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The Solutions of the Classical Relativistic Two-Body Equation

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Abstract

With the relativistic Kepler problem, a two-body system is studied for positronium, the hydrogen atom and $m_2 >> m_1$ cases.

1. Introduction

It is well known that the relativistic two-body problem has a long history and it contains some difficulties. There are two ways to understand this system: First, one is via field theory, which the second involves action-at-a-distance theory [1] (both of which are discussed by Havas [2]).

However, there have been several attempts to obtain the orbit solution for a classical relativistic two-body system interacting electromagnetically. If it is solved exactly, then the relativistic bound state problem in quantum mechanics will be well understood with it will be possible to start from corresponding problem in the classical mechanics and quantize it [3]. Nevertheless, it must be emphasized that these approaches are almost restricted to circular orbits.

In Schid's work [4], the concentric circular motion of two classical relativistic point charges interacting electromagnetically had been described. The similar discussion was also made by Anderson-Baeyer [5] and Cordero-Ghirardi [6].

It is known that the classical Kepler problem is a typical two-body problem [7] and it was first discussed relativistically by Cawley [8].

On the other hand it must be pointed out that the orbital discussion of relativistic two-body system is not only consisted of the circular orbits. Recently, a more general solution was obtained by Barut-Craig [9] in which the beginning action is same with our action [10].

In this work the relativistic Kepler problem, which is a special case of our action, will be studied, and the most general orbital solution will be discussed for the attractive and repulsive potential. Then the case of the positronium ($m_1 = m_2$) and the hydrogen atom ($m_2 \to \infty$) will be discussed for the relativistic two body system interacting ($e_1 e_2/r$) potential. Finally, the case of $m_2 >> m_1$ will be discussed.

2. The Relativistic Kepler Problem

Our one-time parameter action for interacting electromagnetic two-body system is given as follows [10].

$$A = -\int dT \left[m_1 \sqrt{1-\boldsymbol{v}_1^2} + m_2 \sqrt{1-\boldsymbol{v}_2^2} + \frac{e_1 e_2}{|\boldsymbol{r}|}(1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2) \right], \tag{1}$$

where m_1, m_2 are masses corresponding to e_1, e_2 charges, respectively; v_1 and v_2 are vector velocities; and T is centre of mass time. The first two terms are kinetic energy, the third term is the mutual interaction term. On the other hand there are non-dynamical but holonomic constraints on 4-velocity with $x^{\cdot}_{i\mu} x^{\cdot\mu}_{i} = 1$ ($i = 1, 2;\ c = 1$).

This action becomes for one particle:

$$A = -\int dT \left\{ m\sqrt{1-\boldsymbol{v}^2} + \frac{e_1 e_2}{|\boldsymbol{r}|} \right\}. \tag{2}$$

The Hamiltonian is obtained easily in the polar coordinates:

$$H = \sqrt{m^2 + \boldsymbol{p}^2} - \epsilon \frac{K^2}{|\boldsymbol{r}|} = \sqrt{m^2 + p_r^2 + \frac{p_\vartheta^2}{r^2}} - \epsilon \frac{K^2}{r}, \tag{3}$$

where $K^2 = |e_1 e_2|$, and $\epsilon = \pm 1$; from which it is obvious that the potential is the attractive for $\epsilon = 1$ and the repulsive for $\epsilon = -1$.

The equations of motion in polar coordinates are

$$r^{\cdot} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\sqrt{m^2 + p_r^2 + \frac{p_\vartheta^2}{r^2}}} \tag{4a}$$

$$\vartheta^{\cdot} = \frac{\partial H}{\partial p_\vartheta} = \frac{p_\vartheta}{r^2 \sqrt{m^2 + p_r^2 + \frac{p_\vartheta^2}{r^2}}} \tag{4b}$$

$$p_r^{\cdot} = -\frac{\partial H}{\partial r} = \frac{p_\vartheta \vartheta^{\cdot}}{r} - \frac{\epsilon K^2}{r^2} \tag{4c}$$

$$p_\vartheta^{\cdot} = -\frac{\partial H}{\partial \vartheta} = p_\vartheta^{\cdot} = 0, \quad p_\vartheta = l = \text{ Constant }. \tag{4d}$$

