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Medium-dependent metric in electrostatics

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Abstract

The use of differential forms allows the formulation of the principal equations of electrodynamics in a metric-independent way. The metric is needed only for finding the solutions. Various metrics can be introduced, depending on the medium. A special metric, connected with the electric permittivity tensor, allows us to reduce all electrostatic problems in anisotropic media to those in in isotropic one.

Introduction

When asked what is the principle characteristic of the electric field strength **E** and magnetic induction **D** in three-dimensional space we usually answer: they are vectors. We do so because we do not realize that to outer forms (called also differential forms if they depend on position) the attributes of magnitude and direction can also be ascribed. There are arguments showing that (in a three-dimensional, that is, not manifestly relativistic, approach) **E** and **D** are differential forms and they also can be considered as directed quantities.

In the last decades, a way of presenting electrodynamics has been proposed based on a broad use of differential forms; see Refs [1-8]. Such a formulation represents a deep synthesis of formulae and simplifies many deductions. Most authors concentrate on algebraic definitions of the outer forms: nice exceptions are found in Refs. [2], [7] and [9] where visualisations by geometric images are shown. Not all presentations, also, put enough care to the use of pseudoforms. In Refs. [1-4], **D** is claimed to be a two-form. Only Schouten [5], Frankel [6], Ingarden and Jamiolkowski [8] applied pseudoforms in electrodynamics, under the names: *covariant W-p-vectors* [5], *twisted forms* [6,7] *or odd forms* [8]. The named authors admit that **D** is a pseudo-two-form.

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Table 1.

features	vector	pseudovector	bivector	pseudo-bivector
attitude	straight line	straight line	plane	plane
orientation	arrow on	curved arrow around	curved arrow on	arrow piercing
magnitude	length	length	area	area
	l	**l***	**S**	**S***
	one-form	pseudo-one-form	two-form	pseudo-two-form
attitude	plane	plane	straight line	straight line
orientation	arrow piercing	curved arrow on	curved arrow around	arrow on
magnitude	inverse length	inverse length	inverse area	inverse area
	E	**H**	**B**	**D**

The forms and pseudoforms are necessary to formulate electrodynamics in a scalar product independent way. We call it *premetric electrodynamics.* It turns out that only the principal equations of this theory can be tackled in this manner. When one seeks their solutions, that is, specific electromagnetic fields as functions of position, a scalar product is needed for writing, among others, the constitutive equations involving electric permeability and magnetic permittivity. The scalar product allows us to replace the outer forms and pseudoforms by vectors and pseudovectors. A special scalar product can be introduced in the case of anisotropic dielectric, for which the vectors $\vec{\mathbf{E}}$ and $\vec{\mathbf{D}}$ are parallel. In this manner the medium can be treated analogously to the isotropic one. Then the counterpart of the Coulomb field and the fields for many electrostatic problems can be found in a very natural way.

Directed quantities

The list of directed quantities in three-dimensional space consists of multivectors, pseudomultivectors, forms and pseudoforms. Table 1 collects eight quantities which, in the presence of a metric, can be replaced by vectors or pseudovectors. For their more systematic introduction see Refs. [10,11]. The upper part contains multivectors and pseudomultivectors, the lower part contains their duals. The term *attitude*, occurring in Table 1 needs an explanation. Each *directed quantity* has a separate *direction* which consists of *attitude* and *orientation.* For the well known *vector*, depicted as a directed segment, the *direction* consists of a straight line (on which the vector lies), after Lounesto [12] called an *attitude*, and an arrow on that line which is called the *orientation.* Two vectors of the same attitude are parallel.

Geometric images of the eight quantities are shown in Figures 1-7.

Physical quantities

We now present a list of physical quantities with their designation as directed quantities along with short justifications.

The most natural vectorial quantity is the *displacement vector* $\mathbf{l}$ which is of the same nature as the *radius vector* $\mathbf{r}$ of a point in space relative to a reference point. Of course, the *velocity* $\mathbf{v}=d\mathbf{x}/dt$, the derivative of $\mathbf{r}$ with respect to a scalar variable t, is also a vector. The same is true of the *acceleration* $\mathbf{a}=d\mathbf{v}/dt$, the *momentum* $\mathbf{p}=m\mathbf{v}$ and the electric dipole moment $\mathbf{d}=q\mathbf{l}$.

