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q-calculus and irreversible dynamics on a hierarchical lattice

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Abstract

On a lattice with equal spacing on the logarithmic scale, a momentum operator that respects the asymmetry of this nonlinear lattice yields a kinetics that can be understood in terms of diffusion on an underlying ultrametric space, if one also identifies the cannonical commutator with the time dilation operator. The motion to which this non-conventional kinetics corresponds is irreversible, with an explicit violation of time reversal symmetry resulting from the spreading with time of a probability distribution over a larger and larger volume of the phase space.

1. Introduction

Recently there has been a great surge of activity associated with q-basic special functions [1-5], the q-deformed commutations relations leading to a generalization of quantum mechanics and non-commutative geometries associated with quantum groups[3]. In previous papers [6,7] we have shown how the generators of fractal and multifractal sets with discrete dilatation symmetries are associated with q-difference operators [1] and how the exact free energy of spin systems on hierarchical (topologically non-uniform) lattices can be written in terms of q-integrals [2]. In this paper we would like to discuss how a kinetics based upon q-differential equations describes diffusion on a (topologically uniform) hierarchical lattice.

The paper is organized as follows. In section 2, we review briefly the definitions of the q-derivatives and integrals, and how they arise in the description of systems with discrete dilatation symmetries. In section 3, we define the "momentum," and time evolution operators and we construct the state space of the quasi–position operator. In section 4 we find the solutions of the "Schrödinger equation" and compute the expectation value of the canonical commutator. A discussion of the results is given in section 5.

2. Scale invariance and q-calculus

We would first like to recall a number of definitions. We use a subscript in order to indicate the variable with respect to which a q-derivative is to be taken and we use a superscript to indicate the dilatation factor q, in parenthesis. Thus we define the q-derivative as

$$\partial_x^{(q)} f(x;y;\ldots) \equiv \frac{f(qx;y;\ldots) - f(x;y;\ldots)}{(q-1)x} \ . \tag{1}$$

In contrast to the usual derivative, which measures the rate of change of the function in terms of an incremental *translation* of its argument, the q-derivative measures its rate of change with respect to a *dilatation* of its argument by a factor of q. A homogeneous function f of degree ϕ satisfies

$$\partial_x^{(q)} f(x) = \frac{q^\phi - 1}{q-1}\frac{f(x)}{x} \quad \equiv [\phi]_q \frac{f(x)}{x} \tag{2}$$

where the q-basic number ϕ has been defined via the second equality. It is useful to note that the solution F_q of the q-differential equation

$$\partial_x^{(q)} F_q(x) = 0 \tag{3}$$

is either a constant or a function that is periodic in $\ln x$, with period $\ln q$, such that $F_q(qx) = F_q(x)$. With the modified product rule

$$\partial_x^{(q)} g(x) f(x) = g(qx)\partial_x^{(q)} f(x) + f(x)\partial_x^{(q)} g(x) \ \ , \tag{4}$$

one has, besides the arbitrary *additive* constant of integration that appears in the case of ordinary integration, an arbitrary multiplicative *function* of integration satisfying (3), so that the general solution to (2) (taking the constant of integration to be zero) is

$$f(x) = F_q(x) x^\phi \ \ . \tag{5}$$

From (4), it also follows that we may define the dilatation operator

$$A_x^{(q)} \equiv [\partial_x^{(q)}, x] \ \ ; \quad \overline{A_x^{(q)}} = A_x^{(q^{-1})} \tag{6}$$

where the brackets denote the ordinary commutator.

The q-integral is defined via

$$\int_0^x f(t)\, D_t^{(q)} \equiv (q-1)x \sum_{n=0}^{\infty} q^n f(q^n x), \tag{7}$$

for $q > 1$, where $D_t^{(q)}$ denotes the q-differential. Weierstrass functions[8] have the obvious generalization to smooth periodic functions $g(x)$ other than the cosine, viz.,

$$f_W(x) = \sum_{n=0}^{\infty} \frac{g(q^n x)}{a^n} \tag{8}$$

where $a = q^{\phi}$. They obey

$$\partial_x^{(q)} f_W(x) = [\phi]_q \frac{f_W(x)}{x} + \frac{q^{\phi} g(x)}{(1-q)x} \tag{9}$$

and may be conveniently expressed in terms of q-integrals of the form,

$$f_W(x) = \frac{x^{\phi}}{q-1} \int_0^x D_t^{(q)} \frac{g(t)}{t^{1+\phi}}. \tag{10}$$

