radiates energy it will lose mass and, in turn, its event horizon surface area will decrease, thereby the area increase theorem violates. The violation of the theorem is not unexpected since the condition $T_{ab} \geq 0$ is not satisfied on the event horizon by taking into account quantum effects. Therefore, it seems that one can have Hawking radiation or the area increase theorem, but not both. Similarly, when a matter drops into a black hole it disappears into a spacetime singularity. So, the entropy initially present in the matter is lost, and no compensating gain of ordinary entropy occurs. This means that the total entropy of matter in the universe decreases which violates the second law of thermodynamics. Bekenstein proposed a way out of this difficulty by defining the generalized entropy S' to be the sum of the entropy outside the black hole S and the entropy of the black hole itself [174]

$$S' \equiv S + S_{bh} = S + \frac{A}{4}. \tag{5.19}$$

In fact, this means that dropping an object into a black hole increases its surface area and that Hawking radiation creates thermal particles outside the black hole with finite entropy. This is known as the generalized second law which is the statement that the total entropy cannot decrease:

$$\Delta S' \geq 0. \tag{5.20}$$

5.2.4. Third law of black hole mechanics

The third law of black hole mechanics states that it is impossible to achieve a surface gravity of zero within a finite number of steps [179]. Since otherwise the black hole would seem unattractive to a distant observer. Moreover, it will be more unphysical if one imagines κ to be negative because in this case, the black hole would seem repulsive to an observer far from it. This law can also be proven by calculating κ explicitly for the Kerr metric. The non-negative nature of surface gravity is then guaranteed by the requirement that the solution does not have any closed-timelike curves [41]. This law corresponds to the weaker form of the third law of thermodynamics which states that the temperature of a real thermodynamic system is always greater than absolute zero. However, the classical third law of black hole mechanics is not analogous to the stronger form of the third law of thermodynamics which asserts that $S \to 0$ as $T \to 0$. Although the temperature of a black hole can be zero at the extremal limit, black holes do not always have zero area in such a limit. Hence, the black hole entropy does not always go to zero at zero temperature. This is an important difference between fluid and black hole thermodynamics. The second difference is that black hole thermodynamics is not extensive [180], meaning that one cannot increase the mass of the black hole in such a way that all conjugate variables remain constant. The third difference is the frequent absence of black hole thermodynamic stability. The study of thermodynamic stability is conducted by calculating heat capacity such that the positivity (negativity) of heat capacity refers to a stable (unstable) state. Negative heat capacity is a fixture of gravitational thermodynamic problems. For instance, the Kerr-Newman black hole thermodynamics is not stable for any set of values of (M, J, Q) [181]. However, there are black holes that can experience a stable state e.g., BTZ black holes are thermodynamically stable for all of their states. In some cases, the stability problem can be resolved by restricting the number of fluctuating variables, adding an AdS background, or altering the assumptions about the black hole's topology. For example, Reissner-Nordström and Kerr black holes are not thermodynamically stable for any thermodynamic state. However, in AdS spacetime both are stable for some values of black hole parameters [182].

5.3. Thermodynamic curvature for black holes

As was already mentioned, black holes have well-defined thermodynamic properties. Black hole thermodynamics leads naturally to thermodynamic fluctuation theory and an information metric

$$g_{\alpha\beta} = -\frac{\partial^2 S}{\partial X^\alpha \partial X^\beta}, \tag{5.21}$$

where $(M, J, Q, ...) = (X^1, X^2, X^3, ...)$ play the role of the conserved thermodynamic variables in black hole thermodynamics. However, instead of $S = S(X^1, X^2, X^3, ...)$, one can use $M = M(Y^1, Y^2, Y^3, ...)$ where $(Y^1, Y^2, Y^3, ...) = (S, J, Q, ...)$. In this case, the thermodynamic metric in the Weinhold energy form, with an additional prefactor $1/T$ is obtained. The information metric (5.21) produces the thermodynamic curvature $\mathcal{R}$. The concept of thermodynamic curvature of black holes was first studied by Ferrara et al. They applied thermodynamic curvature to calculate critical behavior in moduli spaces [183]. Afterward, Cai and Cho [184] investigated a connection between phase transitions in BTZ black holes to diverging $\mathcal{R}$. They also identified a correspondence with $\mathcal{R}$ for the Takahashi gas which suggests that an appropriate black hole statistical model might be a system of hard rods. Then, Aman, Bengtsson, and Pidokrajt [135] evaluated $\mathcal{R}$ for Reissner-Nordstrom, Kerr, and Kerr-Newman black holes with a nonzero cosmological constant. An extensive review was also conducted in the context of five-dimensional black holes and black rings by Arcioni and Lozano-Tellechea [185] who connected phase transitions to both diverging $\mathcal{R}$ and diverging second fluctuation moments. Afterward, Aman and Pidokrajt [186] investigated thermodynamic curvature for Reissner-Nordstrom and Kerr black holes in higher dimensions spacetime and noticed that patterns in four dimensions continue to higher dimensions. In Ref. [187], authors also studied thermodynamic curvature for a general class of BTZ black holes, including quantum corrections to the entropy.

The thermodynamic curvature of black holes has been studied with various thermodynamic criteria, each of which has been associated with success and failure. We mentioned some of them here.

5.3.1. Weinhold metric

In black hole thermodynamics, as well as in the Weinhold method, the mass of black holes can be considered as a thermodynamic potential with appropriate extensive parameters, such as entropy S, electric charge Q, and angular momentum J, along with their related intensive quantities - temperature, electric potential, and angular velocity.

For example, for a static charged black hole the denominator of Weinhold Ricci scalar is obtained as

$$denom(\mathcal{R}_W) = \left(M_{SS}M_{QQ} - M_{SQ}^2\right)^2 M^2\,(S,Q)\,, \tag{5.22}$$

where $M_{QQ} = \left(\frac{\partial^2 M}{\partial Q^2}\right)_S$ and $M_{SQ} = \frac{\partial^2 M}{\partial S \partial Q}$. As we know, the heat capacity is given by

$$C_Q = \frac{M_S}{M_{SS}}, \tag{5.23}$$

where $M_S = \left(\frac{\partial M}{\partial S}\right)_Q$ and $M_{SS} = \left(\frac{\partial^2 M}{\partial S^2}\right)_Q$ are regular functions. For consistency with the heat capacity results, the roots of the Eq. (5.22) should coincide with divergencies of the heat capacity. As can be seen from Eq. (5.22), only in special case $M_{SQ} = 0$ and nonzero M_{QQ}, the divergence points of the heat capacity coincide with divergencies of the Weinhold Ricci scalar. It is evident that for the case of $M_{SS} = \frac{M_{SQ}^2}{M_{QQ}}$, extra divergencies can be found for $\mathcal{R}_W$ which are not related to any phase transition points of the heat capacity. Therefore, the structure of this part of the denominator can provide extra divergencies that do not correspond to any phase transition points of the heat capacity. A curious observation is that the Weinhold geometry of the Kerr black holes is actually flat.