The *angular momentum* $\mathbf{L}=\mathbf{r}\wedge\mathbf{p}$, as the outer product of two vectors $\mathbf{r}$ and $\mathbf{p}$, is the bivector. The best physical model of a bivector is a flat electric circuit. Its magnitude is just the area encompassed by the circuit; its attitude is the plane of the circuit and orientation is given by the sense of the current. This bivector could be called a *directed area* $\mathbf{S}$ of the circuit. **A** connected bivectorial quantity is then the *magnetic moment* $\mathbf{m}=I\mathbf{S}$ of the circuit, where I is the current.

A one-form quantity is the *electric field strength* $\mathbf{E}$, since we consider it to be a linear map of the infinitesimal vector $d\mathbf{x}$ into the infinitesimal potential difference: $-dV=\mathbf{E}.d\mathbf{x}$. The *magnetic induction* $\mathbf{B}$ is an example of a two-form quantity, since it can be treated as a linear map of the directed area bivector $d\mathbf{S}$ into the magnetic flux: $d\phi=\mathbf{B}.d\mathbf{S}$. The Stokes theorem $\int\mathbf{B}.d\mathbf{S}=\oint\mathbf{A}.d\mathbf{l}$, in which $\mathbf{A}$ is the so called *vector potential,* says that the product $\mathbf{A}.d\mathbf{l}$ is also the magnetic flux. Thus, $\mathbf{A}$ is a one-form rather than a vector.

Now for some examples of pseudoquantities; the area $\mathbf{S^*}$ of a surface, through which a flow is measured, is the first one. The side of the surface from which a substance *mass, energy, electric charge, etc.* passes is important. Hence, the orientation of $\mathbf{S^*}$ can be marked as an arrow not parallel to the surface. This is situation depicted in Figure 8. We claim that the *area of a flow* is a pseudo-bivector quantity. Accordingly, the *flux density* $\mathbf{j}$ (or the *current density* in case of the electric current flowing) has to be a pseudo-two-form quantity. It corresponds to the linear map $dI=\mathbf{j}.d\mathbf{S^*}$ into the electric current dI.

The electric induction $\mathbf{D}$ has a similar nature. We present here a prescription of its measurement quoted from Ref. [13], p. 68: "Take two identical discs each made of very thin sheet metal, and each with an isolated handle. Place one disc on top of the other, holding them by the handles, electrically discharge them and then place them in the presence of a field. As you separate the discs, the charges induced on them (one positive, the other negative) are also separated. Now measure one of them with the aid of a Faraday cage. It turns out that for a small enough disc the charge is proportional to its area."[1] One will agree that the *disc area* $d\mathbf{S}^\star$ is a pseudo-bivector quantity since its magnitude is the area, its attitude is the plane and its orientation is given by an arrow showing which disc is to be connected with the Faraday cage; see Figure 8. Because of the proportionality relation $d\mathbf{Q}=\mathbf{D}.d\mathbf{S^*}$, we ascertain that the *electric induction* is a linear map of the pseudo-bivectors into scalars, i.e. it is a pseudo-two-form. Notice that $\mathbf{D}$ is of the same directed nature as the electric current density. This is reflected in the fact that $\mathbf{D}$ is called a *displacement current.*

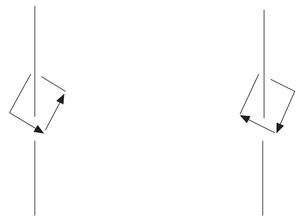

Figure 1. Two pseudovectors of the same attitude and opposite orientations depicted by oriented parallelogram.

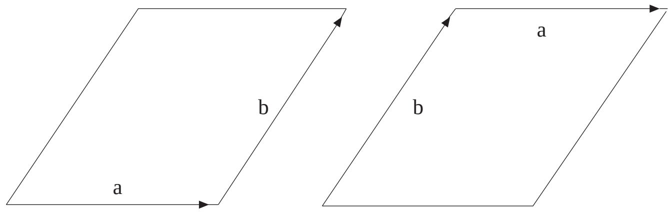

Figure 2. Bivectors represented as outer products of two vectors. Two bivectors **a**∧**b** and **b**∧**a**, opposite to each other.