Generalized Weierstrass-Mandelbrot functions [8], where the lower limit of the sum in Eq.(8) is taken to be $(-\infty)$, and obey Eq.(2), may be written in terms of the"definite integral"

$$\begin{aligned} \int_{0x}^{\infty x} v(t) D_t^{(q)} &\equiv \int_0^x v(t) D_t^{(q)} + \int_x^{\infty} v(t) D_t^{(q)} \\ &= (q-1) \sum_{k=-\infty}^{\infty} q^k v(q^k\, x) \ . \end{aligned}$$

For $v(t) = g(t) t^{-(1+\phi)}$, with the requirement that the first n_0 derivatives of $g(t)$ vanish at the origin, $n_0 \geq$ the integer part of ϕ one has the generalized Weierstrass-Mandelbrot function,

$$f_{WM}(x) = \frac{x^{\phi}}{q-1} \left[\int_{0x}^{\infty x} D_t^{(q)} \frac{g(t)}{t^{1+\phi}} \right]. \tag{11}$$

Equations (10) and (11) make the scaling behaviour of these forms transparent, and provide closed form expressions for their oscillatory amplitudes (5). Note that the generalized Weierstrass–Mandelbrot functions have fractal graphs whose fractal (graph) dimension [8] is given by $d_f = 2 - \phi$. It should be evident that the functions $f_W(x)$ and $f_{WM}(x)$ are nowhere differentiable functions of their arguments, which are yet well behaved under the q-difference operator, as can be seen from (2) and (9). In the limit of $q \to 1$, $[\phi]_q \to \phi$, and although the functions themselves are blowing up at this limit, the logarithmic derivative of $f_{WM}(x)$ is well behaved and finite,

$$\partial_x^{(q)} f_{WM}(x) / f_{WM}(x) = \phi / x \ . \tag{12}$$

Such generalized Weierstrass (or Weierstrass-Mandelbrot) functions arise endemically in systems with discrete dilatation symmetries [9]. We have already discussed the case of the free energy of a spin system on a hierarchical lattice [6]; another example is the autocorrelation function of a random walker[10] on an ultrametric space[11], which we will discuss in the remainder of this paper.

3. Kinetics on a hierarchical lattice

Dimakis and Müller-Hoissen have demonstrated that [12] that q-calculus [1,2] can be obtained from discrete calculus on a lattice, where $x = na$, by an exponential coordinate transformation,

$$y = q^{\frac{x}{q-1}} = q^n \quad . \tag{13}$$

Under this transformation discrete translations go over to discrete dilatations. The q-deformed commutation relations obeyed by the transformed variables and their q-derivatives lead to q-deformed quantum mechanics [12,13]. In this way, q-deformed quantum mechanics has been given an interpretation in terms of quantum mechanics on a lattice. The transformed momentum and Hamiltonian operators remain hermitian. However, they satisfy Heisenberg's equation of motion for the momentum operator, with the ordinary definition of the commutator and not with the deformed definition, namely, $[H, P_y] = 0$. On the other hand, $i[H, Y] \neq P_y$, and it can be easily verified that the cannonical commutation relation is violated both in the ordinary and the q-deformed definition. To be able to give an intepretation of the physics, one has to transform back to the linear lattice.

We would now like to propose [14] a different choice for the momentum operator. Notice that there is a kind of democracy between the right and left difference operators on a lattice, which makes it natural for the (self-adjoint) momentum operator on the discrete lattice to be defined [12] as their average, but this democracy does not hold between $\partial_y^{(q)}$ and $\partial_y^{(q^{-1})}$ which describe processes at *different scales*. On the linear chain, exactly one unit is added to an interval everytime a step is made to the right wherever one may be on the chain. However, when ℓ is increased by unity in y space, the size of the interval which is certain to include the origin increases by $(q-1)q^\ell$.

We will therefore deliberately take into account the q v.s. q^{-1} asymmetry and allow the momentum not to be an observable. This gives us the freedom to associate the momentum operator directly with the q-derivative (1),

$$P_q \equiv -i \, \partial_y^{(q)} \quad . \tag{14}$$

This is clearly a crucial step, which makes the motion of the phase point along the nonlinear chain ballistic, and in the increasing y direction. Below, we will discuss how this motion can be understood in terms of an associated process, that of a diffusing particle on an underlying hierarchical lattice, and therefore leads to nontrivial results.