It can be seen easily that the energy is also conserved as the angular momentum p_ϑ;

$$E = \sqrt{m^2 + p_r^2 + \frac{p_\vartheta^2}{r^2}} - \epsilon \frac{K}{r} = \text{ Constant } . \tag{5}$$

Now we find the orbit equation. It is obvious that it is easily obtained from Eqns. (4a) and (4b):

$$\frac{dr}{d\vartheta} = \frac{r^2 p_r}{l} \tag{6}$$

$$-d\vartheta = \frac{du}{\sqrt{\left[\frac{K^4}{l^2} - 1\right] u^2 + \frac{2E\epsilon K^2}{l^2} u + \frac{E^2 - m^2}{l^2}}} \tag{7}$$

where $u = (1/r)$ and after integration it is found

$$u = \frac{1}{r} = \frac{\epsilon E K^2}{l^2 - K^2} \left[1 + \sqrt{1 + \frac{(E^2 - m^2)(l^2 - K^2)}{E^2 K^4}} \cos \sqrt{1 - \frac{K^4}{l^2}} (\theta - \theta_0) \right] . \tag{8}$$

We take $c = 1$. In order to obtain the correct result we have to replace $l \to cl$ and $m \to mc$ at the Eq. (8) or $m \to mc^2$ and $v \to (v/c)$ at the Eq. (2). Then it can be seen that it is $(1/r) = (1/metre)$ in MKS.

If we introduce

$$C = \frac{\epsilon E K^2}{l^2 - K^4}, \quad \beta = \sqrt{1 - \frac{K^4}{l^2}}, \quad e = \sqrt{1 + \frac{(E^2 - m^2)(l^2 - K^4)}{E^2 K^4}}, \tag{9}$$

where the β and e are dimensionless constant and C has dimension of m^{-1}. Then Eq. (8) becomes

$$\frac{1}{r} = C[1 + e \cos \beta(\vartheta - \vartheta_0)], \tag{10}$$

which can be also written as

$$\frac{1}{r} = \underline{C}[-1 + ech\underline{\beta}(\vartheta - \vartheta_0)], \tag{11}$$

where

$$\underline{C} = \frac{\epsilon E K^2}{K^4 - l^2} = -C, \quad \underline{\beta} = \sqrt{\frac{K^4}{l^2} - 1} = i\beta, \quad \cos i\beta = ch\beta. \tag{12}$$

It is obvious that Eqs. (10) and (11) are similar to the general conic equations. If it is chosen $\beta = 1$ then these equations are reduced to the classical Kepler's equation [7].

Now let's examine the orbits for the different values of e and for attractive potential $(\epsilon = 1)$.

I- $e > 1$:

a- When $(E^2 - m^2) > 0$, $\frac{K^4}{l^2} < 1$ the solution is Eq. (10) which is similar to the nonrelativistic motion. The orbit is a hyperbola when $\cos\beta(\vartheta - \vartheta_0) > 0$ and the minimum distance that the particle can approach is $r_{\min} = [1/C(1+e)]$. When $\cos\beta(\vartheta - \vartheta_0) < 0$ the solution is unphysical.

b- When $(E^2 - m^2) < 0$, $\frac{K^4}{l^2} > 1$ the solution is relativistic and given by Eq. (11). The orbit is a spiral which comes from $r_{\max} = [1/\underline{C}(-1+e)]$ and falls to the origine. The number of it's revolution depends on $''e''$.

II- $e < 1$:

a- When $(E^2 - m^2) < 0$, $\frac{K^4}{l^2} < 1$ the solution is Eq. (10) which is similar to the nonrelativistic solution. The orbit is a precessing ellipse by about $\Delta\vartheta \approx \frac{K^4}{l^2}\pi$ per revolution.

b- When $(E^2 - m^2) > 0$, $\frac{K^4}{l^2} > 1$ the solution is Eq. (11), which is the relativistic solution.

III- $e = 1,\ E^2 - m^2 = 0$:

a- When $\frac{K^2}{l^2} < 1$ the solution is Eq. (11), corresponding to the nonrelativistic motion:

$$\frac{1}{r} = C[1 + \cos\beta(\vartheta - \vartheta_0)]. \tag{13}$$

The orbit is a parabola and the min distance that the particle can come is $r_{\min} = (1/2C)$.

b- When $\frac{K^4}{l^2} > 1$ the relativistic solution is

$$\frac{1}{r} = \underline{C}[-1 + ch\underline{\beta}(\vartheta - \vartheta_0)]. \tag{14}$$

The orbit is a semiparabola, originating from ∞ and falling to the origin.