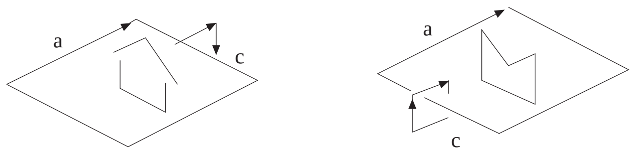

Figure 3. Pseudobivectors represented as outer products of vector **a** and pseudovector **c**. Two pseudobivectors **a**∧**c** and **c**∧**a**, opposite to each other.

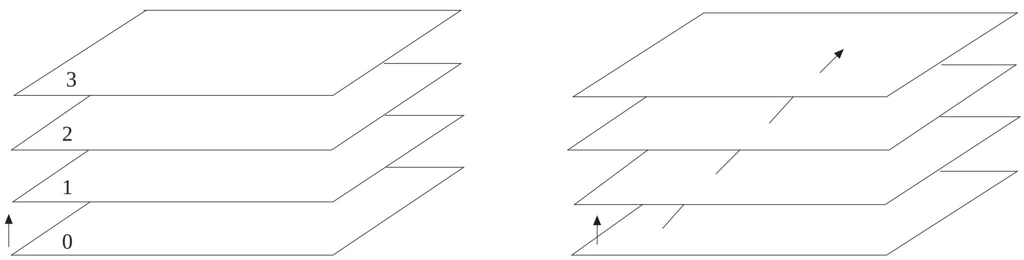

Figure 4. Family of parallel planes representing a one-form. Orientation depicted as the straight arrow. By counting pierced planes we ascribe a number to a vector.

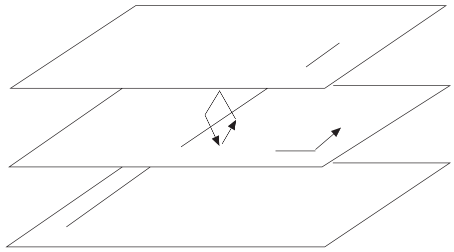

Figure 5. Family of parallel planes representing a pseudo-one-form. Orientation depicted as curved arrow on one of planes.

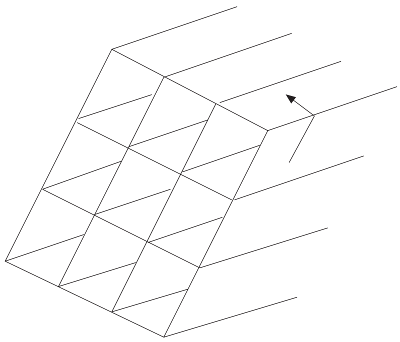

Figure 6. Family of parallel pipes representing two-form. Orientation depicted as a curved arrow around a pipe.

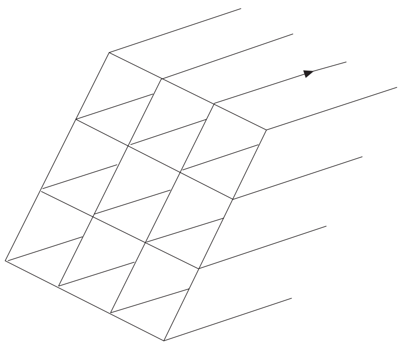

Figure 7. Family of parallel pipes representing two-form. Orientation depicted as a straight arrow along a pipe.

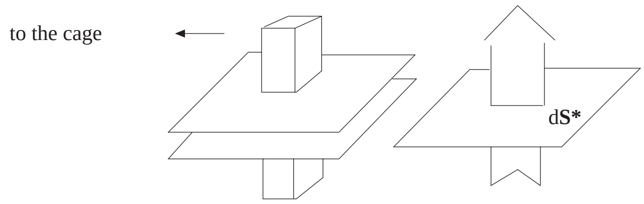

Figure 8. Operational definition of **D**. THe pseudobivector $d\mathbf{S*}$ corresponding to the disc.