If we compute the ordinary commutator of Y and P_q we find that the canonical commutation relation becomes,

$$[P_q, Y] = -i \, A_y^{(q)} \quad . \tag{15}$$

We still have to make a choice of an independent operator for the Hamiltonian. In this, we will be guided by the fact that in going from one dimensional continuous space to the periodic lattice, the canonical commutator goes over from the c-number $-i$ to the operator $-i(1-a^2H)$, where a is the lattice spacing and H is the Hamiltonian operator, which has a rather obvious form proportional to the product of the right and left difference operators [12]. For the choice of the time increment $\Delta t = -ia^2$, one sees that $(1-a^2H)$ is the expansion up to first order in Δt of the time translation operator, so that $[P, X] = -iT$.

Let us therefore make the ad hoc choice, in our case, that $[P_q, Y] = -iT$, where T is now the time *dilation* operator. This is equivalent to the statement,

$$T = A_y^{(q)} \quad . \tag{16}$$

Clearly, the dilatation factor for time, q_t, need not be equal to q; in fact we may define a "dynamical exponent" ζ via the relation,

$$q_t = q^\zeta \quad . \tag{17}$$

Then,

$$T f(y,t) \equiv f(y, q_t\, t) \tag{18}$$

To be consistent in our use of the difference operators, we would like to write the deformed "Schrödinger equation" also in terms of the Jackson derivative, in this case with respect to time. We have $\partial_t^{(q_t)} \equiv (T-1)/[(q_t-1)t]$. The Schrödinger equation then becomes

$$i\, \partial_t^{(q_t)} f(y,t) = H_q f(y,t) \quad , \tag{19}$$

which defines our Hamiltonian operator. From (16) and (19), we find

$$H_q = i\, \frac{T-1}{(q_t-1)t} \quad , \tag{20}$$

or,

$$H_q = \frac{i(q-1) Y \partial_y^{(q)}}{(q_t-1)t} \quad . \tag{21}$$

The constant prefactor in Eq.(21) may be written as the inverse of a "basic number" [4]

$$[\zeta]_q \equiv \frac{q^\zeta - 1}{q-1} \quad , \tag{22}$$

where ζ is the dynamical exponent defined in (17). (This dynamical exponent ζ, which tells us how time scales with the distance, takes the value of 2 on Euclidean space; we expect it to be equal to the random walk dimension [10] on the hierarchical lattice.) With these definitions, the Hamiltonian operator becomes

$$H_q = -\,\frac{1}{[\zeta]_q} \frac{Y\, P_q}{t} \quad , \tag{23}$$

which has the right "dimensions" for being an energy. Note that this "Hamiltonian" is non-hermitian, as is our "momentum," so that the energy is not an observable, neither is it a constant of the motion; H_q depends explicitly on time. Since $[H, P_q] \neq 0$, the momentum is not conserved either.

With the hermitian operator Y we will associate a position-like observable which we will call the "quasi-position," [15] the "quantum numbers" ℓ corresponding to the highest level so far attained by the phase point on the y-lattice. The expectation value of the quasi-position operator is to be computed using the definition of the scalar product [12]

$$\begin{aligned}\langle a, b\rangle_{y_0} &\equiv \int_{0y_0}^{\infty y_0} \frac{a(y)\, b(y)}{y} D_y^{(q)} \\ &\equiv (q-1) \sum_{k=-\infty}^{\infty} a(q^k y_0)\, b(q^k y_0)\end{aligned}$$

Here y_0 serves as the origin of this hierarchical lattice, and could be chosen equal to unity.

In order for this expectation value to converge at both ends of the infinite sum, the unnormalized state functions corresponding to the pure states $|\ell\rangle$ of the quasi-position operator must have the following exponential form, again up to multiplication by functions doubly periodic in $\ln y$ and $\ln t$ with periods $\ln q$ and $\ln q_t$,

$$\epsilon_\ell(y,t) = \exp\Big\{-\frac{1}{2}\Big[\frac{(y/y_0)^\zeta}{\tau_\ell} t\Big]^\lambda\Big\} \tag{24}$$

where $\tau_\ell = R^\ell$, and $\lambda > 0$ is arbitrary. For simplicity, we shall choose $\lambda = 1$, but this does not at all affect the subsequent discussion. By (16), $T\epsilon_\ell(y,t) = A_y^{(q)}\epsilon_\ell(y,t)$. Thus, one must have

$$\epsilon_\ell(y, q_t t) = \epsilon_\ell(qy, t) \ . \tag{25}$$

From (24), one finds that if $R \equiv q_t$, then we also have,

$$A_y^{(q)}\epsilon_\ell(y,t) = \epsilon_{\ell-1}(y,t) \tag{26a}$$

and

$$\overline{A}_y^{(q)}\epsilon_\ell(y,t) = \epsilon_{\ell+1}(y,t) \ . \tag{26b}$$

