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The gauge freedoms of enlarged Helmholtz theorem and Debye potentials; their use in the multipole expansion of conserved current

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Abstract

We discuss gauge freedom within the scope of the enlarged Helmholtz theorem and Neumann-Debye decomposition and then demonstrate its realization for the multipole expansion of a conserved electromagnetic current. The exact solution to the latter problem was obtained in 1974, but answers to some purely mathematical questions raised by I.B. French and Y. Shimamoto [1] about 40 years ago are given in this paper for the first time.

1. Introduction

In considering problems of mathematical physics with definite spatial and/or dynamical symmetries, one commonly uses various decompositions of vector fields over scalar potentials. These decompositions supplement the famous Helmholtz theorem and reduce its "gauge freedoms".

Let $\exists\, \mathbf{V}: \ \mathbf{r} \in R^3 \to \mathbf{V}(\mathbf{r}) \in R^3$ with all "good" properties. We may represent the given field $\mathbf{V}$ in different ways:

$$\mathbf{V} \to \{V_x, V_y, V_z\} \to \{\varphi, \mathbf{A}\}_{\mathrm{div}\mathbf{A}=0} \to \{\varphi, \psi, \chi\}.$$

In any case, it is necessary to constrain superfluous components, if they take place, by introducing conditions similar to $\mathrm{div}\mathbf{A} = 0$.

The last variant, diffeomorphic scalarization of a vector field, is the most economical and convenient approach to vector boundary-value problems of mathematical physics.

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But to use this approach, we have to be able to invert decomposition formulas, i.e. to deduce the integral representations of scalar potentials through the original vector field.

Approach. To obtain a vector function from (pseudo)scalar function set $\varphi(\mathbf{r}), \psi(\mathbf{r})$ and $\chi(\mathbf{r})$, one must act on them by some vector operator $\hat{F}(\mathbf{r}, \boldsymbol{\nabla})$. In addition to $\mathbf{r}$ and $\boldsymbol{\nabla}$ themselves, we may from them construct three simple operators: $\mathbf{L} := -\mathbf{r}\times\boldsymbol{\nabla}$, $\mathbf{N} := \boldsymbol{\nabla}\times\mathbf{L}$ and $\mathbf{M} := -\mathbf{r}\times\mathbf{L}$. One can see their symplectic nature because of their correspondence to the frames of reference in phase space: $\mathbf{r}$, $\mathbf{k}$, $\mathbf{r}\times\mathbf{k}$, $\mathbf{r}\times(\mathbf{r}\times\mathbf{k})$ and $\mathbf{k}\times(\mathbf{k}\times\mathbf{r})$. The *trio* of vectors $\mathbf{k}$, $\mathbf{L}$, $\mathbf{N}$ and $\mathbf{r}$, $\mathbf{L}$, $\mathbf{N}$, being immersed into the spaces $R^3_{\mathbf{r}}$ and $R^3_{\mathbf{p}}$, respectively, form orthogonal bases in them, which are important for different applications.

One may verify the following projection and commutation properties of $\hat{F}$:

$$
\begin{aligned}
&[\mathbf{L}, r^2] = [\mathbf{L}, p^2] = 0\\
&[L^2, r] = i[\mathbf{r}\times\mathbf{L}] - i[\mathbf{L}\times\mathbf{r}]\\
&[L^2, k] = i[\mathbf{k}\times\mathbf{L}] - i[\mathbf{L}\times\mathbf{k}]\\
&\mathbf{r}\cdot\mathbf{L} = \boldsymbol{\nabla}\mathbf{L} = \mathbf{r}\cdot\mathbf{N} = \mathbf{L}\cdot\mathbf{N} = \mathbf{L}\cdot\mathbf{M} = \mathbf{M}\cdot\mathbf{L} = 0, \mathrm{curl}\mathbf{N} = -\mathbf{L}\triangle;\\
&[\mathbf{L}, \triangle] = 0,\ [\mathbf{N}, \triangle] = 0,\ [\mathbf{M}, \triangle] = -6\boldsymbol{\nabla}, \text{ etc.}\\
&[r_i, \nabla_k] = -\delta_{ik}, \quad [r_i, L_k] = -\epsilon_{ikj} r_j, \quad [\nabla_i, L_k] = -\epsilon_{ikj}\nabla_j,\\
&[\nabla_i, N_k] = \nabla_i\nabla_k - \triangle\delta_{ik}, \quad [r_i, M_k] = r_i r_k - r^2\delta_{ik}\\
&[L_i, M_k] = \epsilon_{ikj} r_j - r^2\epsilon_{ikj}\nabla_j.
\end{aligned}
$$