5.3.2. Ruppeiner geometry

Black hole thermodynamics exhibits unfamiliar features which are generic to systems with long-range interactions and self-gravitating systems, making Ruppeiner's arguments not directly applicable to black holes. The first issue encountered is negative specific heats, resulting in non-concave entropy functions. The second issue is the absence of extensive variables, causing the Ruppeiner metric not to be positive definite or have any null eigenvectors. In other words, the canonical ensemble does not exist, and choosing a physical dimension for the metric becomes difficult. In spite of these challenges, it is believed that the Ruppeiner geometry of black holes can provide interesting insights in black hole physics. For further observations on the role of the Ruppeiner metric in black hole physics, see Ref. [183]. For background information on black hole thermodynamics, see [188].

Based on investigations conducted in the context of the thermodynamic geometry of black holes, the Weinhold metric is proportional to the metric on the moduli space for supersymmetric extremal black holes with zero Hawking temperature, while the Ruppeiner metric governing fluctuations naively diverge, consistent with the argument that the thermodynamic description breaks down near the extreme [189, 190]. The Ruppeiner approach has been used for RN black holes, BTZ black holes, Kerr black holes, and Kerr-Newman black holes. It was found that when considering the entropy as a function of mass and other extensive variables, the Ruppeiner metric for RN and BTZ black holes is always flat, with zero scalar curvature. In contrast, for Kerr and Kerr-Newman black holes, the Ruppeiner metric is curved, and the scalar curvature diverges at the extremal limit. This reveals that the Ruppeiner metric for RN and BTZ black holes is quite different from that of Kerr black holes.

A remarkable point is that in anti-de Sitter spacetime, the curvature scalar of the Ruppeiner metric diverges for RN black holes. Here, we try to review this case in more detail based on Ref. [135].

The thermodynamics of these black holes is now defined by the fundamental relation

$$M = \frac{\sqrt{S}}{2}\left(1+\frac{Q^2}{S}\right) \ . \tag{5.24}$$

The Hawking temperature is

$$T = \frac{1}{4\sqrt{S}}\left(1-\frac{Q^2}{S}\right). \tag{5.25}$$

The extremal limit, beyond which the singularity becomes naked, happens at

$$Q^2 = M^2 \quad \Leftrightarrow \quad \frac{Q^2}{S} = 1 \ . \tag{5.26}$$

In its natural coordinates the Weinhold metric becomes

$$ds_W^2 = \frac{1}{8S^{\frac{3}{2}}}\left(-\left(1-\frac{3Q^2}{S}\right)dS^2 - 8QdQdS + 8SdQ^2\right) \ . \tag{5.27}$$

Evidently, the component g^W_{SS} vanishes and changes sign at $\frac{Q^2}{S} = \frac{1}{3}$, indicating that the specific heat C_Q diverges and changes sign there. Introducing the new coordinate

$$u = \frac{Q}{\sqrt{S}} \ ; \qquad -1 \le u \le 1 \ . \tag{5.28}$$

the Weinhold metric has a diagonal form as

$$ds_W^2 = \frac{1}{8S^{\frac{3}{2}}}\left(-(1-u^2)dS^2 + 8S^2du^2\right) \ . \tag{5.29}$$

In these coordinates, the Ruppeiner metric is given by

$$ds^2 = \frac{1}{T}ds_W^2 = -\frac{dS^2}{2S} + 4S\frac{du^2}{1-u^2} \ . \tag{5.30}$$

One can also introduce new coordinates as

$$\tau = \sqrt{2S} \qquad \sin\frac{\sigma}{\sqrt{2}} = u \ , \tag{5.31}$$

which by using, the Ruppeiner metric takes the following form

$$ds^2 = -d\tau^2 + \tau^2 d\sigma^2 \ , \tag{5.32}$$

which is flat and is recognizable as a timelike wedge in Minkowski space when described by Rindler coordinates. The state space of the Reissner-Nordström black holes is depicted in Figure 6.

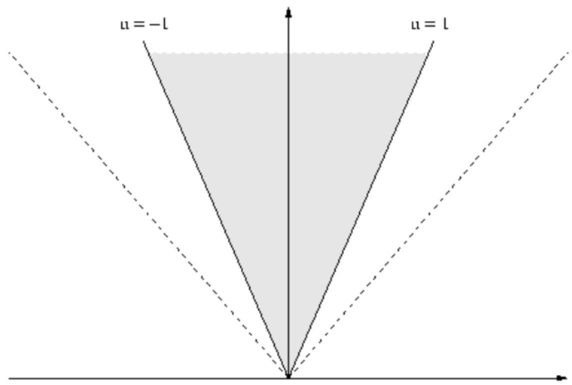

Figure 6: The state space of the Reissner-Nordström black holes shown as a wedge in a flat Minkowski space [135].

In the presence of cosmological constant, the mass is given by

$$M = \frac{\sqrt{S}}{2}\left(1 + \frac{S}{l^2} + \frac{Q^2}{S}\right), \tag{5.33}$$

and the Hawking temperature is obtained as

$$T = \frac{1}{4\sqrt{S}}\left(1 + \frac{3S}{l^2} - \frac{Q^2}{S}\right), \tag{5.34}$$

where the extremal limit occurs when $\frac{Q^2}{S} = 1 + \frac{3S}{l^2}$. Using the above information, the Weinhold metric will be obtained as follows

$$ds_W^2 = \frac{1}{8S^{\frac{3}{2}}}\left(-\left(1 - \frac{3S}{l^2} - \frac{3Q^2}{S}\right)dS^2 - 8QdSdQ + 8SdQ^2\right). \tag{5.35}$$

Making use of the same coordinate transformation as above, one can diagonalize the Weinhold metric and conformally relate it to the Ruppeiner metric as

$$ds^2 = \frac{1}{1 + \frac{3\tau^2}{2l^2} - u^2}\left(-\left(1 - \frac{3\tau^2}{2l^2} - u^2\right)d\tau^2 + 2\tau^2 du^2\right). \tag{5.36}$$

In this case and with non-zero cosmological constant, the state space will be as Figure 7. The curvature scalar of the Ruppeiner metric is obtained as

$$R = \frac{9}{l^2}\frac{\left(\frac{3S}{l^2} + \frac{Q^2}{S}\right)\left(1 - \frac{S}{l^2} - \frac{Q^2}{S}\right)}{\left(1 - \frac{3S}{l^2} - \frac{Q^2}{S}\right)^2\left(1 + \frac{3S}{l^2} - \frac{Q^2}{S}\right)}\ . \tag{5.37}$$

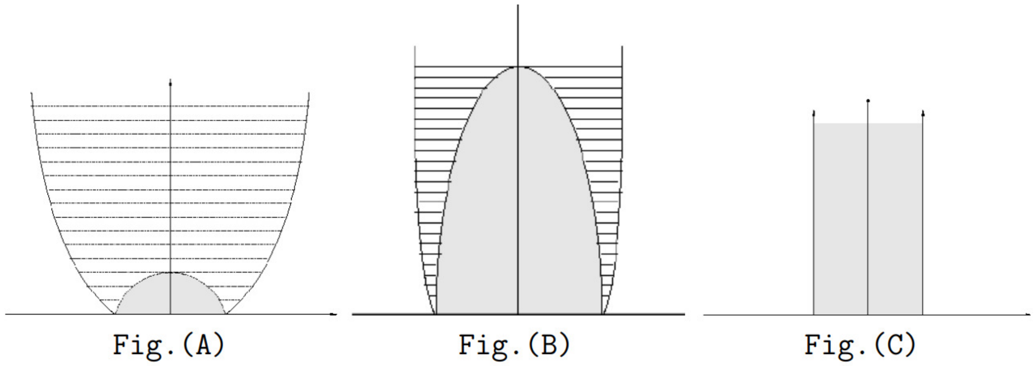

Figure 7: The state space for Reissner-Nordström black holes in the presence of a cosmological constant; the coordinates are u and S and Λ decreases from A to C [135].