Defining $\langle\epsilon_\ell(y,t), y\,\epsilon_\ell(y,t)\rangle_{y_0} \equiv Q_\ell(t)$, we have,

$$Q_\ell(t) = (q-1)y_0 \sum_{k=-\infty}^{\infty} q^k e^{-\,q_t^{k-\ell} t} \ . \tag{27}$$

Notice that $Q_{\ell+1}(t) = Q_\ell(q_t^{-1}t)$ or $Q_\ell(t) = T\, Q_{\ell+1}(t)$. By (27) we also have,

$$Q_{\ell+1}(t) = (q-1)y_0 \sum_{k=-\infty}^{\infty} q^k e^{-\,q_t^{k-1-\ell} t} \ , \tag{28}$$

which, upon redefining the dummy index to be $k' = k-1$ gives,

$$Q_{\ell+1}(t) = q\, Q_\ell(t) \ .$$

Thus, clearly, $Q_\ell(t) = q^\ell Q_0(t)$ and the $\epsilon_\ell(y,t)$ span a representation of the algebra generated by the $\partial_y^{(q)}$, $\overline{\partial}_y^{(q)}$ and Y.

Moreover, we may readily extract the scaling behaviour of $Q_\ell(t)$ from the above; one easily finds, say from (27) that

$$Q_\ell(t) = U_{\ell,q_t}(t)t^{-1/\zeta} \tag{29}$$

where $U_{\ell,q_t}(t)$ is a function logarithmically periodic in t with period $\ln q_t$. It should be noted that the scaling behaviour of $Q_\ell(t)$ does not depend on ℓ, and is thus completely self similar with respect to the choice of the level of coarse graining.

4. Solutions of the Schrödinger equation and the spreading of the probability distribution

The solutions of the "Schrödinger equation" (19) can be found by making a seperation of variables. Taking $f(y,t) = g(y)h(t)$ and using (23), one has

$$\frac{t\partial_t^{(q_t)}h(t)}{h(t)} = \frac{1}{[\zeta]_q}\frac{Y\partial_y^{(q)}g(y)}{g(y)} \quad .$$

Setting both sides of the equation equal to a constant, C, gives,

$$\frac{t\partial_t^{(q_t)}h(t)}{h(t)} = C \quad , \tag{30}$$

and

$$\frac{Y\partial_y^{(q)}g(y)}{g(y)} = [\zeta]_q C \quad . \tag{31}$$

The solutions to these equations are given in terms of homogeneous functions, namely power laws, up to multiplication by oscillatory functions,

$$h(t) = F_{q_t}(t)t^{\psi} \quad , \tag{32}$$

$$g(y) = F_q(y)y^{\chi} \quad . \tag{33}$$

From (17) and (22), we find $[\zeta]_q[\psi]_{q_t} = [\zeta\,\psi]_q$. On the other hand, from (31) and (33), we have $[\chi]_q = [\zeta]_q[\psi]_{q_t}$, whence, $\chi = \zeta\psi$. For finiteness as $t \to \infty$, we must have $\chi,\ \psi < 0$. Finally, the solutions of Eq.(19) can be written,

$$f_\psi(y,t) = F_q(y)F_{q_t}(t)(y^\zeta t)^\psi \quad . \tag{34}$$

Now we would like to show that the kinetics imply that a probability distribution initially localized within an interval q^ℓ of the origin will spread in time in such a way that the uncertainty in the position becomes precisely as large as the whole phase space available at that energy. This means that the probability distribution is essentially uniform over the available phase space at any given time.

The absolute value of the uncertainty in the simultaneous determination of the "momentum" and "position" operators can be found as usual from the canonical commutation relation. In our case, from (14,15) we have

$$|\langle \Delta Y \Delta P_q\rangle| \geq |\langle [Y, P_q]\rangle| = |\langle iA_y^{(q)}\rangle| = |\langle -(q-1)YP_q + i\rangle| \quad . \tag{35}$$

This tells us that the product of the uncertainty in the value of the position and the momentum operators is larger than the expectation value of their product in absolute value. Heuristically, one may say that, if $Y \sim vt^{1/\zeta}$ where v is some effective diffusivity, then the uncertainty $| < \Delta Y \Delta P_q > | > |vt^{1/\zeta} p_q|$, where p_q is the average momentum for this Y eigenstate. Thus, the uncertainty in the position is as large, and increases with time in the same way, as the interval over which the particle or the phase point has travelled within the time t, i.e., it is equally likely to be found anywhere within the phase space volume it is energetically allowed to explore.