It is taken into account that

$$
\mathbf{N} = -\mathbf{r}\triangle + \boldsymbol{\nabla}(\mathbf{r}\boldsymbol{\nabla}) + \boldsymbol{\nabla}, \quad \mathbf{r}\times\mathbf{N} = -(1+\mathbf{r}\boldsymbol{\nabla})\mathbf{L} = -\mathbf{L}(1+\mathbf{r}\boldsymbol{\nabla}), \quad \mathrm{curl}\mathbf{N} = -\mathbf{L}\triangle, \text{ etc.}
$$

Note that in the space $R^3_{\mathbf{r}}$, $\mathbf{L}$ and $\mathbf{N}$ form the following algebra of differential operators:

$$
[L_i, L_j] = \epsilon_{ijk} L_k,\ [L_i, N_j] = \epsilon_{ijk} N_k,\ [N_i, N_j] = -\epsilon_{ijk} L_k \triangle.
$$

After the rescaling of $\mathbf{N} \to \mathbf{N}/\sqrt{\triangle}$ over a supporting function space, we acquire the $O(3,1)$ Lie algebra representation in terms of the $\hat{F}(\mathbf{r}, \boldsymbol{\nabla})$ operators in $R^3_{\mathbf{r}}$. They act in the $O(3)\times O(3)$ function space suitable for arranging the multipole phenomenology in electromagnetic theory (see e.g. [2], [3], [6]). The operators $\mathbf{L}$ and $\mathbf{M}$ obey the same algebra in the space of wave vectors $R^3_{\mathbf{k}}$ that is Fourier-conjugate to $R^3_{\mathbf{r}}$. In the preceding paper [7] the emphasis has been made on the inversion of different formulas for decomposition of vector fields in the mathematical aspect.

In the physical aspect, two vector-potential dual formulations of the theory of continuous media, taking into account both magnetic and electric toroid polarizations [8] (see also [9], part II), are published for the first time. Here we consider the mathematical underlying reason of uniqueness of the division of the transverse electric distribution density $E_{lm}(\mathbf{k}^2,\, t)$ into two independent multipole specimen [3]:

$$
E_{lm}(\mathbf{k}^2,\, t) = \dot{Q}_{lm}(0,\, t) + \mathbf{k}^2 T_{lm}(\mathbf{k}^2,\, t), \tag{1}
$$

where $\dot{Q}_{lm}(0,\, t)$ are the time-derivative of the Coulomb (charge) multipole $l-$moment and $T_{lm}(\mathbf{k}^2,\, t)$ are the toroid multipole form factor of $l-$order. Note that a secularized relation when neglecting toroid contributions and known as the Siegert theorem may be correct for low-symmetric electromagnetic systems only. A attempt will be made in this paper to clarify and explain the mathematical condition and subsequent question addressing the possibility of identifying multipole moments (the leading ones for each given l) in the transverse and longitudinal parts of current raised in [1]. From our representation theorem (see also remark II) it follows that definitions of multipole moments are unique.