As the above relation shows, the curvature scalar diverges both in the extremal limit and along the curve where the metric changes signature, i.e. where the thermodynamic stability properties are changing.

It is worth pointing out that the same results were not obtained in the investigation of the thermodynamic geometry of black holes using the Weinhold and Ruppeiner methods. See Table 3 [135] for a detailed comparison. It is now well established that these inconsistent results are a consequence of the fact that Weinhold and Ruppeiner metrics are not Legendre invariant, meaning that their properties depend on the thermodynamic potential used in their construction. As we mentioned before, this property make them inappropriate for describing the geometry of thermodynamic systems.

Table 3: Comparison between the results obtained from the Weinhold and Ruppeiner metrics for different black hole solutions.

Black hole family	Ruppeiner	Weinhold
RN	Flat	Curved, no Killing vectors
RNadS	Curved, no Killing vectors	Curved, no Killing vectors
Kerr	Curved, no Killing vectors	Flat
BTZ	Flat	Curved, no Killing vectors
Kerr–Newman	Curved	Curved

• ***Shortcoming of Ruppeiner metric:***

Although the Ruppeiner metric was able to describe phase transition and the nature of interactions between black hole microstructures for some types of black holes [191–193], the investigations conducted in the context of thermodynamic geometry showed that this approach cannot be a suitable method for describing the phase transition of all black holes. Because there are extra divergencies for Ruppeiner's Ricci scalar which are not related to any phase transition of the heat capacity. To achieve a better insight into understanding this issue, we examine the denominator of the Ricci scalar of the Ruppeiner metric for a charged case. The denominator of the Ricci scalar for this case is

$$denom(\mathcal{R}_R) = \left(M_{SS}M_{QQ} - M_{SQ}^2\right)^2 T(S,Q)M^2\left(S,Q\right). \tag{5.38}$$

According to the above equation, the phase transitions of the heat capacity correspond to the singularities of the Ruppeiner Ricci scalar only when $M_{SQ}^2 = 0$ and $M_{QQ} \neq 0$. On the other hand, due to the zeroes of $\left(M_{SS}M_{QQ} - M_{SQ}^2\right)$, some divergence points of $\mathcal{R}_R$ may appear which cannot coincide with phase transition points. In the case of $M_{SQ}^2 = 0$, phase transition points of the heat capacity ($M_{SS} = 0$) are covered by divergencies of the Ricci scalar of the Ruppeiner metric. But in this case, one may encounter with extra divergencies related to the roots of $M_{QQ} = 0$ which are not related to any phase transition points of the heat capacity. Moreover, for the case of $M_{SQ}^2 \neq 0$, the possible real roots of $M_{QQ} = \frac{M_{SQ}^2}{M_{SS}}$ lead to the same extra divergencies observed in the Weinhold metric.

5.3.3. Fisher-Rao metric

The Fisher-Rau (FR) metric is an alternative approach to classical statistical mechanics that is used to analyze the geometry of thermodynamic systems. This metric is a choice of Riemannian metric in the space of probability distributions. In a statistical mechanical context, the probability density distribution $p(x|\vartheta)$, which is related to the partition function ($Z(\vartheta)$) of the corresponding system, takes the form of the Gibbs measure [136]

$$p(x|\vartheta) = \exp\left[-\vartheta_i H_i(x) - \ln Z(\vartheta)\right]. \tag{5.39}$$

Where $H_i(x)$ represents Hamiltonian functions, x is a random variable, and $\vartheta = (\vartheta_1, ..., \vartheta_n) \in R^n$ is an $n-$vector continuous parameter characterizing the statistical model under consideration, the FR metric can be derived from the information contained in the partition function. A probability density function $p(x)$ can be associated with a vector in a real Hilbert space $\mathcal{H}$ by taking the square-root $\psi(x) = \sqrt{P(x)}$, which is denoted by a vector ψ^a in $\mathcal{H}$ [194]. Hence, the Hilbert space embodies the state space of the system, and the properties of the statistical model under consideration can be described by embedding of $p(x|\vartheta)$ in $\mathcal{H}$. The geometry resulting from the embedding has a natural Riemannian metric, the FR metric in the classical case or the Fubini-Study metric in the quantum case. For the classical one, Eq. (5.39) takes the following form

$$g^{FR} = \frac{\partial^2 \ln Z(\vartheta)}{\partial\vartheta^\mu \partial\vartheta^\nu} d\vartheta^\mu d\vartheta^\nu \ . \tag{5.40}$$

The geometric features of the manifold described by metric (5.40) were employed for various statistical models. For the van der Waals fluid, we can consider the parameters as $\vartheta^1 = 1/T$, and $\vartheta^2 = P/T$. With such a choice, the internal energy U (volume V) can be considered as the corresponding Hamiltonian function to ϑ^1 (ϑ^2). Hence, the partition function $Z(\vartheta)$ is a function of temperature and pressure. The scalar curvature obtained by this two-dimensional manifold diverges at the critical points and the scaling exponent of the curvature near the transition points coincides with that of the correlation volume. The investigation in the context of the ideal gas indicated that the scalar curvature vanishes in this case and the manifold is flat. Such behavior has also been observed by Ruppeiner's and Weinhold's geometry in these particular cases. As was shown in Ref. [22], these two metrics are related to the FR metric by using Legendre transformations of the corresponding variables, indicating that the FR metric is not invariant under Legendre transformations.