In another place of the same book [13], p. 347, one may find an operational definition of the *magnetic field strength:* “Take a very small wireless solenoid prepared from a superconducting material. Close the circuit in a region of space where the magnetic field vanishes. Afterwards, introduce the circuit into an arbitrary region in the field. A superconductor has the property that the magnetic flux enclosed by it is always the same; a current will be induced to compensate for this external field flux. Now measure the current dI flowing through the superconductor. It turns out to be proportional to the solenoid length: dI=$\mathbf{H}.d\mathbf{l*}$”. The *solenoid length* $d\mathbf{l*}$ in this experiment is apparently a pseudovector, hence the *magnetic field strength* **H** is a pseudo-one-form.

To my knowledge, the above mentioned solenoid length is the only example of a physical quantity with the true pseudovector nature. Many other quantities hitherto called pseudovectors (like angular momentum, magnetic moment, magnetic induction, magnetic field strength) turn out to be-as mentioned previously - bivectors, pseudo-one-forms or two-forms.

Typical examples of geometric and electromagnetic quantities are added in Table 1.

Scalar product

The well known theorem, called Gram-Schmidt orthogonalization, sounds as follows. *For any scalar product in a vector, a basis exists which is orthonormal.* We present its inverse:

Theorem *For any basis in a vector space, a scalar product exists such that the basis is orthonormal.*

Proof. A scalar product in an n-dimensional vector space is a *bilinear form* which is symmetric, positive definite, and nondegenerate. When we have a basis $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n$, in our vector space, then we have, for all vectors **v, w,** the decomposition $\mathbf{v} = \sum_{i=1}^n v^i$ $\mathbf{m}_i$, $\mathbf{w} = \sum_{i=1}^n w^i \ \mathbf{m}_i, v^i, w^i \epsilon R$. It is easy to check that the expression

$$g_m(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{n} v^i w^i \qquad (1)$$

satisfies all the demanded properties. Thus (1) can be considered as a scalar product. It yields $(\mathbf{m}_i, \mathbf{m}_j)$=$\delta_{ij}$ which means that vectors $\mathbf{m}_i$ form an orthonormal basis. □

In this manner, we see that for any basis one can choose a scalar product such that it is an orthonormal basis. One should stress here that such a scalar product is strongly basis dependent. For different bases, different scalar products are obtained by this prescription.

We now agree that a scalar product, a norm and a metric determined by it are not unique for a given vector space. Therefore, there is a need to consider vector spaces that are devoid of a metric. The use of differential forms allows the formulation of the equations of electrodynamics in a way independent of any scalar product. The scalar product is necessary to formulate the constitutive equations relating **E** to **D** and to find the solutions.

After a (nondegenerate) scalar product is established in the linear space, a natural mapping of linear forms into vectors is easy to define. So, for a linear form **f**, there exists one and only one vector $\vec{\mathbf{f}}$ such that

$$\mathbf{f}(\mathbf{r}) = (\vec{\mathbf{f}}, \mathbf{r}) \quad for \quad each \quad vector \quad \mathbf{r}$$

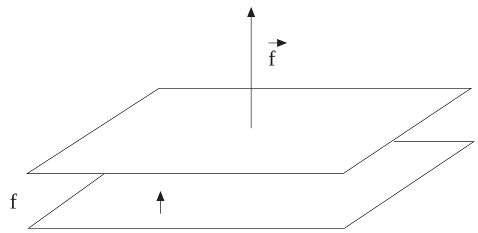

Figure 9. Replacement of a one-form **f** by a vector $\vec{f}$.

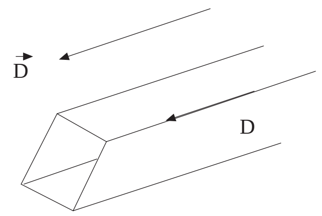

Figure 10. Replacement of a pseude-two-form **D** by a vector $\vec{\mathbf{D}}$.

where (.,.) denotes the scalar product. Of course, $\vec{\mathbf{f}}$ is perpendicular to the planes forming the attitude of **f;** see Figure 9. (Recall that the word "perpendicular" makes sense only when the scalar product is present.) Vector $\vec{\mathbf{f}}$ inherits its orientation from the form **f,** but its attitude is perpendicular to that of **f**.

