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A New Formalism for Nonextensive Physical Systems: Tsallis Thermostatistics

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Abstract

Although Boltzmann-Gibbs (BG) statistics provides a suitable tool which enables us to handle a large number of physical systems satisfactorily, it has some basic restrictions : (i) the range of the microscopic interactions must be small compared to the linear size of the macroscopic systems (short-range interactions) , (ii) the time range of the microscopic memory must be small compared to the observation time (Marcovian processes) and (iii) the system must evolve in an Euclidean-like space-time. In the case of a breakdown in one and/or the others of these restrictions, BG statistics fails. More precisely, the situation could be classified in a general manner as follows: (i) For an Euclidean-like space-time, if the forces and/or the memory are long-ranged, as far as we are interested in an equilibrium state, the BG statistics is weakly violated, therefore BG formalism can be used. On the other hand, whenever a meta-equilibrium state is considered, the BG statistics is strongly violated, hence another formalism must be needed. (ii) For a (multi)fractal space, BG formalism is strongly violated again and a new formalism is needed. The way out from these problems seems to be Nonextensive Statistical Thermodynamics which must be a generalization of the BG statistics in a manner that allows a correct description of the nonextensive physical systems as well. Recently a nonextensive thermostatistics has been proposed by C.Tsallis to handle the nonextensive physical systems and up to now, besides the generalization of some of the conventional concepts, the formalism has been prosperous in some of the physical applications. In this study, our effort is to introduce Tsallis thermostatistics in some details and to emphasize its achievements on physical systems by noting the recent developments on this line.

1. Boltzmann-Gibbs Statistics and Its Restrictions

Boltzmann-Gibbs (BG) statistics provides a suitable and powerful tool which enables us to handle a large number of standard physical systems, more precisely, systems where

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thermodynamical extensivity (additivity) holds. By stating this, indeed, we come to the fact that BG statistics has some basic restrictions, which can be itemized as follows:

- the range of the microscopic interactions must be small compared to the linear size of the macroscopic systems, which, in other words, means that the interactions must be short-ranged
- the range of the microscopic memory must be small compared to the observation time, namely the processes must be Marcovian
- the system must evolve in an Euclidean-like space-time.

Although these facts are straightforward, they are almost never stated in standard textbooks of Statistical Physics, except a few books of some authors. We quote here some statements from them in order to make the situation more concrete :
i) P.T. Landsberg states [1] "...The presence of long-range forces causes important amendments to thermodynamics, some of which are not fully investigated as yet..."
ii) L.G. Taff states [2] "...This means that the total energy of any finite collection of self-gravitating mass points does not have a finite, extensive lower bound. Without such a property there can be no rigorous basis for the statistical mechanics of such a system..."
iii) Once again P.T. Landsberg states [3] "...In the case of systems with long-range forces and which are therefore nonextensive, some thermodynamical results do not hold..."
iv) S.H. Strogatz states [4] "...Most of our everyday life is nonextensive, and the principle of superposition fails spectacularly. If you listen to your two favorite songs at the same time, you won't get double the pleasure..."

Having said all these related quotations, we can conclude that BG statistics fails whenever the system includes long-range forces and/or long-memory effects and/or evolves in a (multi)fractal-like space-time. Here, "to fail" is used to imply the divergences of the standard sums or integrals appearing in the expressions of the relevant thermodynamical quantities such as internal energy, partition function, etc. This means that we have no well-behaved expressions to obtain finite values for the response functions which provide comparisons with experimental data, which are always finite.

These kinds of violations are met for a long time in gravitational systems [5], magnetic systems [6], Levy-like anomalous diffusions [7] and some surface tension problems [8]. Moreover, the same or analogous type of difficulties might be present in long-range Casimir-like systems [9], granular matter [10] and two-dimensional turbulence [11]. In order to overcome these problems, nonextensive formalisms keep growing nowadays in Physics along two apparently different lines: Quantum Groups [12] and Tsallis Thermostatistics (TT). Although it is shown that [13] these two formalisms are related to each other, since the first one is out of the scope of this manuscript we will focus our attention to TT in the reminder of the paper.