For two well-known cases, RN and Kerr black holes, the components of the FR metric $g_{ij}^{FR}(\vartheta)$ are usually chosen as the "inverse" of the thermodynamic variables: $\vartheta^1 = 1/T$, $\vartheta^2 = P/T$, etc. Since the relations $\theta^\mu = \theta^\mu(E^a)$ must allow the inverse transformation, it can be shown that the FR metric is written as $g_{ab}^{FR} = \partial^2 \ln Z(E)/\partial E^a \partial E^b$ for the corresponding coordinates. The partition function is obtained as

$$Z = \exp\left[-\frac{1}{T}(M - TS - \Omega_H J - \phi Q)\right] \; . \tag{5.41}$$

For RN black holes, the partition function is determined as $Z = \exp(-M'_R/T)$ and for Kerr black holes is $Z = \exp(-M'_K/T)$. The thermodynamic potentials M'_R and M'_K are related to the following mass representations

$$\begin{aligned} M'_R &= M - TS - \phi Q \; , \\ M'_K &= M - TS - \Omega_H J. \end{aligned} \tag{5.42}$$

Thus, the components of the FR metric for black holes are essentially given by

$$g_{ab}^{FR} = -\partial^2 (M/T)/\partial E^a \partial E^b. \tag{5.43}$$

As already mentioned, the FR metric is not Legendre invariant for black holes. Therefore, it is not a suitable method for solving the problem contradictory results arising from the application of Weinhold's and Ruppeiner's approaches.

Before ending this part, it is worth mentioning some points regarding the issue that how the representation based on Riemannian geometry, particularly as applied by Rao in 1945, which used entropy as a thermodynamic potential, relates to the information loss paradox, which often arises in the context of thermodynamics and statistical physics. Indeed, by utilizing Riemannian geometry, Rao established a framework where entropy serves as a thermodynamic potential. This approach allows for the quantification of statistical distances between probability distributions, providing a measure of information loss or retention during thermodynamic processes. In this regard, the Fisher-Rao metric and subsequent Hessian metrics play crucial roles in shedding light on the information loss paradox within thermodynamic geometry. The Fisher-Rao metric, derived from the Fisher information matrix, quantifies the distinguishability and separability between probability distributions. It enables the assessment of how much two system states differ in terms of their information content. This metric, together with Hessian metrics that capture the local curvature of the manifold, facilitates the analysis of stability, critical points, and phase transitions in thermodynamic potentials. By incorporating the Fisher-Rao and Hessian metrics into thermodynamic geometry, researchers gain valuable tools to quantify and characterize the distribution and transformation of information during thermodynamic processes. This, in turn, contributes to addressing the information loss paradox and deepening our understanding of complex systems in the realm of thermodynamics and statistical physics.

However, a key challenge of using Riemannian geometry, as applied by Rao in 1945 using entropy as a thermodynamic potential, lies in its practical implementation and applicability to diverse systems. The approach requires making assumptions about the underlying statistical distributions and their functional forms, which may not always accurately capture the complexity of real-world systems. Furthermore, accurately estimating the metric tensor in Riemannian geometry can be computationally demanding and may pose difficulties in high-dimensional spaces. On the other hand, the main limitation of the Fisher-Rao metric and subsequent Hessian metrics is their sensitivity to the choice of coordinate systems or parameterizations. Different parameterizations can lead to varying metric structures, potentially affecting the interpretation of distances or curvature. Additionally, in systems with limited available data, accurately estimating the metric tensors using finite samples may introduce uncertainties and biases in the computed metrics, which can impact the reliability of results obtained through thermodynamic geometry methods.

At the end of this discussion, it is necessary to mention the idea that remnants of black holes can be suggested as a resolution of the black hole information paradox. In one version of this idea [195–197], the black hole stops evaporating when its dimension approaches the Planck scale. But this remnant quiescent object retains the vast amount of information that was stored in the black hole after its initial formation. A common argument for remnants as information repositories is that they can hold large information despite their small dimension since they have a very large internal space in the form of a throat or horn. But Bekenstein proposed the existence of a universal bound on the entropy S of any object of maximal radius R and total energy E [198]:

$$S \leq \frac{2\pi RE}{\hbar c} \tag{5.44}$$

and showed that if a remnant has an information capacity well above that specified by bound (5.44) in terms of its external dimensions, it will violate the generalized second law if it falls into a large black hole [199]. Indeed, the black hole remnants cannot resolve the information paradox (see Ref. [200] for more information in this regard).

5.3.4. Quevedo metric

First, we review the formulation of general relativistic black hole thermodynamics in the language of geometrothermodynamics. As we mentioned earlier, a rotating stationary black hole in general relativity is completely described by three parameters (M, Q, J), which are related to each other through the first law of black hole thermodynamics $dM = TdS + \phi dQ + \Omega_H dJ$. For a given fundamental equation $M = M(S, J, Q)$, there are the following conditions for thermodynamic equilibrium [136]

$$T = \frac{\partial M}{\partial S}\ , \quad \Omega_H = \frac{\partial M}{\partial J}\ , \quad \phi = \frac{\partial M}{\partial Q}\ . \tag{5.45}$$

Here, $\mathcal{T}$ is a 7-dimensional manifold characterized by $Z^A = \{M, S, J, Q, T, \Omega_H, \phi\}$, whereas the submanifold $\mathcal{E}$ is 3-dimensional with coordinates $E^a = \{S, J, Q\}$, and is defined by the following mapping

$$\varphi : \{S, J, Q\} \longmapsto \left\{M(S, J, Q), S, J, Q, \frac{\partial M}{\partial S}, \frac{\partial M}{\partial J}, \frac{\partial M}{\partial Q}\right\}\ . \tag{5.46}$$

The mass M is considered as the thermodynamic potential dependent on the extensive variables S, J, and Q. It is noteworthy that by using Legendre transformations, other thermodynamic potentials can be also introduced as different combinations of extensive and intensive variables. For instance

$$
\begin{aligned}
M_1 &= M - TS \ , \\
M_2 &= M - \Omega_H J \ , \\
M_3 &= M - \phi Q \ , \\
M_4 &= M - TS - \Omega_H J \ , \\
M_5 &= M - TS - \phi Q \ , \\
M_6 &= M - \Omega_H J - \phi Q \ , \\
M_7 &= M - TS - \Omega_H J - \phi Q \ .
\end{aligned} \tag{5.47}
$$

Here, one can define the mapping φ for each case independent of the chosen thermodynamic potential. Since $\mathcal{T}$ and $\mathcal{E}$ are invariant under Legendre transformations, the properties of the underlying geometry for an ordinary thermodynamic system do not depend on the thermodynamic potential. This shows that it is consistent with standard thermodynamics.