More precisely, the expectation value of $[Y, P_q]$, taken with respect to the solutions of the Schrödinger equation, normalized by their scalar product, yields,

$$\langle [Y, P_q] \rangle = i\Big[(q-1)\frac{q^{\zeta\psi}-1}{q-1} + 1\Big] = iq^{\chi} \quad . \tag{36}$$

With $q^{\zeta} = q_t$, this yields,

$$| < \Delta Y \Delta P_q > | \geq q^{\chi} = q_t^{\psi} \quad . \tag{37}$$

On the other hand, taking the expectation value of the canonical commutator between the states $|\ell\rangle$ and using (15,16) and (26) gives us $\langle \epsilon_\ell, \epsilon_{\ell-1} \rangle$, which may be interpreted as a transition probability between the states $|\ell-1\rangle$ and $|\ell\rangle$. This is again consistent with the fact that uncertainty is a function of the leakage of the phase point to larger and larger regions of the phase space, as time goes by.

5. Discussion and connection with q-statistics

In statistical physics, hierarchical lattices have arisen recently in the anomalous relaxation of spin glasses [1], transport in random media [17] and fully developed turbulent media [18] as realizations of ultrametric spaces [11]. They consist of a hierarchy of nested intervals, and one may associate a geometrical progression of spatial (and/or temporal) scales with the different levels of the hierarchy. Diffusion on ultrametric spaces have been thoroughly studied (see[16-20] and references therein) by other methods, including the renormalization group [18,20].

Consider a lattice on which to each successive quasi-position indexed by the quantum number ℓ there corresponds a geometrical increase in the number of microstates. To proceed from the ℓ'th level of the hierarchy to the next, assume the particle has to surmount an energy barrier of hight R^ℓ, with the probabilities $\exp(-t/R^\ell)$. (Note that the manner in which these probabilities decrease could have been chosen differently, if we had restricted our scalar product to a finite integral, and therefore the sum (27), only to positive integers k). If one compares the expectation value $Q_\ell(t)$ we have found for the quasiposition operator with the exact solution of Schreckenberg [19] for diffusion on such an ultrametric, heirarchical lattice, one immediately sees that $Q_\ell(t)$ is the probability for a random walker to be found at time t within an ultrametric distance ℓ from the origin (The comparison is facilitated by realizing that since the sum in (27) ranges from $-\infty$ to ∞, one may change the signs of all the k indices appearing inside the sum). $Q_0(t)$ corresponds to the autocorrelation function of a particle starting out at the 0'th level. For

$\ell \neq 0$, the Q_ℓ may be thought of as ℓ times coarse grained autocorrelation functions. The inverse of the ℓ-coarse grained autocorrelation function is the ℓ-coarse grained volume explored by the particle on the average, within a given time t, and grows like $\sim t^{1/\zeta}$. Clearly this volume is the average number of distinct ℓ-clusters visited after time t.

It might be noted that our Schrödinger equation (19) involves, on the RHS, only the first derivative with respect to position (see (21)), in accordance with the fact that diffusion on the hierarchical lattice corresponds to simply a drift with respect to the quasi-position. This makes the Schrödinger equation resemble the Fokker-Planck equation rather than the diffusion equation.

It is useful to recall [21] that the quantum mechanical expectation of the transition of a free particle between two different space points on ordinary space can be associated with a weighted sum over all possible paths of a classically diffusing particle between these two points. The path integral over *quasi-positions*, however, is trivial - once the phase point has progressed to some level ℓ, the paths of the diffusing particle which go back and explore sites within the phase space volume already broached (at levels $\leq \ell$) simply do not contribute. Therefore the time dilation leads deterministically to an increase in ℓ, and to irreversibility [22]. On the other hand, the probability distribution for finding the particle at some level $\leq \ell$ at a given time t is *not* trivial, as has been shown above.

Finally, we would like to make a connection with recent work on random sets and q-distributions. It has been remarked by Arık et al. [23] that the basic number $[n]_q$ with $q = (1-1/M) < 1$ is the average number of distinct elements in a set which is contructed in n steps by making random draws from a source set with infinitely many elements of which there are M distinct kinds. In our case, $q > 1$, which is complementary to that considered by Arık et al., and we have a source set which is hierarchically constructed so that it consists of nested subsets, or families, with a nonuniform probability of draws. The draws have a clustering property, so that the probability for first making a draw from another cluster that has the ultrametric distance ℓ to the one we start out with, decreases with ℓ. Then, the distinct number of ℓ-level clusters that have been accessed grows with time as power law in t. Such systems arise naturally in evolutionary models [24]. Further work relating to hierarchies with a finite number of levels is presently in progress.

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