Recall that the procedure of multipole decomposition of the current and field densities in electrodynamics in fact corresponds to the description of properties of a system by a set of numerical characteristics which are assigned to a point, which becomes the "center" of the density distribution of the system considered. In this case, for the poloidal and potential parts of the current an additional connection arises between its longitudinal and transverse components [3] due to degeneracy of boundary conditions of longitudinality and transversality in **r**-space at the self-similar shrinkage of the definition domain of current to the chosen center. To prove this statement, we could probably use the transfer technique of boundary conditions [4], [5] and it might have lead to a separation of multipole moments called in [3] the toroid ones [1]. But ways of this kind are very difficult in the general framework of distribution theory. Here, we use a simple concrete approach.

The main feature of the multipole expansion procedure is a special choice of basis functions which, in actual practice, ensues the rapid convergence of multipole series. This circumstance forces us to weaken requirements of the usual Helmholtz theorem and, respectively, to take account of a gauge freedom extension. In sections 1 and 2 we discuss gauge freedoms in the Helmholtz and Neumann-Debye decompositions. In section 3 we turn to their realizations within the multipole expansion of the electromagnetic current.

2. The enlarged Helmholtz theorem

We begin our consideration with the Helmholtz decomposition:

$\triangleleft$ $\forall \mathbf{V}$ with properties of single-valuedness, continuity, boundedness or covergent,

$$|\mathbf{V}| < \frac{k}{r^{1+\epsilon}},\ \epsilon > 0 \text{ at } r \to \infty$$

in the whole space may be represented in the form [see e.g. [12]]:

$$\mathbf{V} = \boldsymbol{\nabla}\varphi + \mathrm{curl}\mathbf{A} \qquad \text{with} \qquad \mathrm{div}\mathbf{A} = 0.$$

Indeed, the theorem's requirements are proved to be sufficient in order to reexpress tautologically the given vector field $\mathbf{V}$ through its divergence and vorticity. The explicit realization of the theorem could be attained due to the following operations:

$$\mathrm{div}\mathbf{V}^{\|} = \triangle\varphi, \qquad \varphi = \triangle^{-1}\mathrm{div}\mathbf{V}, \qquad \text{where} \qquad \triangle^{-1} := -\int_{\Omega\subseteq R^3} \frac{d^3r'}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

$$\mathrm{curl}\mathbf{V}^{\perp} = \mathrm{curl}\ \mathrm{curl}\mathbf{A} = -\triangle\mathbf{A}, \qquad \mathbf{A} = -\triangle^{-1}\mathrm{curl}\mathbf{V}.$$

[1] In fact, their exact title should be the poloidal moments, see; [6], [9].

So we have

$$\mathbf{V} \equiv \mathbf{V}^{\|} + \mathbf{V}^{\perp} = \boldsymbol{\nabla}\triangle^{-1}\mathrm{div}\mathbf{V} - \mathrm{curl}\triangle^{-1}\mathrm{curl}\mathbf{V}. \qquad \triangleright \tag{2}$$

Form (1) often produces misunderstanding (e.g. [13]) that the representation of div$\mathbf{V}$ and curl$\mathbf{V}$ is equivalent to the representation of $\mathbf{V}$ itself. We discuss here to what extent is this representation single-valued. In fact, (1) contains the evident "gauge freedom":

$$\varphi = \varphi + \omega, \quad (\triangle\omega = 0), \qquad \mathbf{A} = \mathbf{A} + \boldsymbol{\nabla}w.$$

If we remove the demands of topological triviality Ω and/or the boundedness of functions ω and w then, as an example important for physical application, we may represent the gauge freedoms in the form of special additional functions to $\mathbf{V}$:

$$0 \neq \mathbf{V}_{\mathbf{N}} = \mathrm{curl}\mathbf{L}\begin{pmatrix} r^l \\ 1/r^{l+1} \end{pmatrix} Y_{lm} \equiv \begin{pmatrix} -(l+1)\boldsymbol{\nabla}r^l Y_{lm} \\ l\boldsymbol{\nabla}r^{-l-1}Y_{lm} \end{pmatrix},$$

which have a nonzero finite value over all space except $r \to \infty$ and $r \to 0$, respectively, as far as

$$\mathrm{div}\mathbf{V}_{\mathbf{N}} = \mathrm{curl}\mathbf{V}_{\mathbf{N}} \equiv 0, \quad \text{in} \quad R^3/S^2_{r\to\infty} \quad \text{and} \quad R^3/\{0\}.$$

Therefore these functions [2] cannot be represented by the usual Helmholtz decomposition, and manifest the gauge freedom of its enlarged formulation.