2. Tsallis Thermostatistics

TT has been proposed by C. Tsallis as an appropriate tool for studying nonextensive physical systems [14,15]. This new formalism is basically based on two axioms :

- The generalized entropy is proposed to be

$$S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \qquad \left[\sum_i p_i = 1\right] \tag{1}$$

  where k is a positive constant, q is a real number, W is the total number of microscopic configurations and $\{p_i\}$ are the probabilities of the microstates of the system. It is easily seen that this entropy is nonextensive and recovers the standard, extensive Shannon entropy $S_1 = -k_B \sum_i^W p_i \ln p_i$ if and only if $q = 1$.

- The q-expectation value of an observable O is given by

$$\langle O \rangle_q \equiv \sum_{i=1}^{W} p_i^q O_i \tag{2}$$

  from which the generalized internal energy follows :

$$U_q = \sum_{i=1}^{W} p_i^q \epsilon_i \tag{3}$$

  where $\{\epsilon_i\}$ is the energy spectrum.

Besides these axioms, it has been verified that this nonextensive entropy has some properties such as positivity, concavity, pseudo-additivity, Shannon additivity ; moreover it has been shown that within this formalism the canonical ensemble equilibrium distribution can be found by optimizing S_q with appropriate constraints (namely, $\sum_i p_i = 1$ and $U_q = \sum_i p_i^q \epsilon_i$) as follows :

$$p_i = \frac{1}{Z_q} = [1-(1-q)\beta\epsilon_i]^{1/(1-q)} \tag{4}$$

where the generalized partition function is defined to be

$$Z_q \equiv \sum_{i=1}^{W} [1-(1-q)\beta\epsilon_i]^{1/(1-q)} \ . \tag{5}$$

At this point it is worth noting that the Legendre-transform structure of thermodynamics is invariant for all values of q, indicating that the entire formalism of thermodynamics can be extended to be nonextensive. In addition to this, it is proved that thermodynamical stability as well as the Ehrenfest theorem hold for all q values. A complete discussion of the properties with a detailed review of the formalism can be found elsewhere [16].

Before closing this Section, it must be noted that in the last decade various concepts of statistical thermodynamics have been successfully generalized within TT. Here, instead of enumerating all of these works, we draw the reader's attention to the bibliography of the formalism available on the Internet, where a complete list of works on TT with a suitable classification can be found [17].

3. Verifications of Tsallis Thermostatistics

Besides the efforts of generalizing the conventional concepts of thermostatistics within this formalism, TT has also been successfully applied to some of the physical systems where BG formalism is known to fail. Since these achievements can be thought as the verifications of TT, the reminder of this paper is devoted to a brief review of these works with the aim of clarifying the impotance of the formalism.

3.1. Stellar Polytropes

One of the problems appearing in Newtonian gravitation so far is the stellar polytropes where the use of BG formalism fails to give a finite mass. In order to solve this problem, Plastino and Plastino studied the exact time-dependent solutions of Vlasov equations. In their first paper [18], they found that for stellar polytropes q must differ from unity and then Boghosian [19] showed that stellar polytropes cannot be described thermodynamically unless $q < 7/9$ values are used. Another improvement on this line came from Plastino and Plastino again [20], where, by using the q-expectation value appearing in eq.(2), they managed to verify that the exact solutions correspond precisely to $q = -1$. This can be considered as the first verification of the formalism since the value of q which yields correct solution is found to be very distinct from unity, indicating that there exist some physical systems in Nature which need $q \neq 1$ statistics !

It is plausible to remark here that, for $d = 3$ gravitation, $q = -1$ result has recently been reobtained by Chimento [21] in a completely different manner, namely within Einstein's general relativity.

3.2. Levy-like Anomalous Diffusion

Another achievement of TT is to provide a solution for an old problem, namely how to obtain Levy distributions from an entropy principle with auxiliary conditions. The first attempt on this line is the work of Alemany and Zanette [22], where they considered one diffusive step and found the one step distribution. After this work, Tsallis et al [23] extended the one step case to N-step case where they also established a connection between q and the fractal dimension. (See also ref.[24]). Hence, this work may also be thought as a first step towards the understanding of the physical meaning of q, still standing as an open problem which must be solved by determining on which other parameters of a physical system q depends. On the other hand, before summarizing the results of these works, it is worth mentioning that Tsallis et al [25] and Costa et al [26] have recently clarified the connection between q and fractal dimension.