In the mass representation, the fundamental Gibbs 1-form is given by

$$\theta = dM - TdS - \Omega_H dJ - \phi dQ, \tag{5.48}$$

whereas, in the entropy representation with the fundamental equation $S = S(M, J, Q)$, it can be chosen as

$$\theta_S = dS - \frac{1}{T}dM + \frac{\Omega_H}{T}dJ + \frac{\phi}{T}dQ \ . \tag{5.49}$$

In this case, the space of equilibrium states can be defined by the following harmonic mapping

$$\varphi_S : \{M, J, Q\} \longmapsto \{M, S(M, J, Q), J, Q, T(M, J, Q), \Omega_H(M, J, Q), \phi(M, J, Q)\} \ , \tag{5.50}$$

with

$$\frac{1}{T} = \frac{\partial S}{\partial M} \ , \quad \frac{\Omega_H}{T} = -\frac{\partial S}{\partial J} \ , \quad \frac{\phi}{T} = -\frac{\partial S}{\partial Q} \ . \tag{5.51}$$

As has already been explained, components of Weinhold's metric g^W are given as the Hessian of the internal thermodynamic energy (the mass representation), while the components of Ruppeiner's metric g^R are defined as the Hessian of the entropy (the entropy representation). The simplest way to reach the Legendre invariance for Weinhold's metric is to apply a conformal transformation, with the thermodynamic potential as the conformal factor. Hence, the simplest Legendre invariant generalization of g^W is as

$$g = Mg^W = M\frac{\partial^2 M}{\partial E^a \partial E^b}dE^a dE^b \ , \tag{5.52}$$

which can be written in terms of the components of Ruppeiner's metric as

$$g = MTg^R = -M\left(\frac{\partial S}{\partial M}\right)^{-1}\frac{\partial^2 S}{\partial F^a \partial F^b}dF^a dF^b \ , \tag{5.53}$$

where $E^a = \{S, J, Q\}$ and $F^a = \{M, J, Q\}$. Using the mass representation, the nondegenerate metric G on the phase space $\mathcal{T}$ is defined as

$$G = (dM - TdS - \Omega_H dJ - \phi dQ)^2 + (TS + \Omega_H J + \phi Q)(dTdS + d\Omega_H dJ + d\phi dQ). \tag{5.54}$$

In Ref. [136], Quevedo employed the Legendre invariant generalizations of Weinhold's and Ruppeiner's metrics and analyzed the geometry of the RN and Kerr black hole thermodynamics, by using Legendre invariant thermodynamic metrics. He noticed that the obtained results are geometrically consistent for both cases. For the Reissner-Nordström solution, there was an agreement with the results of standard black hole thermodynamics, whereas, for Kerr black holes, the simplest Legendre invariant metrics could not reproduce the corresponding phase transition structure.

• ***Shortcoming of Quevedo metric:***

Although the divergencies of the Quevedo Ricci scalars coincide with the divergencies of specific heat [201, 202], studies in this regard showed that the Quevedo metric cannot be a suitable candidate to describe phase transition points of black holes. It is due to the fact that they suffer from some extra divergencies. For example, for charged black holes, the Quevedo metric can be written as [28]

$$ds_Q^2 = \Omega\left(-M_{SS}dS^2 + M_{QQ}dQ^2\right), \tag{5.55}$$

where the (conformal) function Ω has one of the following forms

$$\Omega = \begin{cases} SM_S + QM_Q, & \text{case I} \\ SM_S, & \text{case II} \end{cases}. \tag{5.56}$$

To obtain the curvature singularity of the Quevedo metric, one needs to calculate the Ricci scalar. Since the calculated results of the Ricci scalar are too large, we avoid writing the Ricci scalar here and just write down its denominator with the following explicit forms

$$denom(\mathcal{R}_{Q1}) = \left(SM_S + QM_Q\right)^3 M_{SS}^2 M_{QQ}^2, \tag{5.57}$$

$$denom(\mathcal{R}_{Q2}) = S^3 M_S^3 M_{SS}^2 M_{QQ}^2. \tag{5.58}$$

Evidently, the divergencies of the heat capacity and Quevedo Ricci scalars coincide with together due to the existence of M_{SS} in the denominator of both Quevedo Ricci scalars. It should be noted that there exists an additional function M_{QQ}^2, and its roots provide extra singular points for both cases of Quevedo Ricci scalars. Another divergence point can also appear for nonzero M_Q and for a special choice of $M_S = -\frac{Q}{S}M_Q$.

According to Ref. [203], this problem comes from the fact that in black hole thermodynamics, the thermodynamic potential is not a homogeneous first-order function of its natural extensive variables contrary to classical thermodynamics. In fact, the natural extrinsic thermodynamic variables expressing the first law of thermodynamics are not the same variables in which thermodynamic potentials are homogeneous. This is the difference between black hole thermodynamics and classical ones. To work out this problem, the modified extensive variables should be considered instead of the natural ones (see Ref. [203] for more detail).

5.3.5. HPEM metric

Hendi et al. introduced a new metric, which is known as the HPEM metric, to eliminate the problems of previous thermodynamical metrics by removing extra singular points in the thermodynamical Ricci scalar that do not coincide with phase transitions. HPEM metric is defined as [204]

$$ds^2_{HPEM} = S\frac{M_S}{M^3_{QQ}}\left(-M_{SS}dS^2 + M_{QQ}dQ^2\right). \tag{5.59}$$

The remarkable point, here, is that the HPEM metric is defined in the same way as the Quevedo metric but with a different conformal function. In this metric, the total mass serves as a thermodynamical potential with entropy and electric charge acting as extensive parameters. The numerator and the denominator of the thermodynamical Ricci scalar for this new metric is, respectively,

$$\begin{aligned} num(\mathcal{R}) = {} & 6S^2M_S^2M_{QQ}M_{SS}^2M_{QQQQ} - 6SM_S^2M_{QQ}^2M_{SS}M_{SSQQ} + 2SM_{SQQ}^2M_S^2M_{QQ}M_{SS} \\ & +2\left[SM_SM_{SSS} - \frac{1}{2}M_{SS}\left(SM_{SS} - M_S\right)\right]SM_{QQ}^2M_SM_{SQQ} - 9S^2M_{QQQ}^2M_S^2M_{SS}^2 \\ & +4\left[\frac{1}{4}M_{SQ}M_{SS} + M_SM_{SSQ}\right]S^2M_{QQ}M_SM_{QQQ} + \left[S^2M_S^2M_{SSQ} - S^2M_{SQ}M_{SS}M_SM_{SSQ}\right. \\ & \left. SM_{QQ}M_S\left(SM_{SS} - M_S\right)M_{SS} - 2\left(S^2M_{SS}^3 + M_S^2M_{SS}\right)M_{QQ} + 2S^2M_{SQ}^2M_{SS}^2\right]M_{QQ}^2, \end{aligned} \tag{5.60}$$

and

$$denom(\mathcal{R}) = S^3M_S^3M_{SS}^2. \tag{5.61}$$

The denominator of the thermodynamical Ricci scalar (TRS), Eq. (5.61), ensures that all the phase transition points coincide with divergencies of the mentioned thermodynamical Ricci scalar and there is no extra term that may provide extra divergencies.

It should be noted that, in general, the derivatives of all orders of M (such as M_{QQ}, M_{SS}, M_{SQ}, M_{SSQ} and so on) are independent from each other. For the case of vanishing M_S or M_{SS}, the numerator of TRS has nonzero value, but denominator of TRS vanishes. In the case of $M_S = M_{SS} = 0$, both numerator and denominator vanish, but it should be noted that the denominator approaches zero faster than the numerator. Therefore, one concludes that when M_S and/or M_{SS} go to zero, the thermodynamical Ricci scalar diverges. The validity of the HPEM metric was explored for different types of black holes and was shown that this method is a suitable one for studying the thermodynamic geometry of black holes [71, 205–220].