Remark I. Note that functions $\mathbf{V}_N$ are longitudinal and transverse simultaneously, since they represent the vector solutions of the Laplace equation ($\triangle \equiv \mathrm{grad\,div} - \mathrm{curl\,curl}$). Moreover, in our context it is important to emphasize that they are topologically equivalent to the poloidal (meridional) harmonics on the torus like surface covering the whole space of R^3 except one deleted axis. ■

Thus, under this gauge freedom the Helmholtz decomposition takes the following alternative forms:

$$\mathbf{V} \equiv \mathbf{V}^{\|} + \mathbf{V}^{\perp} =$$

$$= \boldsymbol{\nabla}\left(\triangle^{-1}\mathrm{div}\mathbf{V} - (l+1)\sum_{lm}\left[C_{lm}r^l + C'_{lm}r^{-l-1}\right]Y_{lm}\right) + \mathrm{curl}\triangle^{-1}\mathrm{curl}\mathbf{V} = \tag{3}$$

$$= \boldsymbol{\nabla}\triangle^{-1}\mathrm{div}\mathbf{V} + \mathrm{curl}\left(\triangle^{-1}\mathrm{curl}\mathbf{V} + \mathbf{L}\sum_{lm}\left[C_{lm}r^l + C'_{lm}r^{-l-1}\right]Y_{lm}\right).$$

[2]Note that similar functions may be generated using the common relation

$$\mathrm{curl}(\mathbf{r}\times\nabla)r^{\kappa}Y_{lm} = -(\kappa+1)\nabla r^{\kappa}Y_{lm} + (\kappa - l)(\kappa + l + 1)\mathbf{r}r^{\kappa-2}Y_{lm},$$

and taking into account that

$$\kappa = l \ \text{ and } \ \kappa = -l-1$$

(see [14], Appendix A).

3. The Neumann-Debye decomposition

The well-known mathematical physicist W. M. Elsasser in [15] has already observed that every vector field of the form $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{r}\chi+\boldsymbol{\nabla}\times\mathbf{r}\psi$, where ψ and χ are differentiable scalar functions is solenoidal. In paper [16] it has been shown that if $\mathrm{div}\mathbf{V}=0$ in R^3, then for every choice of origin there exist unique scalars ψ and χ such that $\mathbf{V}=\mathbf{L}\psi+\mathbf{N}\chi$ while ψ and χ average to zero on every spherical surface concentric with the origin. The complete theorem of the possibility of decomposition of $\mathbf{V}(\mathbf{r})$ in terms of scalar functions reads as follows:

Representation Theorem. Given a region $\Omega\subseteq R^3\backslash\{0\}$, with a regular boundary and R^3-vector field, $\mathbf{V}:\ \mathbf{r}\in\Omega\rightarrow\mathbf{V}(\mathbf{r})\in R^3$, there exist three scalar functions $\varphi(\mathbf{r}),\psi(\mathbf{r})$ and $\chi(\mathbf{r})$ on Ω which define this $\mathbf{V}$ [17]. ■

The most used decomposition which we call the Neumann-Debye decomposition , has the form

$$\triangleleft \qquad \mathbf{V}(\mathbf{r}):=\boldsymbol{\nabla}\varphi(\cdot)+\mathrm{curl}\mathbf{r}\psi(\cdot)+\mathrm{curl}\ \mathrm{curl}\mathbf{r}\chi(\cdot)\equiv\boldsymbol{\nabla}\varphi+\mathbf{L}\psi+\mathbf{N}\chi. \tag{4}$$