We formulate the following prescription of changing a given pseudo-two-form **D** into vector $\vec{\mathbf{D}}$: take the direction of **D**, ascribe it to $\vec{\mathbf{D}}$ and use the magnitude of **D** to define the length of *vec***D**. This prescription is illustrated in Figure 10.

Similar prescriptions exist which allow us to replace all bivectors, pseudobivectors, forms and pseudoforms of grade one and two by vectors and pseudovectors. In this manner the number of directed quantities summarized in Table 1 is reduced from eight to two due to the presence of the scalar product.

Anisotropic medium

An anisotropic dielectric is characterized by its electric permittivity tensor ϵ. Let us introduce its square root η:

$$\eta^2 = \epsilon.$$

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a basis orthonormal with respect to ordinary scalar product. We introduce new basis $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$

$$\mathbf{m}_i = \sum_{j=1}^{3} \eta_{ij} \mathbf{e}_j$$

and a second scalar product g_m through prescription (1). We formulate the following

Statement. *For the scalar product (1) determined by the basis* $\mathbf{m}_i$ *the one-form* $\mathbf{E}$ *is perpendicular to the pseudo-two-form* $\mathbf{D}$.

Its proof can be found in Ref. [10]. It follows from this statement that the vectors $\vec{\mathbf{D}}$ and $\vec{\mathbf{E}}$ obtained from the respective forms $\mathbf{D}$ and $\mathbf{E}$ by the prescription described in previous Section are parallel. In this manner the new scalar product g_m and the metric determined by it ensures that the dielectric “looks like” an isotropic one.

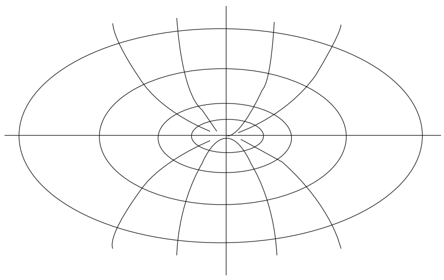

Figure 11. Equipotential surfaces for the Coulomb field.

After this observation one can easily solve counterparts of all electrostatic problems for the anisotropic dielectric medium. For instance, the electric potential ϕ of a single charge Q is

$$\phi(r) = \frac{Q}{4\pi\epsilon_0 (det\eta)|r|_m}$$

where $|r_m|^2 = g_m(r, r)$, see [10]. Such potential can be found in Ref. [16], p. 57. This Coulomb field has the “spherical” symmetry for “spheres” in the metric of g_m. This means that the equipotential surfaces are ellipsoids given by equations $|\mathbf{r}|_m$=const. These ellipsoids are depicted as ellipses in Figure 11. Also curved lines of vector field $\vec{\mathbf{E}}$ are shown in this Figure, if the vectors $\mathbf{E}$ are obtained from the one-form $\mathbf{E}$ through the ordinary scalar product. If, on the other side, we would pass to vector $\vec{\mathbf{E}}'$ with the use of scalar product g_m, the lines of this field would be straight, see Figure 12.

The electric induction vector $\vec{\mathbf{D}}$ (determined with the use of g_m) for the single charge Q is

$$\vec{\mathbf{D}}(r) = \frac{Qr}{4\pi |r|_m^3}.$$

The lines of this field are radially outgoing from the positive charge Q, see Figure 13. Thus the lines of vector field $\vec{\mathbf{D}}$ are straight independently of any scalar product.

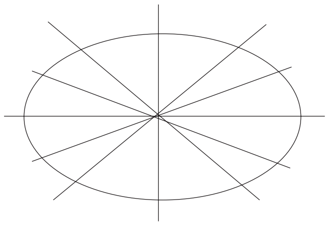

Figure 12. Field lines for the electric vector field in the metric of g_m.

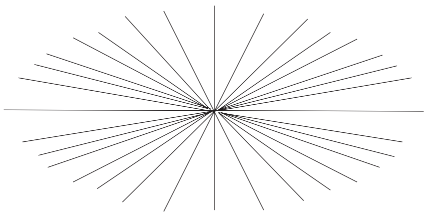

Figure 13. Field lines for the electric induction field of a single charge.

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