IV- $e = 0$:

a- When $\frac{K^4}{l^2} < 1$ the solution is $\frac{1}{r} = C$ and the orbit is a circle as with nonrelativistic motion.

b- When $\frac{K^4}{l^2} > 1$ there is no solution.

On the other hand, we can also discuss the antiparticle solutions. In order to obtain anti-particle solutions, we must replace $E \to -|E|$ in the particle solutions. As it will be easily seen in Table 1, there is a cross relationship between the particle and anti-particle solutions. For example, the particle solution when $e > 1$, $\epsilon = +1$ becomes the anti-particle solution for $e > 1$, $\epsilon = -1$. There are also other similar solutions. This should not come as a surprise as we expect it to conform with quantum mechanics.

If the potential is repulsive $(e = -1)$ we must replace $C \to -C$ and $\underline{C} \to -\underline{C}$ in the above solutions. Thus the all solutions for the relativistic Kepler problem can be ordered as in Table 1. It is obvious that these solutions are more general than Cawley's solutions [8].

3. Two-Body System

Now let's return to the action described by Eqn. (1) and neglect the interacting term between the velocities and the self interaction terms. Eqn. (1) then becomes

$$A = -\int dT \left[m_1\sqrt{1-\boldsymbol{v}_1^2} + m_2\sqrt{1-\boldsymbol{v}_2^2} + \frac{e_1 e_2}{|\boldsymbol{r}|} \right]. \tag{15}$$

It is known that the total momentum of the system is a constant in the COM [10]. If it is chosen identically zero then the Hamiltonian is obtained easily:

$$H = \sqrt{m_1^2 + \boldsymbol{p}^2} + \sqrt{m_2^2 + \boldsymbol{p}^2} - \epsilon\frac{K^2}{|\boldsymbol{r}|}, \tag{16}$$

where $\boldsymbol{p}$ is the relative momentum ($\boldsymbol{p}_1 = \boldsymbol{p}_2 = \boldsymbol{p}$ when $\boldsymbol{p} = \boldsymbol{0}$), r is the relative coordinate $K^2 = |e_1 e_2|$, and $\epsilon = \pm 1$. Binomial expansion of this Hamiltonian contains the Breit Hamiltonian in quantum mechanics [11].

It is obvious that the energy is a constant of motion. In polar coordinates,

$$E = \sqrt{m_1^2 + p_r^2 + \frac{p_\vartheta^2}{r^2}} + \sqrt{m_2^2 + p_r^2 + \frac{p_\vartheta^2}{r^2}} - \epsilon\frac{K^2}{r} = \text{ Constant }. \tag{17}$$

The equations of motion are also obtained easily

$$r^{\cdot} = \frac{\partial H}{\partial p_r} = p_r \left[\frac{1}{\sqrt{m_1^2 + p_r^2 + \frac{p_\vartheta^2}{r^2}}} + \frac{1}{\sqrt{m_2^2 + p_r^2 + \frac{p_\vartheta^2}{r^2}}} \right] \tag{18a}$$

$$\vartheta^{\cdot} = \frac{\partial H}{\partial p_\vartheta} = \frac{p_\vartheta}{r^2} \left[\frac{1}{\sqrt{m_1^2 + p_r^2 + \frac{p_\vartheta^2}{r^2}}} + \frac{1}{\sqrt{m_2^2 + p_r^2 + \frac{p_\vartheta^2}{r^2}}} \right] \tag{18b}$$

$$p_r^{\cdot} = -\frac{\partial H}{\partial r} = \frac{p_\vartheta \vartheta^{\cdot}}{r} - \frac{\epsilon K^2}{r^2} \tag{18c}$$

$$p_\vartheta^{\cdot} = -\frac{\partial H}{\partial \vartheta} = 0, \quad p_\vartheta = l = \text{ Constant }. \tag{18d}$$

As is expected, the total angular momentum is a constant of motion.