Here, ψ and χ are the so-called Debye potentials and φ is the usual (electric) scalar potential. We found fundamental solutions of the inversion problem of (1) in the form [7]:

$$\begin{array}{rcl} \mathrm{div}\mathbf{V}=\triangle\varphi & \rightarrow & \varphi=\triangle^{-1}\mathrm{div}\mathbf{V},\\ \mathbf{L}\mathbf{V}=L^2\psi & \rightarrow & \psi=-L^{-2}\mathbf{L}\mathbf{V}\equiv \mathrm{L}^{-2}\mathbf{r}\ \mathrm{curl}\mathbf{V},\\ \mathbf{r}\mathbf{V}=(\mathbf{r}\boldsymbol{\nabla})\varphi+L^2\chi & \rightarrow & \chi=L^{-2}(\mathbf{r}\boldsymbol{\nabla})\triangle^{-1}\mathrm{div}\mathbf{V}-L^{-2}(\mathbf{r}\mathbf{V}),\end{array} \tag{5}$$

where [18]

$$\mathrm{L}^{-2}:=\int_\sigma\frac{d\omega'}{4\pi}ln(1-\hat r\cdot\hat r'). \quad \triangleright$$

Remark II. Eigenfunctions of the square of the angular momentum operator $iL:\ (iL)^2=-L^2$ are usual the spherical functions satisfying the equation

$$L^2\ Y_{lm}(\hat r)=-l(l+1)Y_{lm}(\hat r).$$

The corresponding Green function for this equation can be found with the help of the known Mercer theorem ([18], v.1) which, in the given case, yields

$$\frac{1}{4\pi}\sum_{l,m}\frac{Y^*_{lm}(\hat r)Y_{lm}(r')}{-l(l+1)}=-\sum_l\frac{2l+1}{l(l+1)}P_l(\hat r\cdot\hat r')=1-ln2+ln(1-\hat r\cdot\hat r'). \quad ■$$

Remark III. It is well-known that the gauge freedom of (5) is the following:

$$\varphi\rightarrow\varphi+C,\quad \psi\rightarrow\psi+\mu(r),\quad \chi\rightarrow\chi+\nu(r).$$

Requiring φ to vanish on the boundary, and $\psi,\ \chi$ not to contain spherically symmetric components

$$\int_{S^2}dw\ \psi\ =\ \int_{S^2}dw\ \chi\ =\ 0,$$

we put these functions in one-to-one correspondence to $\mathbf{V}$. ■

Uniqueness Theorem. If a vector field $\mathbf{V}$ (with the properties determined in theorem (I)) is defined on every S_r^2 in some range $r_0 < r < r_1$; and in that range $V_r = 0$ while $V_\theta(r,\theta,\varphi)$ and $V_\varphi(r,\theta,\varphi)$ are bounded for each fixed r; and is continuously differentiable, except possibly at $\theta = 0$ and $\theta = \pi$; and, further, div$\mathbf{V} = 0$ and curl$\mathbf{V} = 0$, then $\mathbf{V} \equiv 0$.

Our inversion formulas (3) demonstrate these properties immediately (cp. with [16], p.383, where the condition $\mathbf{L}\mathbf{V} = 0$ has been used instead of our curl$\mathbf{V} = 0$). ■

Now we need to reconstruct the representation of χ such that it depends on curl$\mathbf{V}$ and div$\mathbf{V}$ only. Really, the latter quantities have physical meaning but not the radial component of $\mathbf{V}$. Moreover, we may expect χ not to depend on div$\mathbf{V}$ generally because this potential defines the transverse part of the vector field $\mathbf{V}$. Nevertheless, because of gauge freedom, the situation is not so simple as it seems to be.