The results of these works can be summarized as follows: By optimizing S_q with the constraints

$$\int dxp(x) = 1 \tag{6}$$

$$\left\langle x^2 \right\rangle_q = \int dx x^2 \left[p(x)\right]^q < \infty \ , \tag{7}$$

one can obtain

$$p_q = \frac{1}{Z_q}\left[1-\beta(1-q)x^2\right]^{1/(1-q)} \quad , \quad Z_q \equiv \int dx \left[1-\beta(1-q)x^2\right]^{1/(1-q)} \quad . \tag{8}$$

It is easy to notice that the normalization constraint given by eq.(6) can be satisfied for $q \in [-\infty, 3)$, whereas it cannot be satisfied if $q \geq 3$. On the other hand, it can be verified that $\left\langle x^2 \right\rangle_1$ is finite only for $-\infty \leq q < 5/3$ and diverge otherwise, whereas $\left\langle x^2 \right\rangle_q$ is finite for the entire region $-\infty \leq q < 3$.

All of these facts indicate that BG formalism cannot provide Levy distributions within a variational entropic formalism with simple *a priori* constraints, whereas TT can. Needless to say, this achievement of TT is very important since the Levy distributions are ubiquitous in Nature. For a complete review of this subject, the reader should refer to [27].

3.3. Two-dimensional Turbulence

A few years ago Huang and Driscoll found [28] that the shape of the radial vorticity profile of the meta-equilibrium state of the relexation of two-dimensional Euler turbulence of the electron columns cannot be described by maximizing the Shannon entropy under suitable constraints since it yields profiles which are flatter than those observed in experiments. In order to describe these profiles correctly, they considered four different phenomenological theories, one of which, global minimum enstrophy, leads to profiles closer to those observed in experiments, by minimizing the enstrophy (the integral of the square of the vorticity) instead of maximizing the Shannon entropy. On the other hand, since this theory yields negative (namely, unphysical) densities, Huang and Driscoll established another model, restricted minimum enstrophy, by imposing a cut-off radius in such a way that electron density vanishes for all radii above this. With this new model they obtained results which are in good agreement with experiments, however, there is no theoretical explanation why this unusual variational principle yields good results.

Recently, Boghosian showed [19] that the maximization of Tsallis entropy with $q = 1/2$ under the same constraints (but using the q-expectation values) leads to precisely the same profiles of the restricted minimum enstrophy model. However, it should be mentioned that, in this case, the cut-off radius comes out naturally as a consequence of TT for $q < 1$. Therefore, this result can be considered as a strong evidence for the experimental validity of TT. It is worth noting that an extension of this work is now available [29], where Anteneodo and Tsallis showed that the experimental data are also compatible with q slightly above 1/2.

3.4. Peculiar Velocity Distribution of Galaxy Clusters

Very recently, Lavagno et al showed [30] that the observational data concerning the velocity distribution of clusters of galaxies can naturally be fitted by the generalized distribution function

$$p(> v) = \frac{\int_v^{v_{max}} dv \left[1 - (1-q)(v/v_0)^2\right]^{q/(1-q)}}{\int_0^{v_{max}} dv \left[1 - (1-q)(v/v_0)^2\right]^{q/(1-q)}} \tag{9}$$

with

$$v_{max} \equiv \begin{cases} v_0(1-q)^{-1/2} & \text{if } q < 1 \\ \infty & \text{if } q \geq 1. \end{cases} \tag{10}$$

A remarkably good fitting with the data is obtained for $q = 0.23$ (see the Fig.1 of ref.[30]). It is also evident from the Fig.1 of ref.[30] that the efficiency coming from the modification of the statistical formalism (namely, the use of TT instead of BG formalism) is greater and more important than the efficiency obtained from the modification of the cosmological model.

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References

[1] P.T. Landsberg, *Thermodynamics and Statistical Mechanics*, (1990).

[2] L.G. Taff, *Celestial Mechanics*, (Wiley, New York, 1985).

[3] P.T. Landsberg, *J. Stat. Phys.* **35** (1984) 159.

[4] S.H. Strogatz, *Nonlinear Dynamics and Chaos*, (Addison-Wesley, Reading, 1994).

[5] A.M. Salzberg, *J. Math. Phys.* **6** (1965) 158; D. Pavon, *Gen. Rel. Grav.* **19** (1987) 375.

[6] B.J. Hiley and G.S. Joyce, *Proc. Phys. Soc.* **85** (1965) 493.