The orbits equation can be found after little algebra:

$$\frac{d\vartheta}{l} = \frac{2\left(E + \epsilon\frac{K^2}{r}\right) dr}{r^2\sqrt{\left(E+\epsilon\frac{K^2}{r}\right)^4 - \left(E+\epsilon\frac{K^2}{r}\right)^2\left[2(m_1^2+m_2^2) + \frac{4l^2}{r^2}\right] + (m_1^2 - m_2^2)}}. \tag{19}$$

I- When $m_1 = m_2 = m$ (The positronium case)

It will be seen easily that the last term at the denominator is zero. Thus the Eq. (19) becomes

$$-d\vartheta = \frac{du}{\sqrt{\left[\frac{K^4}{4l^2} - 1\right] u^2 + \frac{2E\epsilon K^2}{4l^2} u + \frac{E^2 - 4m^2}{4l^2}}}, \tag{20}$$

where $u = 1/r$. This equation is same with the Eq. (7) when $l = 2l$ and $m = 2m$. Therefore the solutions given in the Table 1 can be adapted to this case easily.

II- When $m_2 \to \infty$ (The case of hydrogen atom):

To calculate the limit of the Eq. (19) when $m_2 \to \infty$ we must replace $E \to E' + m_2$. Atfer some algebra it becomes

$$-d\vartheta = \frac{du}{\sqrt{\left[\frac{K^4}{l^2} - 1\right] u^2 + \frac{2E'\epsilon K^2}{l^2} u + \frac{E'^2 - m_1^2}{l^2}}}. \tag{21}$$

It is also obvious that this equation is same as the Eq. (7) when $E = E'$.

III- When $m_2 >> m_1$:

After we replaced $E \to E' + m_2$ at the Eq. (19), Eq. (22) is obtained with little algebra:

$$\frac{d\vartheta}{l} = \frac{2dr}{r^2\sqrt{(m_2 + V)^2 - 2(m_1^2 + m_2^2) - \frac{4l^2}{r^2} + \frac{(m_1^2 - m_2^2)^2}{(m_2 + V)^2}}} \tag{22}$$

where $V = E' + e\frac{K^2}{r}$. If we ommit the second and higher order terms with respect to $(1/m^2)$, and after expanding the denominator of the last term in the square root in binomial series, we obtain:

$$-d\vartheta = \frac{du}{\sqrt{\left(\frac{K^4}{l^2} - 1\right) u^2 + \left(\frac{2E'\epsilon K^2}{l^2} + \frac{m_1^2 \epsilon K^2}{m_2 l^2}\right) u + \frac{E'^2 - m_1^2}{l^2} + \frac{m_1^2 E'}{m_2 l^2}}}, \tag{23}$$

where $u = 1/r$. As is seen, this integral contains the correction terms according to one particle system, which are

$$\frac{m_1^2 \epsilon K^2}{m_2 l^2} u + \frac{m_1^2 E'}{m_2 l^2}.$$

Nevertheless, it is obvious that this integral is similar to Eq. (7) if the constants in the square root are redefined. Consequently orbits are also similar to the relativistic Kepler orbits.

Table 1. The relativistic Kepler solution of particle and antiparticle for attractive and repulsive potential where (*): Nonrelativistic solutions, (#): Unphysical solutions, (x): Relativistic solutions