Indeed, we may substitute the Helmholtz decomposition (the last expression in (3)) into the term with $\mathbf{rV}$ and see that χ takes the form

$$\chi = L^{-2}\mathbf{L}\triangle^{-1}\mathrm{curl}\mathbf{V}\sum_{lm} L^{-2}\big[C_{lm}\nabla r^l + C'_{lm}\nabla r^{-l-1}\big]Y_{lm}. \tag{6}$$

Further, our vector field corresponding to the gauge freedom may be transformed as

$$\begin{aligned} &\mathrm{curl}\mathbf{L}L^{-2}(r\nabla)r^lY_{lm} = (l+1)\mathrm{curl}r^l\mathbf{L}L^{-2}Y_{lm} = \\ &= (l+1)\mathrm{curl}r^l\mathbf{L}L^{-2}\frac{L^2}{l(l+1)}Y_{lm} = \frac{1}{l}\mathrm{curl}\mathbf{L}r^lY_{lm} = -\frac{l+1}{l}\boldsymbol{\nabla} r^lY_{lm}. \end{aligned} \tag{7}$$

So, we find fundamental solutions of the inversion problem of (5) in the form

$$\begin{aligned} \mathrm{div}\mathbf{V} = \triangle\varphi \quad &\rightarrow \quad \varphi = \triangle^{-1}\mathrm{div}\mathbf{V}, \\ \mathbf{LV} = L^2\psi \quad &\rightarrow \quad \psi = -L^{-2}\mathbf{LV} \equiv \mathbf{L}^{-2}\mathbf{r}\ \mathrm{curl}\mathbf{V}, \\ \mathbf{rV} = (\mathbf{r}\boldsymbol{\nabla})\varphi + L^2\chi \quad &\rightarrow \quad \chi = L^{-2}(\mathbf{r}\boldsymbol{\nabla})\triangle^{-1}\mathrm{div}\mathbf{V} - L^{-2}(\mathbf{rV}), \\ &\qquad = L^{-2}\mathbf{L}\triangle^{-1}\mathrm{curl}\mathbf{V} \\ &\qquad + \sum_{lm} L^{-2}\big[C_{lm}\nabla r^l + C'_{lm}\nabla r^{-l-1}\big]Y_{lm} \end{aligned} \tag{8}$$

having taking into account the gauge freedom in (3). ■

4. The multipole expansion of the electromagnetic current

Now we compare our abstract exercises and the procedure for multipole expansion of electromagnetic current on the basis of the vector Helmholtz equation solutions constructed through the Neumann-Debye decomposition (8). The multipole representation of current $\mathbf{J}(\mathbf{r})$ by (5) may be obtained by the standard expansion of three scalar densities into series form:

$$\phi \sim \sum_{lmk} j_l(kr)Y_{lm}\dot{Q}_{lm}(k^2,t);$$

$$\psi \sim \sum_{lmk} j_l(kr) Y_{lm} M_{lm}(k^2, t); \tag{9}$$
$$\chi \sim \sum_{lmk} j_l(kr) Y_{lm} E_{lm}(k^2, t).$$

Therefore the multipole representations of the transverse part of current ($\mathrm{div}\,\mathbf{J} = 0$) are determined by the magnetic form factors $M_{lm}(k^2, t)$, and the transverse electric contributions $E_{lm}(k^2, t)$, and the scalar part of current ($\mathrm{curl}\,\mathbf{J} = 0$) are expressed due to the 4−current conservation law ($\mathrm{div}\,\mathbf{J} = -\dot{\rho}$) through the Coulomb (charge) multipole moments $Q_{lm}(0, t)$ and their mean $2n$−power radii:

$$Q_{lm}(k^2, t) = Q_{lm}(0, t) + \sum_{n=1}^{\infty} \frac{k^{2n}}{n!} Q_{lm}^{(2n)}(0, t).$$