5.3.6. Other approaches in the context of thermodynamic geometry

A significant effort in the context of thermodynamic geometry was made by Mansouri et al. They explored a formulation of thermodynamic geometry for different types of black holes and proved that their formulation can be applied to an arbitrary thermodynamic system. In studying RN black holes, they noticed that singularities of the scalar curvature $R(S,Q)$ match to phase transition points of the heat capacity C_Φ at a constant electric potential, whereas phase transition points of the heat capacity C_Q at a constant electric charge correspond to the singularities of the scalar curvature $\overline{R}(S,\Phi)$. Here, $\overline{R}(S,\Phi)$ is obtained by defining $\overline{M}$ as a new conjugate potential of $M(S,Q)$ in following form

$$\overline{M}(S,\Phi) = M(S,Q) - \Phi Q \tag{5.62}$$

In fact, the scalar curvature $R(S,Q)$ is not able to explain the properties of the phase transitions of C_Q but it can describe phase transitions C_Φ. Regarding black holes with three parameters (Kerr-Newman black holes), the singularities of the scalar curvature $R(S,Q,J)$ coincide with divergencies of heat capacity at a constant electric potential and angular velocity $C_{\Phi,\Omega}$ and that also those of $\overline{R}(S,\Phi,\Omega)$ correspond to the phase transitions of $C_{Q,J}$. In general, for a black hole with n parameters, the singularities of $R(S,Q_1,...,Q_n)$ correspond to the phase transition points of $C_{\Phi_1,...,\Phi_n}$ and the singularity points of $\overline{R}(S,\Phi_1,\Phi_2,...,\Phi_n)$ coincide with divergencies of $C_{Q_1,Q_2,...,Q_n}$ (see Ref. [221] for more details). In Ref. [222], they employed the Nambu brackets and obtained a simple representation of the conformal transformations that connect different thermodynamic metrics to each other. Using the bracket approach, they studied the relationship between singularities of the scalar curvature in various representations of thermodynamic geometry and phase transition points of different heat capacities. They also investigated the intrinsic, extrinsic, and total curvatures of a certain hypersurface in the thermodynamic geometry of a physical system and explained the relationship between them [223]. Their finding regarding KN-AdS black hole showed that by considering a constant J hypersurface, the curvature scalar is broken down into two parts. One is related to a zero intrinsic curvature (the Ruppeiner curvature of the Reissner-Nordström-AdS black hole) and the other is concerned with an extrinsic part whose divergence points are the singularities of a non-rotating Kerr-Newman-AdS black hole. In Ref. [224], they constructed a new formalism thermodynamic geometry by changing the coordinates of the thermodynamic space by means of Jacobian matrices and showed that the GTD metric is related to their new formalism of the thermodynamic geometry with use of a singular conformal transformation. It is worth mentioning that this new metric removes all extra curvature singularities that appeared in GTD. In fact, there is a one-to-one correspondence between the phase transition points of a black hole and singularities of the curvature associated with this new metric. In Ref. [225], they employed this new metric for black holes in pure Lovelock gravity and found an exact correspondence between thermodynamic Ricci scalar and specific heat C_Q at critical point.

The study of black hole criticality is done with two separate approaches: one approach is related to the behavior of the heat capacity which diverges at the critical point. Another case is related to the AdS background with the cosmological constant, which plays the role of thermodynamic pressure in the extended phase space. In contrast to the usual thermodynamics, $P-V$ criticality, $T-S$ criticality, and also the $Y-X$ criticality (where Y is an external charge and X is its corresponding potential) can be found for the black holes. Some separate attempts were done in this context to explain criticality in a geometrical way based on the divergence point of heat capacity not a point as the inflection point. In Ref. [226] authors defined the relevant Legendre invariant thermogeometrics corresponding to the two criticality conditions and illustrated that the critical point refers to the divergency of the Ricci scalars calculated from these metrics. For the first condition $\left(\frac{\partial P}{\partial V}\right)_{T,Y}=0$, the Helmholtz free energy was considered as a proper thermodynamic quantity to define a thermogeometrical metric. Whereas, for the geometrical description of the second condition (the condition of the point of inflection), the pressure was a proper thermodynamical quantity (instead of F). It is worth mentioning that in Ref. [227], the authors employed this approach to investigate the thermodynamic geometry of charged accelerating black holes and found that the Ricci scalar included an extra divergence point at the critical temperature and was not able to describe the critical point. In other words, this method cannot be a suitable candidate for this kind of black hole. According to their analysis, only the HPEM's metric can provide an appropriate picture of phase transition for accelerating AdS black holes in both extended and nonextended phase space.

6. Conclusion

In this study, we conducted a comprehensive examination of various facets pertaining to geometrical thermodynamics. Our analysis encompassed the exploration of two distinct approaches involving Riemannian metrics: namely, the utilization of *Hessian metrics* and *Legendre invariant metrics* to develop a geometric framework for describing equilibrium thermodynamics. The first approach involves the incorporation of metrics into the equilibrium space, wherein the components of these metrics align with the Hessian matrix associated with specific thermodynamic potentials. This approach enables a direct connection between the geometry of the equilibrium space and thermodynamic properties. Alternatively, the second approach establishes the thermodynamic phase space as a Riemannian contact manifold through a purely geometric perspective. Within this framework, the class of Legendre invariant metrics has been introduced in the formalism of geometric thermodynamics. These metrics serve the purpose of accounting for the fundamental observation that the physical properties of a system in equilibrium thermodynamics remain unchanged regardless of the chosen thermodynamic potential, from a geometric standpoint.

After discussing the two previously mentioned approaches, an examination of the geometric representation of thermodynamics was undertaken for various ordinary systems, including ideal gas and real gas. Additionally, a comprehensive evaluation of these approaches in relation to the issue of black holes was provided. It was noted that black holes can be regarded as thermodynamic systems by equating the parameters of the black hole (mass, surface gravity, and area) to the thermodynamic system's energy, temperature, and entropy, respectively. The four laws of black hole mechanics, which correspond to the four laws of ordinary thermodynamics, were also reviewed. These laws are summarized in the Table 4.

Table 4: Summary of the laws of thermodynamics and their black hole mechanical counterparts.

Law	**Thermodynamics**	**Black Hole Mechanics**
Zeroth Law	The temperature T is constant over a system in thermal equilibrium	The surface gravity κ is constant on the event horizon of a stationary black hole
First Law	$dE = TdS + work\ terms$	$dM = \frac{\kappa}{8\pi G} + work\ terms$
Second Law	$dS_{TD} \geq 0$	$d(S_{outside} + \frac{A}{4}) \geq 0$
Third Law	$dS_{TD} \to 0$ as $T \to 0$	It is impossible to achieve $\kappa = 0$ within a finite number of steps.