			Particle		Antiparticle	
			$\frac{K^4}{l^2} < 1$	$\frac{K^4}{l^2} > 1$	$\frac{K^4}{l^2} < 1$	$\frac{K^4}{l^2} > 1$
$e > 1$ H. bola	$\varepsilon = 1$ (artr.)	$e^2 - m^2 > 0$	(*) $(\cos\beta\varphi' > 0)$ $\frac{1}{r} = c[1 + e\cos\beta\varphi']$	-	(x) $(\cos\beta\vartheta' < 0)$ $\frac{1}{r} = -\lvert C\rvert[1 + \lvert e\rvert\cos\beta\vartheta']$	-
		$e^2 - m^2 < 0$	-	(x) $\frac{1}{r} = \underline{C}[-1 + ech\underline{\beta}\vartheta']$	-	(#) $\frac{1}{r} = -\underline{C}[-1 + ech\underline{\beta}\vartheta']$
	$\epsilon = -1$ (rep.)	$e^2 - m^2 > 0$	(*) $(\cos\beta\vartheta' < 0)$ $\frac{1}{r} = -C[1 + e\cos\beta\vartheta']$	-	(x) $(\cos\beta\vartheta' > 0)$ $\frac{1}{r} = \lvert C\rvert[1 + \lvert e\rvert\cos\beta\vartheta']$	-
		$e^2 - m^2 < 0$	-	(#) $\frac{1}{r} = -\underline{C}[-1 + ech\underline{\beta}\vartheta']$	-	(x) $\frac{1}{r} = \lvert\underline{C}\rvert[-1 + \lvert e\rvert ch\underline{\beta}\vartheta']$
$e < 1$ Elipse	$\epsilon = 1$ (attr.)	$e^2 - m^2 > 0$	-	(x) $\frac{1}{r} = \underline{C}[-1 + ech\underline{\beta}\vartheta']$	-	(#) $\frac{1}{r} = \lvert -\underline{C}\rvert[-1 + \lvert e\rvert ch\underline{\beta}\vartheta']$
		$e^2 - m^2 < 0$	(*) $\frac{1}{r} = C[1 + e\cos\beta\vartheta']$	-	(#) $\frac{1}{r} = -\lvert C\rvert[1 + \lvert e\rvert\cos\beta\vartheta']$	-
	$\varepsilon = -1$ (rep.)	$e^2 - m^2 > 0$	-	(#) $\frac{1}{r} = -\underline{C}[-1 + ech\underline{\beta}\vartheta']$	-	(X) $\frac{1}{r} = -\underline{C}[-1 + \lvert e\rvert ch\underline{\beta}\vartheta']$
		$e^2 - m^2 < 0$	(#) $\frac{1}{r} = -C[1 + e\cos\beta\vartheta']$	-	(X) $\frac{1}{r} = \lvert C\rvert[1 + \lvert e\rvert\cos\beta\vartheta']$	-
$e = 0$ Parabola	$E^2 - m^2 = 0$	$\epsilon = 1$ (attr.)	(*) $\frac{1}{r} = C[1 + \cos\beta\vartheta']$	(x) $\frac{1}{r} = \underline{C}[-1 + ech\underline{\beta}\vartheta']$	(#) $\frac{1}{r} = -\lvert C\rvert[1 + \cos\beta\vartheta']$	(#) $\frac{1}{r} = -\underline{C}[-1 + ch\underline{\beta}\vartheta']$
		$\epsilon = -1$ (rep.)	(#) $\frac{1}{r} = -\lvert C\rvert[1 + \cos\beta\vartheta']$	(#) $\frac{1}{r} = -\underline{C}[-1 + ch\beta\vartheta']$	(x) $\frac{1}{r} = \lvert C\rvert[1 + \cos\beta\vartheta']$	(x) $\frac{1}{r} = \underline{C}[-1 + ch\underline{\beta}\vartheta']$
$e = 0$ circle	$E^2 = m^2(1 - \frac{K^4}{l^2})$	$\epsilon = 1$ (attr.)	(*) $\frac{1}{r} = C$	-	(#) $\frac{1}{r} = -\lvert C\rvert$	-
		$\epsilon = -1$ (rep.)	(#) $\frac{1}{r} = -C$	-	(x) $\frac{1}{r} = \lvert C\rvert$	-

References

[1] A.O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Second Edition Dover, New York 1980) Ch.6.

[2] P. Havas, Phys. Rev., **74** (1948) 456.

[3] A. Degasperis, Phys. Rev. D., **3** (1971) 273.

[4] A. Schild, Phys. Rev., **131** (1963) 2767.

[5] C.M. Andersen and Hans C. von Baeyer, Ann. Phys., **60** (1970) 67.
C.M. Andersen and Hans C. von Baeyer, Phys. Rev., D. **5** (1972) 802.

[6] P. Cordero and G.-C. Ghirardi, J. Math. Phys., **7** (1973) 815.

[7] H. Goldstein, *Classical Mechanics* (Second Edition, Addison-Wesley Publishing Company, 1980) Ch. 3.

[8] Robert G. Cawley, J. Math. Phys., **8** (1967) 2092.

[9] A.O. Barut and G. Craig, Physica A, **197** (1993) 275.

[10] C. Önem, "The Class Rela. Two-Body Prob. with Self Interaction", N. Cimento B, **110** (1995) 110.

[11] H.A. Bethe and E.E. Salpeter, *Quantum Mechanics of One and Two Electron Atoms* (Springer-Verlag, 1957) p.192.