As a result, $\mathbf{J}$ may be represented as [3]

$$\mathbf{J}(\mathbf{r}, t) = (2\pi)^{-3} \sum_{l,m} \int_0^{\infty} dk (-ik)^l \frac{\sqrt{4\pi(2l+1)}}{l(2l+1)!!} \{-ik\, \mathbf{L} f_l(kr) Y_{lm}(\hat{\mathbf{r}}) M_{lm}(k^2, t) -$$
$$ik \mathrm{curl} \mathbf{L} f_l(kr) Y_{lm}(\hat{\mathbf{r}}) E_{lm}(k^2, t) + l\, \boldsymbol{\nabla} f_l(kr) Y_{lm}(\hat{\mathbf{r}}) \dot{Q}_{lm}(k^2, t)\}. \tag{10}$$

We repeat here the procedure used firstly in [3] for the ascertainment of the exact structure of transverse electric contributions $E_{lm}(k^2, t)$. To this end, one should rewrite explicitly our basis functions in terms of vector harmonics:

$$\mathrm{curl} \mathbf{L} f_l(kr) Y_{lm}(\hat{\mathbf{r}}) = (2l+1)^{-1/2} \{f_{l-1}(kr) \sqrt{l+1} \boldsymbol{Y}_{ll-1m}(\hat{\mathbf{r}}) +$$
$$+ f_{l+1}(kr) \sqrt{l} \boldsymbol{Y}_{ll+1m}(\hat{\mathbf{r}})\};$$

$$\boldsymbol{\nabla} f_l(kr) Y_{lm}(\hat{\mathbf{r}}) = (2l+1)^{-1/2} \{f_{l-1}(kr) \sqrt{l} \boldsymbol{Y}_{ll-1m}(\hat{\mathbf{r}}) -$$
$$- f_{l+1}(kr) \sqrt{l+1} \boldsymbol{Y}_{ll+1m}(\hat{\mathbf{r}})\},$$

where $\boldsymbol{Y}_{ll-1m}(\hat{\mathbf{r}})$ is a harmonic polynomial function defined as

$$r^{l-1} \boldsymbol{Y}_{ll-1m}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(2l+1)}} \boldsymbol{\nabla} r^l Y_{lm}.$$

As is obvious, in the wave-length approximation, the leading contributions in the latter expressions are delivered by the vector harmonic functions $f_{l-1} Y_{ll-1m}$ from which it follows that

$$\mathrm{curl} \mathbf{L} f_l(kr) Y_{lm}(\hat{\mathbf{r}}) \approx_{k \to 0} \sqrt{(l+1)/l}\, \boldsymbol{\nabla} f_l(kr) Y_{lm}(\hat{\mathbf{r}}) =$$
$$= \frac{4\pi (ikr)^{l-1}}{(2l+1)!!} \sqrt{\frac{l+1}{l}} \boldsymbol{\nabla} r^l Y_{lm}(\hat{\mathbf{r}}) + O[(kr)^{l+1}]. \tag{11}$$

It is the relation that permits us to identify the leading term in $E_{lm}(k^2,\, t)$ with time-derivatives of $Q_{lm}(0,\, t)$. However, all functionally independent contributions in $E_{lm}(k^2,\, t)$ give the so-called toroid moments and their $2n-$power radii:

$$T^{(2n)}_{lm}(0,\, t) = -\frac{\sqrt{\pi l}}{2l+1}\int r^{l+2n+1}\left[\boldsymbol{Y}^*_{ll-1m}(\mathbf{r}) + \frac{2\sqrt{l/(l+1)}}{2l+3}\boldsymbol{Y}^*_{ll+1m}(\mathbf{r})\right]\mathbf{J}(r,t)d^3r.$$