In addition, we explored the utilization of the thermodynamic metrics mentioned above in the context of black hole thermodynamics. We demonstrated that the Weinhold, Ruppeiner, and Fisher-Rao metrics yield inconsistent outcomes when applied to the same black holes due to their lack of Legendre invariance. While the Quevedo metric resolves the issue of conflicting results from previous metrics, it contains additional divergencies in its Ricci scalar that do not correspond to phase transition points.

To address these challenges, Hendi et al. introduced a novel metric that eliminates the problems encountered with previous thermodynamic metrics by eliminating extra singular points. Furthermore, we examined alternative approaches within the realm of thermodynamic geometry for black holes, which can be applied to any thermodynamic system. One such approach, proposed by Mansouri et al., established a connection between singularities of the scalar curvature at constant charge and phase transition points of the heat capacity at a constant electric potential.

Another approach focused on Legendre invariant thermogeometrics, which encompassed the two criticality conditions. Although this approach yielded consistent results for many black holes, it was not deemed suitable for studying accelerating black holes.

In conclusion, while geometrical thermodynamics approaches have achieved interesting results, this paper highlights several ongoing challenges and open questions that require further investigation. Some of these challenges are outlined below:

1. Incorporating contact geometry and GTD into systems with strong interactions may pose challenges due to the complexity of the underlying dynamics. The nonlinearity and intricate interaction patterns may hinder the identification and calculation of relevant geometric quantities, potentially limiting the applicability of GTD.

2. Extending GTD to non-equilibrium conditions can be intricate, as it requires considering dissipative processes and time-dependent phenomena. Capturing the dynamics and temporal evolution of the system within the geometric framework may require the development of novel mathematical tools and techniques.

3. In systems exhibiting emergent behaviors, where collective phenomena arise from the interactions of individual components, the applicability of GTD may be limited. The geometric framework traditionally relies on the notion of equilibrium and may not fully capture the emergent properties of the system, demanding the formulation of new concepts and theories.

4. Generalizing GTD to incorporate quantum and relativistic effects in a consistent manner represents a significant challenge. Quantum systems may involve noncommutative geometry, entanglement, and uncertainty principles, while relativistic systems require accounting for curved spacetime and the interplay between geometry and dynamics. Bridging GTD with these theories requires careful consideration and potential modifications to accommodate these fundamental principles.

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A. Riemannian geometry

Riemannian geometry is widely used in general relativity to describe spacetime. In fact, based on the geometry we are dealing with, a length element or metric can be defined for the investigated spacetime. Regarding thermodynamic systems, a thermodynamic coordinate surface can be defined using the system's thermodynamic parameters, and for a system with two independent variables, we will have a two-dimensional surface. In what follows, we will review the main elements of Riemannian geometry.

Regarding the Riemannian geometry in two dimensions, manifold or surface is the first constituent element. Based on physics topics, physical quantities can be defined by the points placed on the manifold which typically involve more than one coordinate. In the case of thermodynamic processes, these points denote the thermodynamic states.

The second main constituent part of Riemannian geometry provides a line element's rule for two adjacent points with coordinate difference Δx^α, which is expressed as follows

$$\Delta\ell^2 \equiv g_{\mu\nu}\Delta x^\mu \Delta x^\nu, \qquad \mu,\nu = 1,2 \tag{A.1}$$

in which $g_{\alpha\beta}$ is a positive-definite matrix representing the metric elements. A manifold whose distance between points is defined in Eq. (A.1) is called a Riemannian manifold. It is notable that being independent of the choice of the coordinate system is a requirement for defining the distance between two points. For instance, Euclidean planes are familiar Riemannian surfaces on which the distance is defined by Cartesian coordinates as follows

$$\Delta\ell^2 = (\Delta x^0)^2 + (\Delta x^1)^2. \tag{A.2}$$

Moreover, the above distance can be defined in terms of polar coordinates,

$$\Delta\ell^2 = \Delta r^2 + r^2\,\Delta\theta^2, \tag{A.3}$$

where r indicates the radial coordinate and θ is the angular coordinate. Another example of two-dimensional Riemannian manifold is a sphere of radius a in spherical coordinates (θ,ϕ)

$$\Delta\ell^2 = a^2\Delta\theta^2 + a^2\,\sin^2\theta\,\Delta\phi^2. \tag{A.4}$$

Now, let us find the rule of transformation for the metric elements. To this end, one can consider some different coordinates (x'^1, x'^2) in which the metric elements are $g'_{\alpha\beta}$. Since

$$\Delta x^\alpha = \frac{\partial x^\alpha}{\partial x'^\mu}\Delta x'^\mu, \tag{A.5}$$

we get

$$\Delta\ell^2 = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\Delta x'^\mu \Delta x'^\nu \equiv g'_{\mu\nu}\Delta x'^\mu \Delta x'^\nu. \tag{A.6}$$

Therefore, the new metric elements read

$$g'_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}, \tag{A.7}$$

which is the transformation law for a 2^{nd} order tensor [140, 228].

The third essential elements of a Riemannian geometry which is Riemannian curvature examines the issue that *if the curvatures at given points of two different coordinate systems are the same, they* *will describe the same manifold.* In order to find this scalar invariant, one can employ the following conventions

$$\begin{aligned}
\Gamma^{\sigma}{}_{\mu\nu} &= \frac{1}{2}g^{\sigma\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}), &\text{(A.8)}\\
R^{\sigma}{}_{\rho\mu\nu} &= \partial_\nu \Gamma^{\sigma}{}_{\rho\mu} - \partial_\mu \Gamma^{\sigma}{}_{\rho\nu} + \Gamma^{\delta}{}_{\rho\mu}\Gamma^{\sigma}{}_{\delta\nu} - \Gamma^{\delta}{}_{\rho\nu}\Gamma^{\sigma}{}_{\delta\mu}, &\text{(A.9)}\\
R_{\mu\nu} &= R^{\sigma}{}_{\mu\sigma\nu}, &\text{(A.10)}\\
\mathcal{R} &= g^{\mu\nu}R_{\mu\nu}, &\text{(A.11)}
\end{aligned}$$

where $g^{\mu\nu}g_{\mu\nu} = 1$. For a 2-dimensional space, the Riemannian scalar curvature is obtained as follows

$$\begin{aligned}
\mathcal{R} = &- \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial x^0}\left(\frac{g_{01}}{g_{00}\sqrt{g}}\frac{\partial g_{00}}{\partial x^1} - \frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial x^0}\right)\right.\\
&+ \left.\frac{\partial}{\partial x^1}\left(\frac{2}{\sqrt{g}}\frac{\partial g_{01}}{\partial x^1} - \frac{1}{\sqrt{g}}\frac{\partial g_{00}}{\partial x^1} - \frac{g_{01}}{g_{00}\sqrt{g}}\frac{\partial g_{00}}{\partial x^0}\right)\right], \qquad \text{(A.12)}
\end{aligned}$$

which for a diagonal one, it will be simplified to

$$\mathcal{R} = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial x^0}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial x^0}\right) + \frac{\partial}{\partial x^1}\left(\frac{1}{\sqrt{g}}\frac{\partial g_{00}}{\partial x^1}\right)\right]. \qquad \text{(A.13)}$$

B. Contact manifolds

In this appendix, we aim to review the main properties of contact geometry.