[7] M.F. Shlesinger and B.D. Hughes, *Physica* **A109** (1981) 597; E.W. Montroll and M.F. Shlesinger, *J. Stat. Phys.* **32** (1983) 209.

[8] J.O. Indekeu, *Physica* **A183** (1992) 439; J.O. Indekeu and A. Robledo, *Phys. Rev.* **E47** (1993) 4607.

[9] P.G. de Gennes, *C.R. Acad. Sci. Paris* **II292** (1981) 701; T.W. Burkhardt and E. Eisenriegler, *Phys. Rev. Lett.* **74** (1995) 3189.

[10] M. Ammi, T. Travers and J.P. Troadec, *J. Phys.* **D20** (1987) 424; C.H. Liu and S.R. Nagel, *Phys. Rev. Lett.* **68** (1992) 2301.

[11] R.H. Kraichnan and D. Montgomery, *Rep. Progr. Phys.* **43** (1980) 547.

[12] L.C. Biedenharn, *J. Phys.* **A22** (1989) L873; P.P. Kulish and E.V. Damaskinsky, *J.Phys.* **A23** (1993) L415; M. Arık, *From Q-Oscillator to Quantum Groups*, in B. Gruber ed., *Symmetries in Science VI*, (Plenum Press, New York, 1993).

[13] C. Tsallis, *Phys. Lett.* **A195** (1994) 329; S. Abe, *Phys. Lett.* **A224** (1997) 326; A. Erzan, *Phys. Lett.* **A225** (1997) 235.

[14] C. Tsallis, *J. Stat. Phys.* **52** (1988) 479.

[15] E.M.F. Curado and C. Tsallis, *J. Phys.* **A24** (1991) L69; Corrigenda: *A24* (1991) 3187; **A25** (1992) 1019.

[16] C. Tsallis, *Physica* **A221** (1995) 277; in *New Trends in Magnetism, Magnetic Materials and Their Applications*, eds. J.L. Moran-Lopez and J.M. Sanchez (Plenum Press, New York,1994), page 451; F. Büyükkılıç, U. Tırnaklı and D. Demirhan, *Tr.J.Phys.* **21** (1997) 132.

[17] http://tsallis.cat.cbpf.br/biblio.htm

[18] A.R. Plastino and A. Plastino, *Phys. Lett.* **A174** (1993) 384.

[19] B.M. Boghosian, *Phys. Rev.* **E53** (1996) 4754.

[20] A.R. Plastino and A. Plastino, *Phys. Lett.* **A193** (1994) 251.

[21] L.P. Chimento, *J. Math. Phys.* **38** (1997) 2565.

[22] P.A. Alemany and D.H. Zanette, *Phys. Rev.* **E49** (1994) R956.

[23] C. Tsallis, S.V.F. Levy, A.M.C. de Souza and R. Maynard, *Phys. Rev. Lett.* **75** (1995) 3589; Erratum: *Phys. Rev. Lett.* **77** (1996) 5442; C. Tsallis, A.M.C. de Souza and R. Maynard, in *Levy Flights and Related Phenomena in Physics*, eds. M.F. Shlesinger, U.Frisch and G.M. Zaslavsky (Springer, Berlin, 1995), page 269.

[24] D.H. Zanette and P.A. Alemany, *Phys. Rev. Lett.* **75** (1995) 366; M.O. Caceres and C.E. Budde, **77** (1996) 2589; D.H. Zanette and P.A. Alemany, **77** (1996) 2590.

[25] C. Tsallis, A.R. Plastino and W.-M. Zheng, *Chaos,Solitons and Fractals* **8** (1997), in press.

[26] U.M.S. Costa, M.L. Lyra, A.R. Plastino and C. Tsallis, *Phys. Rev.* **E56** (1997) 245.

[27] C. Tsallis, *Physics World* **10** (July 1997) 42.

[28] H.-P. Huang and C.F. Driscoll, *Phys. Rev. Lett.* **72** (1994) 2187.

[29] C. Anteneodo and C. Tsallis, *J. Mol. Liq.* **71** (1997) 255.

[30] A. Lavagno, G. Kaniadakis, M. Rego-Monteiro, P. Quarati and C. Tsallis, *"Nonextensive Thermostatistical Approach of the Peculiar velocity function of galaxy clusters"*, *Astrophy. Lett. Comm.* (1997), in press.