As the toroid moments have a distinct geometrical meaning (diverse details and representations are given in [3], [4], [9], [10] and see also [11]), the rejection of $T_{lm}(t)$ is generally invalid like it was done in the Siegert theorem $E_{lm}(k^2,\, t) \to_{k\to 0} \dot{Q}_{lm}(0,\, t)$. Neglect of $T_{lm}(t)$ in comparison with $Q_{lm}(t)$ is analogous to the neglect of a higher multipole moment (contributions of highest symmetries of a given system) in comparison with the lower ones, which is of course permissible only when the lower moments (low symmetries) of this system do exist. So, the strict theorem determining the electric part structure has the form (1) $E_{lm}(\mathbf{k},\, t) = \dot{Q}_{lm}(0,\, t) + \mathbf{k}^2 T_{lm}(\mathbf{k},\, t)$, and its validity and uniqueness rely on the gauge freedom which has been obtained for the enlarged Helmholtz theorem and transferred to Debye potentials (compare (8) and (11)).

Moreover, by using the exact relation (1), it is not so hard to find an expression for the complete parameterization of the current in terms of generalized functions [3] [9]:

$$\mathbf{J}(\mathbf{r},\, t) = \sum_{l=1}^{\infty}\sum_{m=-l}^{m=l}\sum_{n=0}^{\infty}\frac{(2l+1)!!}{2^n n! l(2l+2n+1)!!}\sqrt{\frac{4\pi}{2l+1}}\{M^{(2n)}_{lm}(t)(\mathbf{r}\times\boldsymbol{\nabla})\Delta^n\delta_{lm}(\mathbf{r})$$
$$+\big[\dot{Q}_{lm}(t)\delta_{n,0}\Delta^{-1} - T^{(2n)}_{lm}(t)\big]\mathrm{curl}(\mathbf{r}\times\boldsymbol{\nabla})\Delta^n\delta_{lm}(\mathbf{r}) - l\,\dot{Q}^{(2n)}_{lm}(t)\boldsymbol{\nabla}\Delta^{n-1}\delta_{lm}(\mathbf{r})\}. \quad (12)$$

Remark IV. However, the expansion via of spherical harmonics emerges rapidly convergent. We remind (see e.g. [2], p.806) that if the field decreases very slowly for large distances slower than r^{-2}, the divergence and curl of the vector field considered are assigned arbitrary independent values. Conversely, if we know that $\mathbf{J}$ vanishes identically outside some source radius R, $\boldsymbol{\nabla}\mathbf{J}$ and $\boldsymbol{\nabla}\times\mathbf{J}$ are no longer independent of each other. As far as the expansion via spherical harmonics is rapidly convergent, it is realized in the latter representation (12) immediately.

Thus, we strictly demonstrated that the gauge freedom in division of the electromagnetic current into the transverse and longitudinal parts leads to the fact that the multipole contributions to the transverse part of current $E_{lm}(k^2, t)$ are represented in the form (1) and its leading terms may be identified with $\dot{Q}_{lm}(0\, t)$ from the longitudinal part of current for all l. Note that, since the coefficient C_l does not depend on the wave number k, we can use hereafter the Lorentz gauge condition in the calculation of the vector potential.

5. Conclusion

The representation of χ in the Neumann-Debye scalarization already assumes that the prohibition of the electric type of radiation imposes some conditions both on curl$\mathbf{J}$

[3]The contribution of $l = 0$, $n = 0$ to the last term is forbidden by total charge conservation, whereas other terms contain no contribution of $l = 0$, formally, owing to $(\mathbf{r}\times\nabla)\delta(\mathbf{r}) \equiv 0$.

and div**J**. But we can not reveal the ones due to their non-division in the scalarization mentioned. Exploitation of the enlarged Helmholtz theorem for this operation has inserted the extended gauge freedom (let us recall that the Neumann-Debye representation the gauge freedom reduce to functions of the scalar argument $|\mathbf{r}|$ only). It is just this freedom, consideration of which made it possible to identify the coefficients of leading order of the expansion of transverse and longitudinal electric parts of the current!

The form of the expression (1) shows the possibility of compensation of the electric type radiation if the toroid and charge moments are switched on as “anti-phase” ones [3] (see also [19]).

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