Consider a $(2n+1)-$dimensional differential manifold $\mathcal{T}$ and its tangent manifold $T(\mathcal{T})$. It can be shown that for an arbitrary family of hyperplanes $\mathcal{D} \subset T(\mathcal{T})$ there exists a non-vanishing differential one-form θ on the cotangent manifold $T^*(\mathcal{T})$ such that

$$\mathcal{D} = \ker(\theta). \qquad \text{(B.1)}$$

The Frobenius integrability condition $\theta \wedge d\theta = 0$ requires that the hyperplanes $\mathcal{D}$ is completely integrable. By contrast, $\mathcal{D}$ is non-integrable if $\theta \wedge d\theta \neq 0$. In the limiting case which θ satisfies the condition

$$\theta \wedge (d\theta)^n \neq 0, \qquad \text{(B.2)}$$

the hyperplane $\mathcal{D}$ becomes maximally non-integrable and a contact structure can be defined on $\mathcal{T}$. Introducing a set of local coordinates $\{\phi, x_a, y^a\}$ with $a = 1, ..., n$, it is possible to express θ in its canonical form,

$$\theta = d\phi - x_a dy^a. \quad \text{(Darboux theorem)} \qquad \text{(B.3)}$$

The pair $(\mathcal{T}, \mathcal{D})$ determines a contact manifold [229] and it is often denoted as $(\mathcal{T}, \theta)$ to emphasize the role of the contact form θ. In addition, the set $(\mathcal{T}, \theta, G)$ defines a Riemannian contact manifold in which G is a nondegenerate Riemannian metric on $\mathcal{T}$.

C. Legendre transformations

The Legendre transformation is a convenient procedure in theoretical physics that acts an important role in classical mechanics [230, 231] as well as statistical mechanics and thermodynamics [232–234]. As two examples of the initial and important applications of these transformations in physics, the following can be mentioned:

- In classical mechanics, where it yields the relation between the Hamiltonian $\mathcal{H}(p)$ and the Lagrangian $\mathcal{L}(\dot{q})$ to switch from Hamiltonian to Lagrangian dynamics, and conversely.
- In statistical mechanics where it provides connection between the internal energy and the various sorts of the thermodynamic potentials like enthalpy, Gibbs and Helmholtz free energies.

In a nutshell, a Legendre transformation aims to convert a function of one group of variables (such as velocity, pressure, or temperature) to another function of a *conjugate* set of variables (momentum, volume, and entropy, respectively). This transformation can be performed for a function with any number of variables. For simplicity, we explain its concept for a two variable function.

Consider a function $f(x, y)$ with the differential

$$df = \frac{\partial f}{\partial x}\Big|_y dx + \frac{\partial f}{\partial y}\Big|_x dy \tag{C.1}$$

in which x and y are two independent variables. Making use of the definition $\xi = \frac{\partial f}{\partial x}\Big|_y$ and $\eta = \frac{\partial f}{\partial y}\Big|_x$, the above relation can be written as follows

$$df = \xi dx + \eta dy. \tag{C.2}$$

Based on the aforementioned definitions, x and ξ as well as y and η are conjugate pair of variables. Herein, the Legendre transformation with respect to the variable x is defined as

$$h = f - x\frac{\partial f}{\partial x} = f - \xi x, \tag{C.3}$$

and hence

$$dh = df - xd\xi - \xi dx = -xd\xi + \eta dy. \tag{C.4}$$

As can be seen h is a function of two independent variables ξ and y, i.e. $h = h(\xi, y)$. Moreover, we have

$$\frac{\partial h}{\partial \xi}\Big|_y = -x, \qquad \frac{\partial h}{\partial y}\Big|_\xi = \eta. \tag{C.5}$$

Therefore, the Legendre transformation exchange the role of one variable with its conjugate, along with a minus sign. Indeed, for a function with two variables, there are 4 possible kinds of the transformed function. If instead we have 3 independent variables, there are 8 different transformed functions. In general, there are 2^n transformed function for a function of n independent variables, as each variable can be represented by either member of a conjugate pair.

In geometric language, Legendre transformations are a special case of contact transformations which leave invariant the contact structure of $\mathcal{T}$. To explain this point in more detail, let us consider the class of maps with the feature of leaving the contact structure invariant. A transformation $f : \mathcal{T} \to \mathcal{T}$ is a diffeomorphism of the contact manifold if f maintains the contact structure, i.e

$$f^{*}(\theta) = \Omega\,\theta \;=\; \bar{\theta} \in [\,\theta\,] \qquad \text{where} \qquad \Omega \neq 0. \tag{C.6}$$

where $f^{\star}$ is the pullback of f. In this case, f is called a contact transformation or a *contactomorphism* [235]. A Legendre transformation belongs to the special class of contactomorphisms in which $f^{*}(\theta) = \theta$. Hence, it represents a symmetry of the contact structure and is defined by the following $2n+1$ equations between the sets of coordinates $\{\phi, x_a, y^a\}$ and $\{\bar{\phi}, \bar{x}_a, \bar{y}^a\}$,

$$\bar{\phi} = \phi - y^i x_i, \quad \bar{y}^i = -x_i, \quad \bar{y}^j = y^j, \quad \bar{x}_i = y^i, \quad \bar{x}_j = x_j\,, \tag{C.7}$$

where $i \cup j$ is any disjoint decomposition of the set of indices $\{1, ..., n\}$. A direct calculation shows that

$$\theta = d\phi - x_a\; dy^a = d\bar{\phi} - \bar{x}_a d\bar{y}^a\,. \tag{C.8}$$

It is notable that since this transformation only exchanges the ith pair of coordinates, it is called a *partial Legendre transformation*. However, the transformation with the property of exchanging each pair of coordinates is called the *total Legendre transformation*.

In the context of the thermodynamics, the importance of the Legendre transformation stems from the fact that it changes the dependence of the energy function from an extensive variable to its conjugate intensive variable, which can usually be controlled more easily in a physical experiment. In equilibrium thermodynamics, a system with n degrees of freedom is fully described by n extensive variables E_i along with their corresponding conjugate intensive variables I_i and a thermodynamic potential Φ which relates them to each other. Taking into account what was mentioned before, the Legendre transformations, which correspond to a redefinition of the thermodynamic potential, can be expressed as follows

$$\tilde{\Phi}_{(i)} \equiv \Phi_{(i)} - I_{(i)} E^{(i)} \quad \text{(No sum over } i\text{)} \tag{C.9}$$

$$\tilde{I}_{(i)} \equiv E^{(i)} \tag{C.10}$$

$$\tilde{E}^{(i)} \equiv -I_{(i)} \tag{C.11}$$

where $I_i = \frac{\partial \Phi}{\partial E^i}$. Besides, for $j \neq i$, we have $\tilde{I}_{(j)} = I_{(j)}$ and $\tilde{E}^{(j)} = E^{(j)}$.

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